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CRITERIA FOR TOTAL PROJECTIVITY

PAUL HILL

1. Introduction. All groups herein are assumed to be abelian. It was not until the 1940's that it was known that a subgroup of an infinite direct sum of finite cyclic groups is again a direct sum of cyclics. This result rests on a general criterion due to Kulikov [**7**] for a primary abelian group to be a direct sum of cyclic groups. If G is p-primary, Kulikov's criterion presupposes that G has no elements (other than zero) having infinite p-height. For such a group G, the criterion is simply that G be the union of an ascending sequence of subgroups H_n where the heights of the elements of H_n computed in G are bounded by some positive integer $\lambda(n)$. The theory of abelian groups has now developed to the point that totally projective groups currently play much the same role, at least in the theory of torsion groups, that direct sums of cyclic groups and countable groups played in combination prior to the discovery of totally projective groups and their structure beginning with a paper by R. Nunke [**11**] in 1967. This paper itself is intended to make a contribution in that direction by providing useful criteria for a primary group to be totally projective. One of the results that we establish is completely analogous to Kulikov's criterion for a primary group to be a direct sum of cyclic groups. In fact, our result generalizes to groups of arbitrary length Theorem A in [**9**], which is called the generalized Kulikov criterion. Thus we generalize still further the Kulikov–Megibben criterion. Our main theorem, however, is the following: If the p-primary abelian group G is the set-theoretic union of a countable number of isotype subgroups G_n, then G is totally projective provided that G_n is totally projective for each n. If the above theorem sounds familiar, the explanation could be that it is exactly the same as a theorem proved by the author [**5**] except that the earlier theorem had the severe restriction that the length of the group G is countable. Incidentally, the applications given in Section 3 (including the generalization already mentioned of the Kulikov–Megibben criterion) should convince one that the present theorem is more than merely an aesthetic improvement over the earlier one although we stand by our statement in [**5**] that few applications exist for those groups of length $\lambda = \omega_1$. Most applications of the more general theorem are found not at ω_1 but beyond at those λ's that are cofinal with $\omega = \omega_0$. For example, one of our results is that G must be totally projective provided that: (i) $G = \bigcup_{n<\omega} H_n$ where $H_n \cap p^{\lambda(n)}G = 0$ with $\lambda(n) < \lambda$, and (ii) for each

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$\alpha < \lambda$ the quotient group $G/p^\alpha G$ is totally projective. Now, even though it is not explicitly so stated, the above conditions imply that G has length λ cofinal with ω; a typical case, for example, would be $\lambda = \omega_\omega > \omega_1$.

We refer to [**1**] for three different and very interesting characterizations of totally projective groups, however, the description of totally projective groups that we find most useful for this paper is the one given in [**2**], which is called the third axiom of countability. It is the following.

Axiom 3. An abelian p-group G has a collection $\mathscr{C}$ of nice subgroups satisfying the following conditions:

(0) $0 \in \mathscr{C}$.
(1) $\mathscr{C}$ is closed with respect to ascending unions.
(2) If $H \in \mathscr{C}$ and if A is any subgroup of G such that $\langle H, A\rangle/H$ is countable, there exists $B \in \mathscr{C}$ such that $B \supseteq \langle H, A\rangle$ and B/H is countable.

Recall that H is a *nice* subgroup of the p-group G if

$$p^\alpha(G/H) = \langle p^\alpha G, H\rangle/H \quad \text{for every ordinal } \alpha.$$

We remark that one formulation of Axiom 3 requires the collection $\mathscr{C}$ to be closed with respect to arbitrary group-theoretic unions not just ascending unions as required in (1) above, but it has been shown that these two versions of Axiom 3 turn out to be equivalent; see [**1**] or [**6**] for details.

2. Countable unions of totally projective groups. It is a well-known elementary result that every abelian group G is the union of a countable sequence of subgroups G_n where G_n is a direct sum of cyclic groups for each n. But it was proved in [**4**] that if the primary group G is the set-theoretic union of a countable number of pure subgroups G_n, then G must be a direct sum of cyclic groups if G_n is for each n. Prufer discovered early the virtue of purity; the subgroup H of a p-group G is pure if $H \cap p^nG = p^nH$ for all $n < \omega$. If the more general condition $H \cap p^\alpha G = p^\alpha H$ is satisfied for all α, then H is said to be isotype in G. More and more, it is becoming evident that the isotype property, being stronger, ought substantially to replace that of purity in the structure theory of primary groups of arbitrary length. As we shall see, the following property, which is abbreviated SUIT for "set-theoretic union of isotype and totally projectives", determines to a large extent the structure of primary groups.

Property SUIT. The group G is the set-theoretic union of isotype and totally projective subgroups.

If G is the set-theoretic union of a countable number of isotype and totally projective subgroups, then G is said to be a countable SUIT. In this language our first theorem would state simply that if a primary group is a countable SUIT then it must be totally projective.

THEOREM 1. *Let the p-group G be the set-theoretic union of a countable collection of isotype subgroups G_n. If G_n is totally projective for each n, then G is totally projective.*

Before beginning the proof of Theorem 1, we establish some notation that will be standard throughout. Adopting the hypothesis of Theorem 1, we let $\mathscr{C}_n$ be a collection of nice subgroups of G_n that satisfies conditions (0)–(2) of Axiom 3. For a subgroup H of G, we consider the following two conditions:

$$\text{(a) } H \cap G_n \in \mathscr{C}_n \quad \text{for each } n < \omega,$$

and

$$\text{(b) } p^\alpha(G/H) \cap \langle G_n, H\rangle/H = p^\alpha(\langle G_n, H\rangle/H) \quad \text{for each } \alpha.$$

The major portion of the proof of Theorem 1 is contained in the next two lemmas.

LEMMA 1. *If the subgroup H of G satisfies condition* (a) *and condition* (b) *for $\alpha < \lambda$, then $p^\alpha(G/H) = \langle p^\alpha G, H\rangle/H$ for $\alpha \leqq \lambda$. In particular, conditions* (a) *and* (b) *(for all α) imply that H is nice in G.*

Proof. Since $H \cap G_n \in \mathscr{C}_n$, we have that

$$p^\alpha(G_n/H \cap G_n) = \langle p^\alpha G_n, H \cap G_n\rangle/H \cap G_n \quad \text{for all } \alpha.$$

Thus

$$p^\alpha(\langle G_n, H\rangle/H) = \phi(p^\alpha(G_n/H \cap G_n))$$
$$= \phi(\langle p^\alpha G_n, G_n \cap H\rangle/H \cap G_n) = \langle p^\alpha G_n, H\rangle/H,$$

where ϕ is the natural isomorphism from $G_n/H \cap G_n$ onto $\langle G_n, H\rangle/H$. Assume that $p^\alpha(G/H) = \langle p^\alpha G, H\rangle/H$ fails for some ordinal $\alpha \leqq \lambda$, and let α be the first such ordinal. Then α is necessarily a limit ordinal. Hence condition (b) is satisfied for α (whether $\alpha < \lambda$ or $\alpha = \lambda$), which implies that

$$p^\alpha(G/H) \cap \langle G_n, H\rangle/H = p^\alpha(\langle G_n, H\rangle/H) = \langle p^\alpha G_n, H\rangle/H.$$

Therefore,

$$p^\alpha(G/H) \subseteq \cup_{n<\omega} \langle p^\alpha G_n, H\rangle/H \subseteq \langle p^\alpha G, H\rangle/H.$$

Since $\langle p^\alpha G, H\rangle/H \subseteq p^\alpha(G/H)$, we have shown that

$$p^\alpha(G/H) = \langle p^\alpha G, H\rangle/H \quad \text{for all } \alpha \leqq \lambda.$$

In the following lemma, we assume the hypothesis of Theorem 1. The preceding notation and terminology are employed in the proof, but we

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need one further definition. An ordinal α is said to be *relevant* for a subgroup H of G if there exists an element in H that has height precisely α in G.

LEMMA 2. *Let A be any countable subgroup of G. There exists a countable subgroup B of G containing A that satisfies conditions* (*a*) *and* (*b*).

Proof. It should be clear that A has a countable extension B that satisfies condition (a) alone. In fact, for a fixed n, this is the essence of condition (2) of Axiom 3 when applied to the group G_n and the collection $\mathscr{C}_n$. Moreover, we can pass from a fixed n to all $n < \omega$ by a simple application of the back-and-forth procedure that has now become a standard technique for infinite abelian groups [**1**]. Thus we shall assume, without loss of generality, that A already satisfies condition (a).

For each element $a \in A$, for each $n < \omega$, and for each ordinal α that is relevant for A in G such that $a \in \langle G_n, p^{\alpha+1}G\rangle$, we choose a single representative $z(a, n, \alpha) \in p^{\alpha+1}G$ that satisfies the relation

$$\text{(R)} \qquad a - z(a, n, \alpha) \in G_n.$$

Observe that there are only a countable number of the representatives $z(a, n, \alpha)$ because A is countable. Set $A_0 = A$. Letting A_1 be a countable extension of $\langle A_0, z(a, n, \alpha)\rangle$ such that $A_1 \cap G_n \in \mathscr{C}_n$ for each n, letting A_1 replace A_0, and letting A_{i+1}, in turn, replace A_i, we define $B = \bigcup_{i<\omega} A_i$.

Obviously, B (when substituted for H) satisfies condition (a) since A_i satisfies condition (a) for each i. Thus we are concerned only with showing that B satisfies condition (b), that is, we need to prove only that $\langle G_n, B\rangle/B$ is isotype in G/B. Toward this endeavor, we first need to observe an important property of B. If $a \in B$ and if $a \in \langle G_n, p^{\alpha+1}G\rangle$ where α is relevant for the subgroup B of G, then the relation (R) is satisfied for some element $z(a, n, \alpha) \in B \cap p^{\alpha+1}G$. The point is simply that we can choose k so that $a \in A_k$ and α is relevant for the subgroup A_k. The desired element $z(a, n, \alpha)$ exists in $A_{k+1} \subseteq B$.

Now, assume that B does not satisfy condition (b) and let $\alpha + 1$ be the smallest ordinal for which it fails (for some n); obviously the failure must occur first at an isolated ordinal. Lemma 1 implies, by the choice of α, that

$$p^{\alpha+1}(G/B) = \langle p^{\alpha+1}G, B\rangle/B.$$

Moreover, due to the choice of α, we have that

$$p^{\alpha}(G/B) \cap \langle G_n, B\rangle/B = p^{\alpha}(\langle G_n, B\rangle/B) = \langle p^{\alpha}G_n, B\rangle/B.$$

Let

$$g_n + B \in p^{\alpha+1}(G/B) \cap \langle G_n, B\rangle/B$$

where $g_n \in G_n$. From the preceding equations, we obtain

$$g_n \in \langle p^{\alpha+1}G, B\rangle \cap \langle p^\alpha G_n, B\rangle.$$

Write

$$g^{\alpha+1} + b_1 = g_n = g_n{}^\alpha + b_2$$

where $g^{\alpha+1} \in p^{\alpha+1}G$, $g_n{}^\alpha \in p^\alpha G_n$, and $b_1, b_2 \in B$. If $b_2 - b_1 \in p^{\alpha+1}G$, then

$$g_n{}^\alpha \in G_n \cap p^{\alpha+1}G = p^{\alpha+1}G_n \quad \text{and}$$

$$g_n + B \in \langle p^{\alpha+1}G_n, B\rangle/B = p^{\alpha+1}(\langle G_n, B\rangle/B).$$

Furthermore, if $b_2 - b_1 \notin p^{\alpha+1}G$, then the height of $b_2 - b_1$ in G is precisely α. This means that α is relevant for the subgroup B of G. From our earlier observation, we know that there exists $z = z(b_1, n, \alpha) \in B \cap p^{\alpha+1}G$ such that $b_1 - z \in G_n$ since $b_1 \in \langle G_n, p^{\alpha+1}G\rangle$. Letting $g_n{}' = b_1 - z$, we have that

$$g_n - g_n{}' = g^{\alpha+1} + z \in p^{\alpha+1}G \cap G_n = p^{\alpha+1}G_n.$$

Hence,

$$g_n + B = g_n - g_n{}' + B \in p^{\alpha+1}(\langle G_n, B\rangle/B).$$

In either case, we have shown that

$$g_n + B \in p^{\alpha+1}(\langle G_n, B\rangle/B)$$

and that

$$p^{\alpha+1}(G/B) \cap \langle G_n, B\rangle/B \subseteq p^{\alpha+1}(\langle G_n, B\rangle/B).$$

Since the above inclusion is tantamount to equality, Lemma 2 is proved.

Proof of Theorem 1. We shall prove that G satisfies Axiom 3. As before, let $\mathscr{C}_n$ be a collection of nice subgroups of G_n that satisfies conditions (0)–(2) of Axiom 3. Define $\mathscr{C}$ to be the collection of subgroups H of G satisfying conditions

(a) $H \cap G_n \in \mathscr{C}_n$ for each $n < \omega$,

and

(b) $p^\alpha(G/H) \cap \langle G_n, H\rangle/H = p^\alpha(\langle G_n, H\rangle/H)$ for each α.

The members of $\mathscr{C}$ are nice subgroups of G by Lemma 1. We propose to show that $\mathscr{C}$ satisfies conditions (0)–(2) of Axiom 3. Trivially, $0 \in \mathscr{C}$ since $0 \in \mathscr{C}_n$ and since G_n is isotype in G for each n. In order to show that $\mathscr{C}$ satisfies condition (1), let H be the union of an ascending chain

$$H_1 \subseteq H_2 \subseteq \ldots \subseteq H_\lambda \subseteq \ldots$$

of subgroups H_λ belonging to $\mathscr{C}$. The verification that H satisfies con-

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dition (a) is elementary since $\mathscr{C}_n$ is closed with respect to ascending unions for each n. The proof that H satisfies (b) is indirect. Suppose that (b) fails for the first time at the ordinal α. As we have already observed, α must be isolated, and $p^\alpha(G/H) = \langle p^\alpha G, H\rangle/H$ by Lemma 1. Therefore,

$$p^\alpha(G/H) \cap \langle G_n, H\rangle/H = \langle p^\alpha G, H\rangle/H \cap \langle G_n, H\rangle/H.$$

Letting $g + H \in p^\alpha(G/H) \cap \langle G_n, H\rangle/H$, we can write

$$g^\alpha + h_1 = g = g_n + h_2,$$

where $g^\alpha \in p^\alpha G$, $g_n \in G_n$, and $h_1, h_2 \in H$. If we choose H_λ so that h_1 and h_2 are both contained in H_λ, then we conclude that

$$g + H_\lambda \in p^\alpha(G/H_\lambda) \cap \langle G_n, H_\lambda\rangle/H_\lambda = p^\alpha(\langle G_n, H_\lambda\rangle/H_\lambda) = \langle p^\alpha G_n, H_\lambda\rangle/H_\lambda.$$

Hence $g + H \in \langle p^\alpha G_n, H\rangle/H$, and H must satisfy condition (b) for all α. This completes the argument that $\mathscr{C}$ satisfies condition (1) of Axiom 3. Finally, we need to show that $\mathscr{C}$ satisfies condition (2) of Axiom 3, the countable extension property.

Let $H \in \mathscr{C}$ and let A be a countable subgroup of G. We make two crucial observations:

(i) $\langle G_n, H\rangle/H \cong G_n/H \cap G_n$ is totally projective since $H \cap G_n \in \mathscr{C}_n$ by condition (a).

(ii) $\langle G_n, H\rangle/H$ is isotype in G/H by condition (b).

Therefore, in view of (i) and (ii), we can apply Lemma 2 to the subgroup $\langle A, H\rangle/H$ of G/H, where the collection

$$\mathscr{C}_n(H) = \{\langle X, H\rangle/H\colon X \in \mathscr{C}_n \text{ with } X \supseteq H \cap G_n\}$$

replaces the collection $\mathscr{C}_n$. Note that $\langle X, H\rangle/H$ is nice in $\langle G_n, H\rangle/H$ since it corresponds to the nice subgroup $X/H \cap G_n$ of $G_n/H \cap G_n$. According to Lemma 2, there exists an extension B of H containing A such that B satisfies conditions (a) and (b) and such that B/H is countable. We should remark that the conditions (a) and (b) for G and for G/H correspond naturally. We have verified that the collection $\mathscr{C}$ satisfies the countable extension property (2), and this completes the proof of Theorem 1.

3. Generalizations of the Kulikov–Megibben criterion. A p-primary group G is called a C_λ-group if $G/p^\alpha G$ is totally projective for each $\alpha < \lambda$. The basic properties of C_λ-groups have been developed by Megibben [**10**]. In order to be totally projective it is absolutely necessary for G to be a C_λ-group where λ is the length of the group G. The fact that the converse is true for a summable group of countable length λ is the Kulikov–Megibben criterion, which is Theorem A in [**9**]. An equivalent

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version of the Kulikov–Megibben criterion that does not specifically require the group to be summable is the following.

Kulikov–Megibben Criterion. Suppose that the p-primary group G has countable length λ and is a C_λ-group. If G is the union of an ascending sequence of subgroups H_n where the heights of the elements of H_n computed in G are bounded by some ordinal $\lambda(n)$ less than λ, then G is totally projective.

We believe that we have given the preferred form of the Kulikov–Megibben criterion for the following reason. Megibben considered the conditions imposed in the first sentence of the preceding criterion to be gratuitous properties in the original Kulikov criterion, where $\lambda = \omega$. Furthermore, the second sentence of the above criterion is an obvious and natural extension of Kulikov's condition stated in our opening paragraph.

The Kulikov–Megibben criterion has now been generalized even further by Linton and Megibben [**8**]. Recall that an ordinal λ is cofinal with ω if λ is the limit of a countable number of smaller ordinals. The criterion due to Linton and Megibben is equivalent to the following.

Linton–Megibben Criterion. Suppose that the p-primary group G has length λ cofinal with ω and is a C_λ-group. If G is the union of an ascending sequence of subgroups H_n where the heights of the elements of H_n computed in G are bounded by some ordinal $\lambda(n) < \lambda$, then G is totally projective.

Our purpose and contribution here is to go one step further in improving the Linton–Megibben criterion. We do not require the subgroups H_n to ascend. The hypotheses imply that the length of G is cofinal with ω.

THEOREM 2. *Suppose that the p-primary group G has length λ and is a C_λ-group. If G is the set-theoretic union of a countable number of subgroups H_n where the heights of the elements of H_n computed in G are bounded by some ordinal $\lambda(n) < \lambda$, then G is totally projective.*

Proof. Let $G = \bigcup_{n<\omega} H_n$ where H_n is isotype of length $\lambda(n) < \lambda$ for each n. If G has countable length, then H_n must be totally projective because

$$H_n \cong \langle H_n, p^{\lambda(n)}G\rangle / p^{\lambda(n)}G$$

is isomorphic to an isotype subgroup of a totally projective group $G/p^{\lambda(n)}G$ having countable length $\lambda(n)$. Thus G is totally projective by Theorem 1 (or by Theorem 1 in [**5**]) if λ is countable. Likewise, if the segment (β, λ) of ordinals is countable for some β less than λ, then $p^\beta G$ is totally projective because $p^\beta G$ has countable length and

$$p^\beta G = \bigcup_{n<\omega} p^\beta H_n$$

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with $p^\beta H_n$ isotype in $p^\beta G$. Therefore, if (β, λ) is countable for some $\beta < \lambda$, the group $p^\beta G$ must be totally projective. Moreover, $G/p^\beta G$ is totally projective by hypothesis since G is a C_λ-group. It follows that G is totally projective [**11**], and the theorem is proved for the case (β, λ) is countable for some $\beta < \lambda$.

We have shown that in the proof of Theorem 2 we may assume that (β, λ) is uncountable for every ordinal β less than the length λ of G. In particular, we may assume that $\beta + \omega < \lambda$ if $\beta < \lambda$. Furthermore, there is no loss of generality in assuming that H_n is maximal with respect to having trivial intersection with $p^{\lambda(n)}G$. Thus we shall suppose that H_n is a $p^{\lambda(n)}$-high subgroup of G. The key point here is that we may assume that $\lambda(n) = \mu(n) + \omega$ for some ordinal $\mu(n)$, for the original $\lambda(n)$ can be replaced by $\lambda(n) + \omega$ and still be less than λ. Since $\lambda(n) = \mu(n) + \omega$ and since H_n is $p^{\lambda(n)}$-high in G, we know that

$$G = \langle H_n, p^{\mu(n)}G\rangle$$

and that

$$G/p^{\mu(n)}G \cong H_n/p^{\mu(n)}H_n \quad \text{for each } n.$$

Therefore, $H_n/p^{\mu(n)}H_n$ is totally projective, but $p^{\mu(n)}H_n$ is also totally projective since it is isomorphic to an isotype subgroup of the totally projective group $p^{\mu(n)}G/p^{\lambda(n)}G$. Hence, H_n is totally projective, and Theorem 1 implies that G is totally projective.

The next result seems interesting, but we do not know if it has any value in its own right or not. It is obviously equivalent to Theorem 1.

THEOREM 3. *If G is any p-group, let $\mathscr{T}(G)$ be the collection of all isotype and totally projective subgroups of G. If H is a subgroup of G then either H belongs to the collection $\mathscr{T}(G)$ or else is so far removed from it that it is not the set-theoretic union of a countable number of subgroups belonging to $\mathscr{T}(G)$.*

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Auburn University,
Auburn, Alabama

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