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ON A SUM OF DIVISORS

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ABSTRACT. Let $l(N, r)$ be the minimum number of terms needed to express r as a sum of distinct divisors of N. Let $l(N) = \max\{l(N, r) : 1 \leq r \leq N\}$. Then for Vose's sequence $\{N_k\}$, $l(N_k) \asymp \sqrt{\log N_k}$, improving the result of M. Vose.

1. **Introduction.** For N a positive integer, we denote by $l(N, r)$ the minimum number of terms needed to express r as a sum of distinct divisors of N. Let $l(N) = \max\{l(N, r) : 1 \leq r \leq N\}$. Then it is not hard to see that $l(N)$ is defined for all N with the property $d_{j+1} \leq \sum_{i=1}^{j} d_i + 1$, where $1 = d_1 < d_2 < \cdots < d_\tau = N$ are the divisors of N. For those N having the above property, we are interested in the behavior of $l(N)$. Note that if $l(N)$ is defined, then $l(N) \leq \log N / \log 2$. First question arises here is the existence of N satisfying $l(N) = o(\log N)$. Erdős [1, 2] answered this by showing for $N = n!$

$$l(N) = l(n!) \leq n = O(\log N / \log_2 N)$$

and conjectured

$$l(n!) = O(\log_2 n!).$$

Furthermore, he asked the existence of N satisfying

$$l(N) = o(\log N / \log_2 N).$$

Vose [5] answered the latter question by constructing a sequence $\{N_k\}$ of positive integers satisfying

$$l(N_k) = O(\sqrt{\log N_k}),$$

and currently this is the best bound known for all sequences $\{N_k\}$ of positive integers. Tenenbaum and the author [4] were able to show that for $N = n!$

$$l(N) = l(n!) = n/(\log n)^{\frac{1}{2}-\varepsilon} = o(\log N / \log_2 N).$$

In this article, we first characterize a necessary condition for N to have $l(N) = o(\log N / \log_2 N)$ and then show the bound $l(N_k) = O(\sqrt{\log N_k})$ for the Vose's sequence $\{N_k\}$ can not be improved.

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2. **Main theorems.** Before stating results, we establish some notation and terminology. We use the standard notations $f = O(g)$ and $f \ll g$ to mean that $|f| < Cg$ for some positive constant C. The expression $f = o(g)$ means that $f/g \longrightarrow 0$, and $f \asymp g$ means that f is of the same order of magnitude as g. As usual, we let $\log_j$ denote the jth-fold iterated logarithm. For real x, $1 < x < N$, we let $d^-(x)$ and $d^+(x)$ be consecutive divisors of N such that $d^-(x) \leq x < d^+(x)$.

By Vose's sequence $\{N_k\}$, we mean by the following: $N_k = 2^{2\alpha k^2} \prod_{l=2}^{k} p_l^2$, where $p_2 < p_3 < \cdots < p_k$ are odd primes such that

(1) $\max_{\sqrt{N_{k-1}}<i<\sqrt{N_k}} \big(\log d^+(i) - \log d^-(i)\big) \ll (2/3)^k$

(2) $\log p_l \asymp l$

(3) α may be any sufficiently large integer.

THEOREM 1. *Let $\{N_k\}$ be a sequence of positive integers satisfying the following conditions:*

1) $N_1 = 1$, $N_k | N_{k+1}$, $(k = 1, 2, \ldots)$, $\log_2 N_k \ll \log k$

2) $\max_{1<i\leq\sqrt{N_{k_0}}} \Big(1 - \frac{d^-(i)}{d^+(i)}\Big) \leq \frac{1}{2}$

3) $\max_{\sqrt{N_k}<i\leq\sqrt{N_{k+1}}} \big(d^+(i) - i\big) \leq i\varepsilon_k$ *for* $k \geq k_0$, *where* $\varepsilon_k = \exp\{-(\log k)^\beta\}$ *with* $0 < \beta < \log_2 N_k / (\log_3 N_k - \log 2)$.

Then

$$l(N_k) \ll \exp\{\log_2 N_k - \beta(\log_3 N_k - \log 2)\}.$$

In Theorem 1, the upper bound of $l(N)$ is heavily dependent on the existence of a divisor of N in the small interval near $\sqrt{N}$. On the other hand, in the following theorem, we obtain the lower bound of $l(N)$ in terms of the number of divisors of N.

THEOREM 2. *For all N that defines $l(N)$,*

$$l(N) \gg \frac{\log N}{\log \tau(N)} \left(1 + \frac{\log_2 N - \log_2 \tau(N)}{2 \log \tau(N)}\right).$$

With Theorem 1, 2, we can show that the bound $l(N_k) = O(\sqrt{\log N_k})$ for Vose's sequence $\{N_k\}$ can not be improved. In other words, the algorithm used to obtain the upper bound of $l(N_k)$ where $\{N_k\}$ is Vose's sequence is best possible.

COROLLARY 1. *Let $\{N_k\}$ be Vose's sequence. Then*

$$l(N_k) \asymp \sqrt{\log N_k}.$$

3. **Proof of theorems.** We start with the proof of Theorem 1. Let r be an integer such that $1 \le r \le N_k$. We will construct a strictly decreasing sequence $d_1 > d_2 > \cdots > d_m$ of divisors of N_k such that $r = \sum_{i=1}^{m} d_i$. Put $r = r_0$, $r_j = r - \sum_{i=1}^{j} d_i$ $(j \ge 1)$. Let $Z = \sqrt{N_{k_0+1}}$. Then r lies in one of the intervals of the form $(1, Z]$, $(Z, \sqrt{N_k}]$, $(\sqrt{N_k}, N_k/Z]$, $(N_k/Z, N_k]$. We will show that $r_b \le N_k/Z$ with $b \ll 1$. Suppose that $r_0 > N_k/Z$. Otherwise put $b = 0$. Let d_1 be the largest divisor of N_k not exceeding r_0. Then by condition (2),

$$r_1 = r_0 - d_1 \le d_1,$$

and the equality is only possible if r_1 itself is a divisor of N_k. If $r_1 < d_1$, we iterate this procedure and obtain

$$r_b = r - \sum_{i=1}^{b} d_i \le \frac{N_k}{Z}.$$

Note that $r_b = r_{b-1} - d_b \le d_b$ and $b \ll 1$.

We will show that $r_h \le \sqrt{N_k}$ with $h - b \ll \exp\{\log_2 N_k - \beta(\log_3 N_k - \log 2)\}$. Suppose that $r_b > \sqrt{N_k}$. Otherwise put $h = b$. Since $Z < N_k/r_b < \sqrt{N_k}$, we let m_1 be the unique integer such that

$$\sqrt{N_{m_1}} < \frac{N_k}{r_b} \le \sqrt{N_{m_1+1}}.$$

Note that $k_0 \le m_1 < k$. Now by condition (3), we have

$$d^+\Big(\frac{N_k}{r_b}\Big) \le \frac{N_k}{r_b}(1 + \varepsilon_{m_1}).$$

Thus

$$(1 - \varepsilon_{m_1})d^+\Big(\frac{N_k}{r_b}\Big) < \frac{N_k}{r_b} < d^+\Big(\frac{N_k}{r_b}\Big).$$

Set $d_{b+1} = N_k/d^+(N_k/r_b)$. Then we see that d_{b+1} is a divisor of N_k since $d^+(N_k/r_b) | N_{m_1} | N_k$ by condition (1). Now

$$\begin{aligned} 0 \le r_{b+1} &= r - \sum_{i=1}^{b+1} d_i = r_b - d_{b+1} \\ &\le r_b - r_b(1 - \varepsilon_{m_1}) \\ &= r_b \varepsilon_{m_1}. \end{aligned}$$

Note that $d_{b+1} < r_b \le d_b$. If $r_{b+1} \le \sqrt{N_k}$, we put $h = b + 1$, otherwise we repeat the application of condition (3) and produce

$$d^+\Big(\frac{N_k}{r_{b+1}}\Big) \le \frac{N_k}{r_{b+1}}(1 + \varepsilon_{m_2})$$

with $d_{b+2} = N_k/d^+(N_k/r_{b+1})$ and $0 \le r_{b+2} = r_{b+1} - d_{b+2} \le r_{b+1}\varepsilon_{m_2} \le r_b \varepsilon_{m_1}\varepsilon_{m_2}$ for some m_2 such that $\sqrt{N_{m_2+1}} > N_k/r_{b+1} > r_b/r_{b+1} > 1/\varepsilon_{m_1}$. Since $\log_2 \sqrt{N_{m_2+1}} \ll \log m_2$ by condition (1), we have

$$\varepsilon_{m_2} = \exp\{-(\log m_2)^\beta\} \le \exp\Big\{-\Big(\frac{1}{c}\log_2\Big(\frac{1}{\varepsilon_{m_1}}\Big)\Big)^\beta\Big\}$$

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for some positive constant c. Moreover $d_{b+2} < d_{b+1}$ for $r_{b+1} - d_{b+1} \leq -(1 - 2\varepsilon_{m_1})r_b < 0 \leq r_{b+1} - d_{b+2}$. Iterating the procedure, we eventually obtain $r_h \leq \sqrt{N_k}$. Since $r_{b+j} \leq r_b \varepsilon_{m_1} \varepsilon_{m_2} \cdots \varepsilon_{m_j}$ and $k_0 \leq m_1 \leq m_2 \leq \cdots \leq m_j \leq k$, we estimate h by using the inequality

$$r_{b+j} \leq r_b \gamma_j,$$

where $\gamma_j = \varepsilon_{m_1} \varepsilon_{m_2} \cdots \varepsilon_{m_j}$ satisfies

$$\gamma_{j+1} \leq \gamma_j \exp\Big\{-\Big(\frac{1}{c} \log_2\Big(\frac{1}{\gamma_2}\Big)\Big)^\beta\Big\}$$

for some positive constant c. A simple computation yields $\log(1/\gamma_j) \gg j(\log j)^\beta$. We note that since $r_b < N_k$, $r_{b+j} \leq \sqrt{N_k}$ provided $\log(1/\gamma_j) \geq \log N_k/2$. Now let

$$j_0 = \exp\{\log_2 N_k - \beta(\log_3 N_k - \log 2)\}.$$

Then

$$\log j_0 + \beta \log_2 j_0 = \log_2 N_k - \beta(\log_3 N_k - \log 2) + \beta \log_3 N_k + \log\Big(1 - \frac{\beta(\log_3 N_k - \log 2)}{\log_2 N_k}\Big).$$

Since $0 < \beta < \log_2 N_k/(\log_3 N_k - \log 2)$ by condition (3), we have

$$\log j_0 + \beta \log_2 j_0 \geq \log_2 N_k.$$

Thus we have $r_{b+j} \leq \sqrt{N_k}$ with $j \ll \exp\{\log_2 N_k - \beta(\log_3 N_k - \log 2)\}$.

We now show similarly that $r_{h+1} \leq Z$ holds with $l \ll \exp\{\log_2 N_k - \beta(\log_3 N_k - \log 2)\}$. Let q_1 be defined by $\sqrt{N_{q_1}} < r_h \leq \sqrt{N_{q_1+1}}$. Then $q_1 < k$. If $q_1 \leq k_0$, then we are done, so suppose otherwise. Then we can apply condition (3) to obtain

$$0 \leq r_{h+1} = r_h - d_{h+1} \leq r_h \varepsilon_{q_1},$$

for some divisor d_{h+1} of $N_{q_1} | N_k$. As before $d_{h+1} < d_h$ since $r_h - d_h < 0$. Iterating, we obtain

$$r_{h+j} = r_h - \sum_{i=h+1}^{h+j} d_i \leq r_h \varepsilon_{q_1} \varepsilon_{q_2} \cdots \varepsilon_{q_j},$$

where $k > q_1 \geq q_2 \geq \cdots \geq q_j \geq k_0$. Suppose that l can be defined by the condition

$$r_h \varepsilon_{q_1} \cdots \varepsilon_{q_l} < Z \leq r_h \varepsilon_{q_1} \cdots \varepsilon_{q_{l-1}}.$$

We reindex q_i according to increasing size by letting $\varepsilon_{q_i} = \varepsilon_{p_{l-i+1}}$. Let $\delta_j = \varepsilon_{p_1} \varepsilon_{p_2} \cdots \varepsilon_{p_j}$, where $k_0 \leq p_1 \leq p_2 \leq \cdots \leq p_j \cdots \leq p_l < k$. Then

$$\begin{aligned}\delta_{j+1} &= \delta_j \varepsilon_{p_{j+1}} \\ &= \delta_j \exp\{-(\log q_{l-j})^\beta\} \\ &\leq \delta_j \exp\{-(\log q_{l-j+1})^\beta\} \\ &\leq \delta_j \exp\Big\{-\Big(\frac{1}{c} \log_2 N_{q_{l-j+1}}\Big)^\beta\Big\}\end{aligned}$$

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for some positive constant c. Since $\sqrt{N_{q_{l-j+1}}} > r_{h+l-j}$ and

$$r_{h+l} \leq r_{h+l-j}\varepsilon_{q_{l-j+1}}\varepsilon_{q_{l-j+2}}\cdots\varepsilon_{q_l} = r_{h+l-j}\varepsilon_{p_j}\cdots\varepsilon_{p_2}\varepsilon_{p_1},$$

we have

$$\delta_{j+1} \leq \delta_j \exp\Big\{-\Big(\frac{1}{c}\log_2\Big(\frac{1}{\delta_j}\Big)\Big)^{\beta}\Big\}$$

for some positive constant c. As above we have $\log(1/\delta_j) \gg j(\log j)^{\beta}$. Since $0 < \beta < \log_2 N_k/(\log_3 N_k - \log 2)$ by condition (3), we have $r_{h+l} \leq Z$ with $l \ll \exp\{\log_2 N_k - \beta(\log_3 N_k - \log 2)\}$.

It remains to show that $r_m = 0$ with $m - (h + l) \ll 1$. Since $r_{h+l} < Z$ and $d_{h+l} > r_{h+l}$, we let d_{h+l+1} be the largest divisor of N_k not exceeding r_{h+l}. Then by condition (2), we have

$$r_{h+l+1} = r_{h+l} - d_{h+l+1} \leq d_{h+l+1} < r_{h+l} < d_{h+l}.$$

We iterate this procedure and obtain in a finite number of steps

$$r_m = r_{h+l} - \sum_{i=h+l+1}^{m} d_i = 0.$$

Thus $m - (h + l) \ll 1$. Therefore

$$l(N_k) \ll \exp\{\log_2 N_k - \beta(\log_3 N_k - \log 2)\}.$$

PROOF OF THEOREM 2. Let $1 = d_1 < d_2 < \cdots < d_{\tau(N)} = N$ be the divisors of N. Then the number of distinct subset sums

$$S(N) := \operatorname{card}\Big\{\sum_{i=1}^{\tau(N)} d_i\varepsilon_i : \varepsilon_i = 0 \text{ or } 1\Big\}$$

is at most $2^{\tau(N)}$. Since $2^{\tau(N)} \geq N$, we must have $\tau(N) \geq \log N/\log 2$. Now let $\sigma(i, N)$ denote the number of distinct subset sums of i distinct divisors of N whose sum is less than N. Suppose that we can express all $r, 1 \leq r \leq N$, as a sum of at most m distinct divisors of N. Then the maximum number of distinct subset sums we can obtain by using at most m distinct divisors of N whose sum is less than N is

$$\sum_{i=1}^{m}\sigma(i, N).$$

Since $\sigma(i, N) \leq \binom{\tau(N)}{i}$ for all $i = 1, 2, \ldots, N$, we have

$$\sum_{i=1}^{m}\binom{\tau(N)}{i} \geq \sum_{i=1}^{m}\sigma(i, N) \geq N.$$

Note that we can assume $\tau(N) > 3m - 1$, otherwise $m \gg \log N$ and there is nothing to prove. Then

$$\sum_{i=1}^{m}\binom{\tau(N)}{i} \leq 2\binom{\tau(N)}{m}.$$

Thus

$$2\binom{\tau(N)}{m} \geq N.$$

Now

$$\begin{aligned}\binom{\tau(N)}{m} &= \frac{\tau(N)\big(\tau(N)-1\big)\cdots\big(\tau(N)-m+1\big)}{m!}\\ &\leq \frac{\big(\tau(N)\big)^m}{m!}\\ &\leq \left(\frac{e\tau(N)}{m}\right)^m \frac{1}{\sqrt{2\pi m}}.\end{aligned}$$

Thus we have

$$\left(\frac{e\tau(N)}{m}\right)^m \geq \frac{N\sqrt{2\pi m}}{2} \geq N.$$

Let

$$m_0 = \frac{\log N}{\log \tau(N)}\left(1+\frac{\log_2 N - \log_2 \tau(N)}{2\log \tau(N)}\right).$$

Then

$$\begin{aligned}&m_0\big(\log\tau(N) - \log m_0 + 1\big)\\ &\quad= \frac{\log N}{\log\tau(N)}\left(1+\frac{\log_2 N-\log_2\tau(N)}{2\log\tau(N)}\right)\big(\log\tau(N)-\log m_0+1\big)\\ &\quad\leq \log N\left(1+\frac{\log_2 N-\log_2\tau(N)}{2\log\tau(N)}\right)\left(1-\frac{\log_2 N-\log_2\tau(N)-1}{\log\tau(N)}\right)\\ &\quad< \log N.\end{aligned}$$

Thus

$$m \gg \frac{\log N}{\log \tau(N)}\left(1+\frac{\log_2 N - \log_2 \tau(N)}{2\log \tau(N)}\right).$$

Therefore

$$l(N) \gg \frac{\log N}{\log \tau(N)}\left(1+\frac{\log_2 N - \log_2 \tau(N)}{2\log \tau(N)}\right).$$

PROOF OF COROLLARY. Since $N_k = 2^{2\alpha k^2}\prod_{l=2}^k p_l^2$, where $p_2 < p_3 < \cdots < p_k$ are odd primes such that

(1) $\max_{\sqrt{N_{k-1}}<i<\sqrt{N_k}}\big(\log d^+(i) - \log d^-(i)\big) \ll (2/3)^k$

(2) $\log p_l \asymp l$

(3) α may be any sufficiently large integer,

we have

$$\log\sqrt{N_k} = \alpha k^2 \log 2 + \sum_{l=2}^k \log p_l \leq 2\alpha k^2 \log 2$$

for sufficiently large α. Thus

$$\sqrt{N_k} \leq 2^{2\alpha k^2}.$$

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Therefore $\max_{1\le i\le\sqrt{N_k}}\big(1-d^-(i)/d^+(i)\big)\le 1/2$ is satisfied by the divisors of $2^{2\alpha k^2}$. We also have

$$2\alpha k^2 \le \log N_k \le 4\alpha k^2.$$

Thus

$$(\log_2 N_k - \log 4\alpha)/2 \le \log k \le (\log_2 N_k - \log 2\alpha)/2.$$

Now by (1)

$$\log\Big(\frac{d^+(i)}{d^-(i)}\Big) \ll \Big(\frac{2}{3}\Big)^k,$$

where $\sqrt{N_{k-1}} < i < \sqrt{N_k}$. Since $-\log x \ge 1-x$, we have

$$1-\frac{i}{d^+(i)} \le \log\Big(\frac{d^+(i)}{i}\Big) \ll \Big(\frac{2}{3}\Big)^k$$

which implies that for some constant c

$$\frac{d^+(i)}{i}-1 \le \frac{c2^k}{3^k-c2^k}.$$

Let $\varepsilon_k = \exp\{-(\log k)^\beta\} = c2^k/(3^k-c2^k)$. Then $(\log k)^\beta = k\log(3/2)+\log\big(1-c(2/3)^k\big)-\log c$. Thus

$$\beta = \frac{\log k + \log\Big(\log(3/2)+\big(\log\big(1-c(2/3)^k\big)-\log c\big)/k\Big)}{\log_2 k}.$$

Since $(\log_2 N_k - \log 4\alpha)/2 \le \log k \le (\log_2 N_k - \log 2\alpha)/2$, we have

$$\beta \ge \frac{\log_2 N_k - \log 4\alpha + \log\Big(\log(3/2)+\big(\log\big(1-c(2/3)^k\big)-\log c\big)/k\Big)}{2(\log_3 N_k - \log 2 + \log(1-\log 2\alpha/\log_2 N_k)}$$

and

$$\beta \le \frac{\log_2 N_k - \log 2\alpha + \log\Big(\log(3/2)+\big(\log\big(1-c(2/3)^k\big)-\log c\big)/k\Big)}{2(\log_3 N_k - \log 2 + \log(1-\log 4\alpha/\log_2 N_k)},$$

yielding

$$\frac{\log_2 N_k - 2\log 4\alpha}{2(\log_3 N_k - \log 2)} \le \beta \le \frac{\log_2 N_k - \log 2\alpha}{2(\log_3 N_k - \log 2 - 3\log 4\alpha/2\log_2 N_k)}$$
$$\le \frac{\log_2 N_k}{2(\log_3 N_k - \log 2)}.$$

Since $N_k | N_{k+1}$, we may apply Theorem 1 to obtain

$$\begin{aligned} l(N_k) &\ll \exp\{\log_2 N_k - \beta(\log_3 N_k - \log 2)\} \\ &\le \exp\Big\{\frac{1}{2}\log_2 N_k + \log 4\alpha\Big\} \\ &\ll \sqrt{\log N_k}. \end{aligned}$$

On the other hand,

$$\log \tau(N_k) = \log\big((2\alpha k^2 + 1)3^k\big) \asymp k$$

and $k \asymp \sqrt{\log N_k}$. Thus by Theorem 2, we have

$$l(N_k) \gg \sqrt{\log N_k}.$$

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