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A Note on Finite Dehn Fillings

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Abstract. Let M be a compact, connected, orientable 3-manifold whose boundary is a torus and whose interior admits a complete hyperbolic metric of finite volume. In this paper we show that if the minimal Culler-Shalen norm of a non-zero class in $H_1(\partial M)$ is larger than 8, then the finite surgery conjecture holds for M. This means that there are at most 5 Dehn fillings of M which can yield manifolds having cyclic or finite fundamental groups and the distance between any slopes yielding such manifolds is at most 3.

1 Introduction

Throughout this note M will denote a compact, connected, orientable 3-manifold whose boundary is a torus and whose interior admits a complete hyperbolic metric of finite volume.

Let r be a *slope* on ∂M, *i.e.*, the isotopy class of an unoriented, essential, simple closed curve on ∂M, and denote by $M(r)$ the manifold obtained by Dehn filling M with a solid torus so that a meridian of the solid torus has slope r. Recall that the distance $\Delta(r_1, r_2)$ between two slopes r_1, r_2 on ∂M is defined to be their minimal geometric intersection number (on ∂M). The following conjecture was first raised in [G].

Finite Surgery Conjecture

There are at most 5 Dehn fillings of M which can yield manifolds having cyclic or finite fundamental groups and the distance between any slopes yielding such manifolds is at most 3.

The distance 3 and number 5 in the conjecture are the best possible bounds that one can expect in that both of them are realized by slopes on the boundary of the figure 8 sister manifold, as was discovered by Jeff Weeks [W] (see [BZ1, Example 10.5]).

Though the conjecture still remains open, much progress has been made toward its solution ([BH], [BZ1]). Indeed in [BZ1] it is shown that there are at most six Dehn fillings of M which can yield manifolds having cyclic or finite fundamental groups and the distance between any slopes yielding such manifolds is at most 5. These bounds are obtained through the use of the Culler-Shalen norm

$$\|\cdot\|\colon H_1(\partial M;\mathbf{R}) \to [0,\infty)$$

which was introduced in [CGLS] and defined through the use of the canonical algebraic curve, in the $SL_2(\mathbf{C})$-character variety of M, which arises from the hyperbolic structure on $\mathrm{int}(M)$.

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Denote by V the 2-dimensional vector space $H_1(\partial M;\mathbf{R})$ and by L the lattice $H_1(\partial M;\mathbf{Z})$ in V. It turns out that the values of $\|\cdot\|$ on L lie in $\mathbf{Z}$ [1]. Set

$$s(M) = \min\{\|\alpha\| \mid \alpha \in L \setminus \{0\}\} \in \mathbf{Z}.$$

The purpose of this note is to prove that the finite surgery conjecture holds for manifolds M with a relatively large $s(M)$.

Theorem 1 *If the minimal Culler-Shalen norm $s(M)$ is larger than 8, then the finite surgery conjecture holds for M.*

It is well-known that the hyperbolic structure on $\operatorname{int}(M)$ is determined up to isometry by its fundamental group, and since $s(M)$ is a measure of the number of homomorphisms of $\pi_1(M)$ to $\mathrm{SL}_2(\mathbf{C})$, it is in this sense a measure of the complexity of the hyperbolic structure on $\operatorname{int}(M)$. Seen in this light the theorem tells us that hyperbolic manifolds which are sufficiently complex satisfy the conjecture. An example of this behaviour is provided in [BMZ], where it is calculated that for the exteriors M_n of those twist knots $K_n \subset S^3$ which are hyperbolic (*i.e.*, $n \neq 0, 1$), $s(M_n) = 2|n|$ for $n \leq -1$ and $s(M_n) = 2n - 1$ for $n \geq 2$. So in particular $s(M_n) > 8$ for all n with $|n| > 5$. It is an interesting problem to understand the topology of the manifolds M with $s(M) \leq 8$. We suspect that such manifolds are less complex topologically; for instance their unknotting tunnel numbers might be bounded above by a small positive integer.

Proof of Theorem 1 First of all we shall fix our notation and describe some concepts.

Set $B = \{v \in V \mid \|v\| \leq s(M)\}$, the $\|\cdot\|$-disc of radius $s(M)$ in V. Note then that by the definition of $s(M)$, $L \cap \operatorname{int}(B) = \{0\}$ while $L \cap B \neq \{0\}$. For each $k \in (0,\infty)$, kB will denote the disc in V of radius $ks(M)$. Fix a homology class $\theta \in L$ (necessarily primitive) such that $\theta \in \partial B$. Now fix any $\tau \in L$ such that θ and τ form a basis of L.

We identify the pair (V, L) with $(\mathbf{R}^2, \mathbf{Z}^2)$ by setting $\theta \equiv (1, 0)$ and $\tau \equiv (0, 1)$. Any class in L of the form $(m, 1)$ is called an *integral class*. A *pair of elements* in V is a $\pm$ couple $\{(a, b), (-a, -b)\}$. In the usual way a slope on ∂M determines and is determined by a pair of primitive elements of L. If r_1 and r_2 correspond to $(p_1, q_1), (p_2, q_2) \in \mathbf{Z}^2$, then

$$\Delta(r_1, r_2) = |p_1 q_2 - p_2 q_1|.$$

When $\alpha \in L$ is one of the primitive classes corresponding to a slope r, we shall use $M(\alpha)$ to denote $M(r)$.

A slope r on ∂M is called a *boundary slope* if there exists an incompressible, ∂-incompressible, properly embedded surface F in M whose boundary is nonempty and consists of curves of slope r. It is called a *strict boundary slope* if such a surface can be found which is not a fibre in a fibration of M over S^1. A *(strict) boundary class* is a primitive element of L whose corresponding slope is a (strict) boundary slope.

We shall call a primitive element $\alpha \in L$ a *finite* (respectively *cyclic*) *filling class* if $M(\alpha)$ has a finite (respectively cyclic) fundamental group. The finite groups which can arise as

[1] In [CGLS] and [BZ1] the $\mathrm{SL}_2(\mathbf{C})$ version of the norm was used. Though equivalent to the $\mathrm{PSL}_2(\mathbf{C})$ version we shall use here (see [BZ2]), a few of its basic properties have a slightly different form. For instance the values of the restriction of the $\mathrm{SL}_2(\mathbf{C})$ norm to L lie in $2\mathbf{Z}$.

the fundamental groups of closed, orientable 3-manifolds lie in six families [Mi] which are referred to in [BZ1] as C-type, T-type, O-type, I-type, D-type, and Q-type groups. If $\alpha \in L$ is a primitive class such that the fundamental group of $M(\alpha)$ is of one of the types listed above, then we shall say that α is a class of that type.

The proof of Theorem 1 depends on the basic properties of the norm given in the following lemma.

Lemma 2

(i) [CGLS, Proposition 1.1.2] The disc B is a finite-sided, convex polygon whose vertices are rational multiples of strict boundary classes in L. Furthermore B is balanced, i.e., $-B = B$.

(ii) [BZ1, Lemma 6.4] If $(a, b) \in B$ then $|b| \leq 2$. If there is some $(a, b) \in B$ with $b = 2$, then $(a, b) \in L$ and B is a parallelogram with vertices $\pm(1, 0)$ and $\pm(a, b)$.

(iii) [CGLS, Corollary 1.1.4] If $\alpha = (m, n) \in L$ is a cyclic filling class but is not a strict boundary class, then $\|(m, n)\| = s(M)$. Hence $\alpha \in \partial B$ but is not a vertex of B.

(iv) [BZ2, Theorem 6.2] If $\alpha = (m, n) \in L$ is a D-type or Q-type class but is not a strict boundary class, then $\|\alpha\| \leq 2s(M)$ and further $\|(m, n)\| \leq \|(m + 2k, n + 2j)\|$ for any element $(k, j) \in L$.

(v) [BZ2, Theorem 6.2] If $\alpha = (m, n) \in L$ is a C, T, O, or I-type filling class which is not a strict boundary class, then $\|\alpha\| \leq s(M) + 4$. ■

We begin by using Lemma 2 to prove a lemma concerning the τ coordinate of a finite or cyclic filling class.

Lemma 3 *Suppose that $s(M) > 8$.*

(i) If $\alpha = (j, k)$, $k \geq 0$, is a finite or cyclic filling class which is not a strict boundary class, then $k \leq 2$.

(ii) If $\alpha = (j, 2)$ is a finite or cyclic filling class which is not a strict boundary class, then α is neither a D-type nor a Q-type class.

(iii) There is at most one $\alpha \in L$ having τ coordinate equal to 2 which is a finite or cyclic filling class but which is not a strict boundary class.

Proof (i) As a first case assume that α is neither a D-type nor a Q-type filling class. According to part (v) of Lemma 2, the inequality $s(M) > 8$ implies that $\|\alpha\| < \frac{3}{2}s(M)$. Then α lies in the interior of $\frac{3}{2}B$ and so by Lemma 2(ii) we have $k < 3$.

Now suppose that α is either a D-type or Q-type filling class. From Lemma 2(iv) we have $\alpha \in 2B$ and thus part (ii) of this Lemma gives $k \leq 3$ and if $k = 3$, then B must contain points whose τ coordinates are greater than 1. Hence assuming that $\alpha = (j, 3)$, then from the definition of B we can find integral classes $(m, 1)$ and $(m + 1, 1)$ of L which are not contained in int(B) but for which there are points $(a, 1) \in B$ with $m < a < m + 1$. By parts (ii) and (iv) of Lemma 2, we see that

$$\frac{3}{2}s(M) < \|(j, 3)\| \leq \max\{\|(m, 1)\|, \|(m + 1, 1)\|\}.$$

But if one of $\|(m, 1)\|$ and $\|(m + 1, 1)\|$ is at least $\frac{3}{2}s(M)$, then the highest τ coordinate of a point in B is at most $\frac{4}{3}$. To see this suppose, for concreteness, that $\|(m, 1)\| \geq \frac{3}{2}s(M)$. Then

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there is a $\lambda \in (0, \frac{2}{3}]$ such that $\lambda\|(m,1)\| = s(M)$. Let A denote the band in V bounded by the lines $\gamma_0 = \{t(m,1) \mid t \in \mathbf{R}\}$ and $\gamma_1 = \{t(m,1) + (1,0) \mid t \in \mathbf{R}\}$. Since $-\theta$ and $\lambda(m,1)$ lie on ∂B, the convexity of B implies that $B \cap A$ is bounded above by the line through $-\theta$ and $\frac{2}{3}(m,1)$. Therefore by our choice of m, the largest τ coordinates of points in $2B$ is at most $\frac{8}{3} < 3$. But this contradicts our assumption that $\alpha = (j,3) \in 2B$. Thus part (i) of the lemma holds.

(ii) There is an integer m such that $\alpha = (2m+1,2)$ and hence as θ is congruent to $(2m+1,2)$ modulo 2, part (iv) of Lemma 2 implies that $\|(2m+1,2)\| = s(M)$, *i.e.*, $(2m+1,2) \in \partial B$. But then part (ii) of this lemma implies that $(2m+1,2)$ is a strict boundary slope, contrary to our assumptions.

(iii) If we suppose that both $(2m+1,2)$ and $(2n+1,2)$ are either finite or cyclic filling classes, then the inequality $s(M) > 8$ combines with part (v) of Lemma 2 to give $\|(2m+1,2)\|, \|(2n+1,2)\| < \frac{3}{2}s(M)$. Hence the convexity of B shows that the cone on the horizontal line segment from $\frac{2}{3}(2m+1,2)$ to $\frac{2}{3}(2n+1,2)$ is contained in the interior of B. But this is impossible as it implies that $\mathrm{int}(B)$ contains an integral class. Thus there is at most one finite or cyclic filling class with τ coordinate equal to 2. ■

Lemma 4 *Suppose that $s(M) > 8$. Then there are at most 5 primitive classes in L (up to the sign) which are either finite or cyclic filling classes but not strict boundary classes. Furthermore there is an integer m such that the collection of such classes having non-negative τ coordinate consists of a subset of either*

(i) $(1,0)$, $(m,1)$, $(m+1,1)$, $(2m+1,2)$ *and one of* $(m-1,1)$, $(m+2,1)$*; or*
(ii) $(1,0)$, $(m-1,1)$, $(m,1)$, $(m+1,1)$, *and* $(m+2,1)$.

Thus the maximal mutual distance between two classes of the set is at most 3.

Proof Let $\mathcal{F}$ denote the set of all finite or cyclic filling classes which are not strict boundary classes and which have non-negative τ coordinate. Recall from Lemma 3 that no finite or cyclic filling class has a τ coordinate larger than 2.

If there exists a class $\alpha \in \mathcal{F}$ of the form $(2m+1,2)$, then Lemma 3 and Lemma 2(v) imply that $(2m+1,2)$ is contained in the interior of $\frac{3}{2}B$. Thus the convexity of B implies that the interior of the triangle with vertices $-\frac{3}{2}\theta = (-\frac{3}{2},0)$, $\frac{3}{2}\theta = (\frac{3}{2},0)$, and $(2m+1,2)$ lies in the interior of $\frac{3}{2}B$ and so $(m,1), (m+1,1) \in \mathrm{int}(\frac{3}{2}B)$. If there is a D-type or Q-type filling class in $\mathcal{F}$, then Lemma 2(iv) implies that its norm is bounded above by $\|(1,0)\| = s(M)$, $\|(m,1)\|$, or $\|(m+1,1)\|$, and hence strictly bounded above by $\frac{3}{2}s(M)$. Thus $\mathcal{F}$ is a subset of $\mathrm{int}(\frac{3}{2}B)$.

Next observe that by our choice of θ, Lemma 2(i) implies that any horizontal line (*i.e.*, $\tau =$ constant) in V intersects B in a segment of length no more than 2 with respect to the Euclidean metric on $V \equiv \mathbf{R}^2$, and thus $\frac{3}{2}B \cap \{y=1\}$ has length at most 3. Hence besides $(m,1)$ and $(m+1,1)$, $\frac{3}{2}B \cap \{\tau = 1\}$ can contain only one of the classes $(m-1,1)$ and $(m+2,1)$ in its interior. Since no finite or cyclic filling class has a τ coordinate larger than 2, it is easy to see that possibility (i) of the lemma holds.

Assume then that there is no element of $\mathcal{F}$ has a τ coordinate larger than 1. If $\mathcal{F} \subset \mathrm{int}(\frac{3}{2}B)$ then the argument from the previous paragraph shows that there is an integer m such that $\mathcal{F} \subseteq \{\theta, (m-1,1), (m,1), (m+1,1)\}$ and we are done. Hence we shall assume

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otherwise. According to Lemma 2(iv) and (v) we may then choose a D or Q-type class $\alpha_1 \in \mathcal{F}$ whose norm satisfies

$$(1) \qquad \frac{3}{2}s(M) \leq \|\alpha_1\| = \max\{\|\alpha\| \mid \alpha \in \mathcal{F}\} \leq 2s(M).$$

As a first case suppose that there is an integral class α_0 such that $\|\alpha_0\| < \|\alpha_1\|$. Choose $\epsilon \in \{\pm 1\}$ such that $\alpha_0 + \epsilon\theta$ lies on the segment $[\alpha_0, \alpha_1]$. From Lemma 2(iv) we see that the inequality $\|\alpha_0\| < \|\alpha_1\|$ implies $\alpha_1 \not\equiv \alpha_0 \pmod{2L}$. The same result now shows us that

$$(2) \qquad \|\alpha_0\| < \|\alpha_1\| \leq \|\alpha_0 + \epsilon\theta\|.$$

But then from our choice of ϵ and the properties of a norm, the function $t \mapsto \|\alpha_0 + t\epsilon\theta\|$ is strictly increasing for $t \geq 1$. This implies that $\alpha_1 = \alpha_0 + \epsilon\theta$ (see Inequality (2)) and $\alpha_0 + k\epsilon\theta \notin \mathcal{F}$ when $k \geq 2$ (see Equation (1)). Another application of Lemma 2(iv) gives $\|\alpha_0 - \epsilon\theta\| \geq \|\alpha_1\| > \|\alpha_0\|$ and a similar argument shows that $\alpha_0 + k\epsilon\theta \notin \mathcal{F}$ when $k \leq -2$. Thus possibility (i) of the lemma holds.

As a final case assume that $\|\alpha_1\| = \min\{\|\alpha\| \mid \alpha \text{ is integral}\}$ and choose integral classes $\beta_1, \beta_1 + k\theta \in \mathcal{F}$, $k \geq 0$, such that $\mathcal{F} \cap \{\tau = 1\} \subseteq \{\beta_1, \beta_1 + \theta, \ldots, \beta_1 + k\theta\}$. Our hypotheses on α_1 imply that

$$(3) \qquad \|\alpha_1\| = \|\beta_1\| = \|\beta_1 + \theta\| = \cdots = \|\beta_1 + k\theta\| \leq 2s(M)$$

Thus $\mathcal{F} \subseteq \left(\frac{\|\alpha_1\|}{s(M)}\right)B \subseteq 2B$.

Now any horizontal line intersects $\left(\frac{\|\alpha_1\|}{s(M)}\right)B$ in a segment of length less than or equal to $2\left(\frac{\|\alpha_1\|}{s(M)}\right) \leq 4$. If this length, for the line $\tau = 1$, is less than 4, then there is an integer m such that $\mathcal{F} \cap \{\tau = 1\} \subseteq \{(m-1,1), (m,1), (m+1,1), (m+2,1)\}$ and possibility (ii) of the lemma arises. On the other hand if the length is 4 then $\|\alpha_1\| = 2s(M)$ and either possibility (ii) holds or $k = 4$ and $\mathcal{F} \cap \{\tau = 1\} = \{\beta_1, \beta_1 + \theta, \beta_1 + 2\theta, \beta_1 + 3\theta, \beta_2 = \beta_1 + 4\theta\}$. Assume that the latter case arises and observe that Relation (3) implies $\partial B \cap \{\tau = \frac{1}{2}\}$ contains the line segment $[\beta_1/2, \beta_1/2 + 2\theta]$. Since this segment has length 2 and $-\theta, \theta \in \partial B$, the convexity and balanced nature of B (Lemma 2(i)) implies that B is a parallelogram with vertices $\pm\beta_1/2, \pm(\beta_1/2 + 2\theta)$. In particular the class $\beta_1 \in \mathcal{F}$ is a strict boundary slope (Lemma 2(i)), contrary to the definition of $\mathcal{F}$. Thus this last case does not arise and so the lemma is proved. ■

Lemma 5 **([BZ1, Lemma 6.7])** *If $\alpha \in L$ is a strict boundary class which is also either a finite or cyclic filling class, then $\Delta(\alpha, \beta) \leq 1$ for any finite or cyclic filling class β.* ■

Lemma 6 *Suppose that there is only one slope on ∂M which is at the same time a strict boundary slope and finite or cyclic filling slope. Also suppose that $s(M) > 8$. Then there are at most five finite or cyclic filling slopes on ∂M and their mutual distance is at most* 3.

Proof It follows from Lemmas 4 and 5 that under the conditions of the lemma, the distance between any two finite or cyclic filling slopes is bounded above by 3. Thus we need only verify that there are no more than five such slopes.

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Suppose that $\alpha = (j, k)$, $k \geq 0$, is the unique class (up to sign) which is a finite or cyclic filling class and is also a strict boundary class. According to Lemma 4 there is an integer m such that the rest of the finite or cyclic filling classes are contained amongst a collection of elements of L of the form

(a) $(1, 0), (m, 1), (m + 1, 1), (2m + 1, 2)$ and one of $(m - 1, 1), (m + 2, 1)$; or
(b) $(1, 0), (m - 1, 1), (m, 1), (m + 1, 1)$, and $(m + 2, 1)$.

We may certainly assume that α is not one of these classes. Considering the cases $k = 1, k = 2, k \geq 3$ it is now a simple matter to verify that there are no more than 4 elements amongst those listed in (a) and (b) above that are of distance 1 from $\alpha = (j, k)$. Hence appealing to Lemma 5 we see that the lemma has been proved. ■

***Lemma 7* ([BZ1, Lemma 7.1])** *Suppose that there are at least two slopes on ∂M which are at the same time strict boundary slopes and finite or cyclic filling slopes. Then there are no more than four slopes on ∂M which are finite or cyclic filling slopes, and the distance between any two such slopes is at most* 2. ■

The proof of Theorem 1 now follows from Lemmas 4, 6 and 7.

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