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A MULTIPLE LINKING MINIMAX PRINCIPLE

DUMITRU MOTREANU

The aim of this paper is three-fold: to fill the gap between different deformation lemmas, to obtain a unifying minimax result where multiple linking situations can occur, and to locate the critical points as solutions of minimisation problems.

1. A UNIFORM DEFORMATION LEMMA

This section is devoted to a sharp version of the deformation lemma due to Du [4] with the following specific features:

(a) the functional $I \in C^1(X,\mathbb{R})$ and the disjoint closed sets A and B are permitted to satisfy

$$I|_B \leqslant c \text{ and } I|_A \geqslant c - \varepsilon$$

for some $c \in \mathbb{R}$ and $\varepsilon > 0$;

(b) the deformation involved has a uniform character with respect to A, and in some sense also relative to B.

Before proceeding we list some notation used in the sequel. For a continuously differentiable functional $I \in C^1(X,\mathbb{R})$ on a real Banach space X and a number $c \in \mathbb{R}$ we denote

$$I_c := \{x \in X : I(x) \leqslant c\} \quad \text{and} \quad I^c := \{x \in X : I(x) \geqslant c\}.$$

The notation I' stands for the differential of I, while $K_c(I)$ designates the critical points of I (that is, the elements of X where I' vanishes) at the level $c \in \mathbb{R}$, so

$$K_c(I) := \{x \in X : I'(x) = 0 \text{ and } I(x) = c\}.$$

If $K_c(I) \neq \emptyset$ the number c is called a critical value of I. The functional $I \in C^1(X,\mathbb{R})$ is said to satisfy the Palais-Smale condition (briefly (PS)) if every sequence (x_n) in X with $I(x_n)$ bounded and $I'(x_n) \longrightarrow 0$ contains a convergent subsequence. If we replace

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in the preceding formulation the boundedness of $I(x_n)$ with $I(x_n) \longrightarrow c$ we say that I satisfies the Palais–Smale condition at the level $c \in \mathbb{R}$ (briefly $(\mathrm{PS})_c$). We quote from Ghoussoub and Preiss [**5**] that $I \in C^1(X,\mathbb{R})$ verifies the Palais–Smale condition around a set $F \subset X$ at the level $c \in \mathbb{R}$ (in short $(\mathrm{PS})_{F,c}$) if every sequence (x_n) in X satisfying $dist(x_n, F) \longrightarrow 0$, $I(x_n) \longrightarrow c$ and $I'(x_n) \longrightarrow 0$ has a convergent subsequence. We further recall that a pseudo–gradient vector field of $I \in C^1(X,\mathbb{R})$ (in short p.g.) means a locally Lipschitz mapping V from the set $\mathcal{R}(I) := \{x \in X : I'(x) \neq 0\}$ of regular points of I into X satisfying on its domain the relations

$$\|V(x)\| \leqslant 2\,\|I'(x)\| \text{ and } I'(x)V(x) \geqslant \|I'(x)\|^2 . \tag{1.1}$$

The existence of such V is well known (see, for example, [**6, 10, 11, 14, 15**]). For any set $S \subset X$ and any number $\varepsilon > 0$ the notation $N_\varepsilon(S)$ represents the (closed) ε–neighbourhood of S in X, that is, $N_\varepsilon(S) := \{x \in X : dist(x,S) \leqslant \varepsilon\}$.

LEMMA 1.1. *Let a functional $I \in C^1(X,\mathbb{R})$ on the Banach space X, a p.g. V of I, a number $c \in \mathbb{R}$ and a closed subset B of X be given such that $(PS)_{B,c}$, $B \cap K_c(I) = \emptyset$ and $B \subset I_c$. Then for every $\overline{\varepsilon} > 0$ there exists an $\varepsilon \in (0,\overline{\varepsilon})$ and a $\delta < c$ such that for each closed subset A of X with $A \subset I^{c-\varepsilon}$ and $A \cap B = \emptyset$ there is a homotopy $\eta_A \in C(\mathbb{R} \times X, X)$ with the properties below*

- (i) *$\eta_A(\cdot,x)$ is the solution of the vector field $V_A = -\varphi_A \min(1,1/\|V\|)V$ with the initial condition $x \in X$ for some locally Lipschitz function $\varphi_A : X \longrightarrow [0,1]$ whose support is contained in $\mathcal{R}(I) \cap (X \setminus A)$;*
- (ii) *$\eta_A(t,x) = x$ for all $t \in \mathbb{R}$ and $x \in A \cup I_{c-\overline{\varepsilon}} \cup I^{c+\overline{\varepsilon}}$;*
- (iii) *for every $\delta \leqslant d \leqslant c$ one has $\eta_A(1, B \cap I_d) \subset I_{d-\varepsilon}$.*

PROOF: The argument is rather standard following the lines in Du [4] (see also Ding [**3**]). Condition $(\mathrm{PS})_{B,c}$ implies the existence of numbers $0 < \delta_1 \leqslant 1$, $\varepsilon_1 > 0$, $\sigma_1 > 0$ such that

$$\|I'(x)\| \geqslant \sigma_1 \quad \text{whenever} \quad x \in N_{3\delta_1}(B) \cap I_{c+3\varepsilon_1} \cap I^{c-3\varepsilon_1}. \tag{1.2}$$

We claim that the lemma holds for every $\varepsilon > 0$ with

$$\varepsilon < \min\{\overline{\varepsilon}, \varepsilon_1, \frac{1}{2}\delta_1 \min(1, 2\sigma_1)\sigma_1\}. \tag{1.3}$$

In order to check the claim let us fix two locally Lipschitz functions φ and ψ from X to $[0,1]$ satisfying

$$\begin{aligned}
\varphi = 1 &\quad \text{on} \quad N_{\delta_1}(B) \cap I_{c+\varepsilon_1} \cap I^{c-\varepsilon_1},\\
\varphi = 0 &\quad \text{on} \quad X \setminus N_{2\delta_1}(B) \cap I_{c+2\varepsilon_1} \cap I^{c-2\varepsilon_1},\\
\psi = 0 &\quad \text{on} \quad I_{c-\overline{\varepsilon}} \cup I^{c+\overline{\varepsilon}},\\
\psi = 1 &\quad \text{on} \quad I_{c+\varepsilon_0} \cap I^{c-\varepsilon_0},
\end{aligned}$$

for some number ε_0 with

$$\varepsilon < \varepsilon_0 < \min(\bar{\varepsilon}, \varepsilon_1). \tag{1.4}$$

The vector field $-\delta_1\varphi\psi \min(1, 1/\|V\|)V$ is well defined on X (being considered 0 outside $\mathcal{R}(I)$ in view of the definition of φ and (1.2)), locally Lipschitz and bounded. Consequently, its global flow $\eta(t,x)$ exists throughout for $(t,x) \in \mathbb{R} \times X$. It is readily seen that

$$B_1 := \eta([0,1] \times B) \text{ is a closed subset in } X. \tag{1.5}$$

Let now $A \subset X$ be a set as in the statement of lemma. Then

$$A \cap B_1 = \emptyset. \tag{1.6}$$

For otherwise we would find $0 < t_0 = t_0(A) \leqslant 1$ and $x_0 = x_0(A) \in B$ with $\eta(t_0, x_0) \in A$. Then one can write

$$c - \varepsilon \leqslant I(\eta(t_0, x_0)) \leqslant I(\eta(t, x_0)) \leqslant I(x_0) \leqslant c, \quad t \in [0, t_0]. \tag{1.7}$$

Hence

$$\eta(t, x_0) \in N_{\delta_1}(B) \cap I_c \cap I^{c-\varepsilon}, \quad t \in [0, t_0].$$

Then from (1.1) and (1.3) we have the estimate

$$\begin{aligned} I(\eta(t_0,x_0)) - I(x_0) \\ = -\delta_1 \int_0^{t_0} \min\left(1, \frac{1}{\|V(\eta(t,x_0))\|}\right) I'(\eta(t,x_0))V(\eta(t,x_0))dt < -\varepsilon. \end{aligned} \tag{1.8}$$

The contradiction between (1.7) and (1.8) proves the claim (1.6). Taking into account (1.5) and (1.6) there is a locally Lipschitz function $\psi_A : X \longrightarrow [0,1]$ such that $\psi_A = 0$ on a neighbourhood of A and $\psi_A = 1$ on B_1. Finally we define the homotopy $\eta_A : \mathbb{R} \times X \longrightarrow X$ as being the global flow of the vector field $V_A = -\varphi_A \min(1, 1/\|V\|)V$ where $\varphi_A = \delta_1 \psi_A \varphi \psi$. The assertions (i) and (ii) are clear from the construction of η_A. We show that (iii) is valid for $\delta = c + \varepsilon - \varepsilon_0$ with ε described by (1.3) and ε_0 by (1.4). To this end we argue by contradiction. Suppose that for some $d \in [\delta, c]$ there exists $x \in B \cap I_d$ such that

$$I(\eta_A(1,x)) > d - \varepsilon. \tag{1.9}$$

It is straightforward to deduce

$$\eta_A(t,x) = \eta(t,x) \in N_{\delta_1}(B) \cap I_d \cap I^{d-\varepsilon}, \quad t \in [0,1].$$

Then by reasoning similar to (1.8) replacing t_0, x_0 by $1, x$, respectively, leads to

$$I(\eta_A(t,x)) - I(x) < -\varepsilon.$$

The contradiction with (1.9) completes the proof. $\square$

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REMARK 1.1. Lemma 1.1. unifies different types of deformation results such as, for example, the classical deformation lemmas (see Corollaries 1.1 and 1.2) and those initiated by Du (see Corollaries 1.3 and 1.4). This question has been raised in Du [4] (as well as in Ding [3]) where one makes use of a proper deformation lemma according to the specific situations. Below we get some known deformation results as special cases of Lemma 1.1.

PROPOSITION 1.1. (Proposition 2.1 in Rabinowitz [14].) *If c is not a critical value of $I \in C^1(X,\mathbb{R})$ satisfying (PS), given any $\overline{\varepsilon} > 0$ there exists an $\varepsilon \in (0,\overline{\varepsilon})$ and $\eta \in C([0,1] \times X, X)$ such that*

$1°$ $\eta(1,x) = x$ *if* $x \in I_{c-\overline{\varepsilon}} \cup I^{c+\overline{\varepsilon}}$,

$2°$ $\eta(1, I_{c+\varepsilon}) \subset I_{c-\varepsilon}$.

PROOF: Fix a positive number $\overline{a} < \overline{\varepsilon}$ such that the interval $[c-\overline{a}, c+\overline{a}]$ contains no critical values of I. We apply Lemma 1.1 with $B_a = I_{c+a} \cap I^{c-a}$ and $c+a$ in place of B and c, for each $a \in (0,\overline{a}]$. Lemma 1.1 yields $\varepsilon_a > 0$, $\delta_a < c+a$ and, with $A = I^{c+\overline{\varepsilon}}$, the homotopy $\eta_a \in C(R \times X, X)$ satisfying the requirements (i)–(iii) for $\varepsilon, \delta, \eta_A$ replaced by $\varepsilon_a, \delta_a, \eta_a$, respectively. Then $1°$ follows from (ii) of Lemma 1.1. The examination of relations (1.2),(1.3) shows that ε_a is bounded away from zero, say $\varepsilon_a \geqslant \widetilde{\varepsilon} > 0$ for $a \in (0,\overline{a}]$. Then we set $d = c + \min(a, \widetilde{\varepsilon})/2$. If $a > 0$ is small enough, d is an admissible value in (iii) of Lemma 1.1 relative to η_a, that is $\delta_a \leqslant d \leqslant c+a$. This happens because ε_0 entering (1.4) can be chosen independently of $a \in (0,\overline{a}]$. Then $2°$ is obtained with $\varepsilon = \min(a,\widetilde{\varepsilon})/2$ by using property (iii) in Lemma 1.1 for B_a and $c+a$ in place of B and c. Indeed, one has $c+\varepsilon = d$ and $d - \varepsilon_a < c - \varepsilon$, so the result follows. ∎

COROLLARY 1.2. (Theorem A' in Pucci and Serrin [13].) *Let $I \in C^1(X,\mathbb{R})$ satisfy (PS), let E be a closed set in X and $c \in \mathbb{R}$ with $K_c(I) \cap E = \emptyset$. Then for all sufficiently small $d > 0$ there is a continuous map $\sigma : E \longrightarrow X$ such that the conditions $x \in E$, $I(x) \geqslant c+d$ imply $I(\sigma(x)) \leqslant c-d$. Moreover we can suppose that $\sigma(x) = x$ whenever $I(x) \leqslant c-2d$.*

PROOF: Condition (PS) insures that $K_{c+a}(I) \cap E = \emptyset$ for any small $a > 0$. Setting $B = E \cap I_{c+a}$ in Lemma 1.1 it is seen from (1.3) that one can put $\overline{\varepsilon} = 3d$ and $\varepsilon = 2d$ with $0 < d < a$ small enough. For $A = I^b$ with $b > c+a$ we define $\sigma = \eta_A(1,\cdot)$. The last assertion in Corollary 1.2 follows from (ii) of Lemma 1.1 because $c+a-3d \geqslant c-2d$. Choose d in (iii) of Lemma 1.1 to be $c+d$, which is possible since $\delta = c+a+d-\varepsilon_0 < c+d < c+a$ if $\varepsilon_0 > a$. Here we used the fact that for a sufficiently small $a > 0$ the number $\varepsilon_0 > 0$ in (1.4) can be supposed to be independent of a. Then one gets that $\sigma(E \cap I_{c+d}) \subset I_{c-d}$. ∎

COROLLARY 1.3. (Lemma 1.1 in Du [4].) *Let $I \in C^1(X,\mathbb{R})$ satisfy (PS), let*

A and B be two closed disjoint subsets of X and let $c \in \mathbb{R}$ such that $B \cap K_c(I) = \emptyset$, $B \subset I_c$ and $A \subset I^c$. Then there exists $\varepsilon > 0$ and a homeomorphism η of X such that

(i) $I(\eta(x)) \leqslant I(x), \quad x \in X$;
(ii) $\eta(x) = x, \quad x \in A$;
(iii) $\eta(B) \subset I_{c-\varepsilon}$.

PROOF: Apply Lemma 1.1 for the set B and the number c. One obtains an $\varepsilon > 0$ and $\eta := \eta_A(1,\cdot) \in C(X,X)$ corresponding to $A \subset I^c \subset I^{c-\varepsilon}$. It is obvious that (i),(ii),(iii) of Corollary 1.3 are derived from (i),(ii),(iii) of Lemma 1.1, respectively, the property (iii) being deduced for $d = c$. ∎

COROLLARY 1.4. (Lemma 2.1 in Ding [3].) *Let $I \in C^1(X,\mathbb{R})$ defined on a real Hilbert space $X = X_1 \oplus X_2$ have the form $I(x) = (1/2)(Lx,x)_X + G(x)$, where $Lx = (L_1x_1, L_2x_2)$ for $x = (x_1,x_2) \in X$, with $L_i : X_i \longrightarrow X_i$, $i = 1,2$, bounded linear and self-adjoint, and $G \in C^1(X,\mathbb{R})$ with G' compact. Then, under the same hypotheses upon the sets A and B as in Corollary 1.3, there exist $\varepsilon > 0$ and $\eta \in C(\mathbb{R} \times X, X)$ such that*

(i) $\eta(t,\cdot)$ *is a homeomorphism of* X, $t \in \mathbb{R}$;
(ii) $I(\eta(t,x)) \leqslant I(x)$ *for* $t \geqslant 0$ *and* $x \in X$;
(iii) $\eta(t,x) = x$ *for all* $t \in \mathbb{R}$ *and* $x \in A$;
(iv) $\eta(1,B) \subset I_{c-\varepsilon}$;
(v) $\eta(t,x) = e^{\Theta(t,x)L}x + W(t,x)$, *where* $\Theta(t,x) = \int_0^t w(\eta(s,x))ds$, $w : X \longrightarrow [0,1]$ *is locally Lipschitz,* $W(0,x) = 0$ *and* W *is compact.*

PROOF: The argument is the same as in Corollary 1.3 with the difference of fixing in Lemma 1.1 a p.g. V of I of the form $V = L + C$ with C a compact mapping from X to X. ∎

2. A GENERAL MINIMAX THEOREM

Our main result is stated below.

THEOREM 2.1. *Let the functional $I \in C^1(X,\mathbb{R})$ and the closed subset B of a Banach space X satisfy $c := \inf_B I > -\infty$ and $(PS)_{B,c}$. Let V be a fixed p.g. of I and let $\mathcal{M}$ be a nonempty family of subsets M of X such that*

$$c = \inf_{M \in \mathcal{M}} \sup_{x \in M} I(x) \tag{2.1}$$

and the following hypothesis holds

(H) *for each $M \in \mathcal{M}$ there exists a closed subset A of X with $A \subset M \setminus B$ such that for each locally Lipschitz function $\varphi_A : X \longrightarrow [0,1]$ with $supp\, \varphi_A \subset (X \setminus A) \cap \mathcal{R}(I)$ the global flow ξ_A of $-\varphi_A \min(1, 1/\|V\|)V$ satisfies $\xi_A(1,M) \cap B \neq \emptyset$.*

Then

(i) *the infimum c of I over B is attained;*

(ii) $K_c(I) \setminus A \neq \emptyset$ *for each set A as in (H);*

(iii) $K_c(I) \cap B \neq \emptyset$.

PROOF: The assertions (i) and (ii) are derived from (iii). In order to check (iii) we argue by contradiction, so suppose $K_{-c}(-I) \cap B = \emptyset$. Since one has $-I|_B \leqslant -c$ we may apply Lemma 1.1 for our set B but with I and c replaced by $-I$ and $-c$, respectively. Let $\varepsilon > 0$ be the number supplied by Lemma 1.1. The minimax characterisation of c in (2.1) ensures that a set $M \in \mathcal{M}$ must satisfy

$$I|_M < c + \varepsilon. \tag{2.2}$$

Corresponding to M, hypothesis (H) yields a set A with the properties there stated. In particular we infer that

$$A \cap B = \emptyset \quad \text{and} \quad A \subset (-I)^{-c-\varepsilon}. \tag{2.3}$$

For $-I$, $-c$ and the p.g. $-V$ of $-I$, Lemma 1.1 implies that there exists $\eta_A \in C(\mathbb{R} \times X, X)$ satisfying among other things

$$\eta_A(1, B \cap (-I)_{-c}) \subset (-I)_{-c-\varepsilon}$$

which gives

$$\eta_A(1, B) \subset I^{c+\varepsilon}. \tag{2.4}$$

Notice from condition (i) of Lemma 1.1 applied to $-I$ and from hypothesis (H) that

$$\xi_A(t, x) = \eta_A(-t, x), \quad (t, x) \in \mathbb{R} \times X. \tag{2.5}$$

By (H) we know that

$$\xi_A(1, M) \cap B \neq \emptyset.$$

Then (2.5) gives

$$\eta_A(1, B) \cap M \neq \emptyset. \tag{2.6}$$

By combining (2.4) and (2.6), we deduce that there exists a point $x_0 \in M$ with $I(x_0) \geqslant c + \varepsilon$. Since this contradicts (2.2) the proof is complete. ∎

The next minimax principle which is a direct consequence of Theorem 2.1 includes all the classical minimax results, for example, the Mountain Pass Theorem of Ambrosetti and Rabinowitz [1], the Saddle Point Theorem (Theorem 4.6 in Rabinowitz [14]), the Generalised Saddle Point Theorem (Theorem 5.3 in Rabinowitz [14]). In these results all the sets A in (H) coincide with the boundary of a prescribed (compact) manifold in X. We relax this framework by not demanding even to have $\eta_A(1, M) \in \mathcal{M}$ whenever $M \in \mathcal{M}$.

Corollary 2.1. *Let $I \in C^1(X,\mathbb{R})$ satisfying (PS) and a family $\mathcal{M}$ of subsets M of X be such that c defined by (2.1) is a real number. Assume that*

(H$'$) *for each $M \in \mathcal{M}$ there exists a closed set A in X with $I|_A < c$ such that for every homeomorphism h of X with $h|_A = id_A$ one has $h(M) \cap I^c \neq \emptyset$.*

Then c in (2.1) is a critical value of I and certainly $K_c(I) \cap A = \emptyset$ for every A as in (H$'$).

Proof: For any p.g. V of I we can apply Theorem 2.1 where $B = I^c$. It is clear that (H$'$) implies (H) because $A \subset M \setminus B$ and $\xi_A(1,\cdot)$ is a homeomorphism of X with $\xi_A(1,\cdot) = id$ on A. Then Theorem 2.1 concludes the proof. ∎

Theorem 2.1 is useful in locating the critical points. We illustrate this aspect by deducing from Theorem 2.1 a result due to Ghoussoub and Preiss [5].

Corollary 2.2. (Theorem 1 bis in Ghoussoub and Preiss [5].) *Let a functional $I \in C^1(X,\mathbb{R})$, two points $u, v \in X$ and the number*

$$c = \inf_{g\in\Gamma} \max_{0 \leqslant t \leqslant 1} I(g(t)),$$

be given, where Γ is the set of all paths $g \in C([0,1],X)$ joining u and v. Suppose F is a closed subset of X such that $F \cap I^c$ separates u and v, that is, u,v belong to disjoint connected components of $X \setminus F \cap I^c$, and condition $(PS)_{F,c}$ is satisfied. Then there exists a critical point of I in F with critical value c.

Proof: We set $\mathcal{M} = \{g([0,1]) : g \in \Gamma\}$, $B = F \cap I^c$ and each A equal to $\{u,v\}$. With the notation of Theorem 2.1 we see that $\xi_A(M) \in \mathcal{M}$ if $M \in \mathcal{M}$, thus hypothesis (H) is true. Theorem 2.1 implies then the result. ∎

Now we point out two (known) relevant special cases of Theorem 2.1 with essentially only one linking situation.

Corollary 2.3. (Theorem 2.1 in Du [4].) *Let Q be a Banach manifold with boundary ∂Q in the Banach space X and let S be a closed set in X that is linking with ∂Q in the Banach space X in the sense that $\partial Q \cap S = \emptyset$ and $f(Q) \cap S \neq \emptyset$ for any $f \in \Gamma := \{f \in C(\overline{Q}, X) : f|_{\partial Q} = id_{\partial Q}\}$. Suppose that $I \in C^1(X,\mathbb{R})$ satisfies (PS), $\sup_Q I < +\infty$ and, for some $\alpha \in \mathbb{R}$,*

$$\partial Q \subset I_\alpha \text{ and } S \subset I^\alpha. \tag{2.7}$$

Defining

$$c := \inf_{f\in\Gamma} \sup_{x \in Q} I(f(x)) \tag{2.8}$$

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we have

(i) $c \geqslant \alpha$;

(ii) $K_c \setminus \partial Q \neq \emptyset$;

(iii) $K_c \cap S \neq \emptyset$ *if* $c = \alpha$.

PROOF: Choose a p.g. V of I and $\mathcal{M} = \{f(\overline{Q}) : f \in \Gamma\}$. If $c > \alpha$ in (2.8), hypothesis (H) applies with $B = I^{c'}$ with $\alpha < c' < c$ and each A equal to ∂Q since $\xi_A(1, M) \in \mathcal{M}$ for $M \in \mathcal{M}$. If $c = \alpha$ in (2.8), hypothesis (H) applies with $B = S$ and each A as above in view of the linking property between ∂Q and S. We get the result from Theorem 2.1. ∎

COROLLARY 2.4. (Theorem 2.1 in Ding [3].) *Let the functional $I \in C^1(X, \mathbb{R})$ on the Hilbert space $X = X_1 \oplus X_2$ be of the form described in Corollary 1.4. Given $S \subset X_2$ and $Q \subset \widetilde{X}$, a subspace of X, Q bounded, S and ∂Q link in the sense of* [3], *we assume there is $\alpha \in \mathbb{R}$ such that (2.7) holds. Let*

$$c := \inf_{h \in \Gamma} \sup_{x \in Q} I(h(1, x)) \tag{2.9}$$

where

(2.10)

$$\Gamma = \{h \in C([0,1] \times X, X) : h(0, x) = x, h(t, x) = x, \ \forall x \in \partial Q,$$
$$h(t, x) = e^{\theta(t,x)L}x + W(t, x), \ \text{where } \theta \in C([0,1] \times X, \mathbb{R}) \text{ and } W \text{ is compact}\Big\}.$$

Suppose I satisfies $(PS)_c$. Then the assertions (i)–(iii) of Corollary (2.3) hold.

PROOF: Consider a p.g. V as in Corollary 1.4. Let the class $\mathcal{M}$ be determined by $\mathcal{M} = \{h(1, \overline{Q}) : h \in \Gamma\}$ where Γ is given by (2.10). For c defined in (2.9) hypothesis (H) of Theorem 2.1 is satisfied with $B = I^{c'}$, $\alpha < c' < c$, if $c > \alpha$ and $B = S$ if $c = \alpha$, and the sets A coinciding with ∂Q. We utilised that $\xi_A|_{[0,1] \times X} \in \Gamma$ for each flow ξ_A. The corollary follows from Theorem 2.1. ∎

Theorem 2.1 allows us to deduce many other critical point theorems, for instance the ones dealing with functionals admitting symmetries, for example even functionals as in Theorem 4 of Clark [4]. As an illustration we indicate how Theorem 2.1 gives rise to an important known such result.

COROLLARY 2.5. (The symmetric version of the Mountain Pass Theorem, see Kavian [6], Rabinowitz [14], Struwe [15].) *Let $I \in C^1(X, \mathbb{R})$ be a functional on an infinite dimensional Banach space X which is even, satisfies (PS) together with*

(i) *$I(0) = 0$ and there exist $R > 0$ and $a > 0$ such that $I(x) \geqslant a$ for $\|x\| = R$;*

(ii) *if $X_1 \subset X$ is a finite dimensional subspace, then the set $\{x \in X_1 : I(x) \geqslant 0\}$ is bounded.*

Then I possesses an unbounded sequence of critical values.

PROOF: Since I is even we can construct an odd p.g. V of I. Denote by S the unit sphere of X and by γ the Krasnoselski genus (see [**6**, **14**, **15**]). We introduce the sequence

$$c_j := \inf_{E \in B_j} \max_{x \in E} I(x), \quad j \geqslant 1, \tag{2.11}$$

where

> $B_j := \{E \subset X : E$ is compact and symmetric such that for every odd homeomorphism h of X with $I|_{h(S)} > 0$ one has $\gamma(E \cap h(S)) \geqslant j\}$.

From assumptions (i) and (ii) one can establish that the sequence (c_j) is unbounded. We omit this part of the argument. Then the proof reduces to justifying that the sequence (c_j) consists of critical values of I. Let us apply Theorem 2.1 with B equal to the sphere in X of radius R and centred at the origin, $c = c_j$ and $\mathcal{M} = B_j$. We have to check hypothesis (H). From (i) and (ii) we easily derive that for each $M \in \mathcal{M}$ there is a closed set A with $A \subset M \setminus B$. It remains to show that

$$\xi_A(1, M) \cap B \neq \emptyset \tag{2.12}$$

for each global flow ξ_A of a vector field as in (H). Denoting by g the standard linear isomorphism of X mapping S onto B, that is, $g(x) = Rx$ for $x \in X$, we see that $h = \xi_A(-1, g(\cdot))$ is an odd homeomorphism of X and by assumption (i) for each $x \in S$ one has

$$I(h(x)) = I(\xi_A(-1, Rx)) \geqslant I(\xi_A(0, Rx)) = I(Rx) \geqslant a > 0.$$

Thus $\gamma(M \cap h(S)) \geqslant j$ which ensures in particular (2.12). Therefore hypothesis (H) is satisfied and Theorem 2.1 implies that each c_j in (2.11) is a critical value of I. $\square$

3. An application to a minimisation problem

In this section we treat by the minimax method of Theorem 2.1 the following abstract minimisation problem: given a closed set B of a Banach space X and a functional $I \in C^1(X, \mathbb{R})$ bounded below on B, find

$$u \in B \quad \text{with} \quad \inf_B I = I(u) \text{ and } I'(u) = 0. \tag{P}$$

A large number of variational problems can be expressed by statement (P) (see, for example, Ambrosetti and Rabinowitz [**1**], Ding [**3**], Kavian [**6**], Motreanu [**7**], Motreanu and Naniewicz [**8**], Motreanu and Panagiotopoulos [**9**], Nirenberg [**10**], Palais [**11**], Panagiotopoulos [**12**], Rabinowitz [**14**], Struwe [**15**]). We are briefly concerned here with a minimax approach in studying the general problem (P).

THEOREM 3.1. *Let B be a closed subset of a Banach space X and let the functional $I \in C^1(X, \mathbb{R})$ satisfy $c := \inf_B I > -\infty$ and $(PS)_{B,c}$. Assume that there exist a minimising sequence $(x_n) \subset B$ of $I|_B$ and a sequence $\mathcal{M} = \{M_n\}_{n \geqslant 1}$ of subsets M_n of X such that*

$$x_n \in M_n, \quad \sup_{M_n} I = I(x_n) \tag{3.1}$$

and the family $\mathcal{M}$ satisfies the intersection property (H) of Theorem 2.1 for some p.g. V of I. Then the problem (P) admits a solution.

PROOF: In view of (3.1) one finds that

$$c = \inf_B I \leqslant \inf_n \sup_{M_n} I \leqslant I(x_n), \quad n \geqslant 1. \tag{3.2}$$

By letting $n \longrightarrow \infty$ in (3.2), it follows that formula (2.1) holds. Hence the hypotheses of Theorem 2.1 are met. Then assertion (iii) of Theorem 2.1 provides an element $u \in K_c(I) \cap B$. This is exactly a solution of problem (P). ∎

We simply present the applicability of Theorem 3.1 by deducing the celebrated Palais Minimisation Theorem. Other existence results for the solvability of various optimisation problems can be derived using the same technique based on Theorem 3.1.

COROLLARY 3.1. (Palais [11].) *Assume that $J \in C^1(E, \mathbb{R})$ is bounded below on the Banach space E and satisfies (PS). Then J attains its infimum on E (at a critical point).*

PROOF: Let us define $X := E \times \mathbb{R}$, $B := E \times 0 \subset X$ and $I : X \longrightarrow \mathbb{R}$ given by

$$I(v,t) = J(v) - t^2, \quad (v,t) \in E \times \mathbb{R} = X. \tag{3.3}$$

A direct computation involving (3.3) shows that I satisfies (PS). Choose a minimising sequence (v_n) of J, so $v_n \in E$ and $J(v_n) \longrightarrow \inf_E J$ as $n \longrightarrow \infty$. For each $n \geqslant 1$ we consider the line segment

$$M_n := \{(v_n, t) : |t| \leqslant 1\}$$

joining the points $(v_n, \pm 1)$ in X. It is seen that (3.1) is satisfied with $x_n = (v_n, 0) \in B$. Let us check hypothesis (H) of Theorem 2.1 for $\mathcal{M} = \{M_n\}_{n \geqslant 1}$ and a fixed p.g. V of I. Clearly we may set $A_n = \{(v_n, \pm 1)\}$, $n \geqslant 1$. Because $\xi_{A_n}(1, \cdot) = id$ on A_n and $B = E \times 0$ disconnects $X = E \times \mathbb{R}$ one obtains $\xi_{A_n}(1, M_n) \cap B \neq \emptyset$, and have (H) holds. Theorem 3.1 assures us the existence of a point $(u,s) \in X = E \times \mathbb{R}$ solving (P). By (3.3) we find that $s = 0$, so $I(u,0) = J(u) = \inf_E J$. The proof is thus complete. ∎

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Department of Mathematics
University of Iasi
6600 Iasi
Romania