representing $\mathscr{T}(\Sigma, \Omega_1)$ and $\mathscr{T}(\Sigma, \Omega_2)$ with suitably chosen 'parameters' Σ, Ω_1, Ω_2 – so that an argument establishing suitability of these parameters remains within the scope of algebraic theories, and hence constitutes the 'algebraic' part of a proof.

Complete separation of arguments on internal properties of algebraic theories from their actual representation in Top (or other categories) allows us to deal with more complex questions concerning clone segments. For instance, the present paper investigates also 'simultaneous representations' of embeddings

$$\mathscr{T}(\Sigma, \Omega) \hookrightarrow \mathscr{T}(\Sigma, \Omega')$$

in various modifications m : Top $\to$ Top, see Section 3.

All needed notions are presented in Section 0. The subsequent two 'algebraic' sections give constructions of algebraic theories representable by topological spaces constructed in 'topological' Sections 3, 4 and 5. Other applications of the 'algebraic' part of this article will appear elsewhere.

0. Preliminaries

We say that a concrete category $(\mathscr{K}, U)$ has *concrete finite products* if $\mathscr{K}$ has finite products and the faithful functor $U : \mathscr{K} \to$ Set preserves them. It has the *transfer property* (also called transportability, see [1]) if for any $\mathscr{K}$-object a and any bijection $f : U(a) \to X$ there exists a $\mathscr{K}$-object b with $U(b) = X$ and also an isomorphism $\beta \in \mathscr{K}(a, b)$ such that $U(\beta) = f$.

Let $(\mathscr{K}, U)$ be a concrete category with the transfer property and concrete finite products. For any $\mathscr{K}$-object a with $U(a) \neq \emptyset$, let $\mathscr{K}_a$ denote the full subcategory of $\mathscr{K}$ determined by all finite powers $a^0, a, a^2, \dots$ of a, and let U_a denote the domain restriction of U to $\mathscr{K}_a$. The concrete category $(\mathscr{K}_a, U_a)$ will be denoted by $\mathrm{Clo}(a)$. Since $(\mathscr{K}, U)$ has the transfer property, we may require that each n-th power a^n of a and its product projections $\pi_i^{(n)} : a^n \to a$ with $i \in \{0, \dots, n-1\} = n$ are chosen so that $U(a^n)$ is the n-th Cartesian power $U(a)^n$ of the set $U(a)$ and, for all $i \in n$, the map $U(\pi_i^{(n)})$ is the Cartesian projection

$$U(\pi_i^{(n)})(x_0, \dots, x_{n-1}) = x_i.$$

With this requirement satisfied, $\mathrm{Clo}(a)$ is what universal algebraists call a clone on the set $U(a)$, see [5]. For any integer $k \geq 1$, the concrete category $(\mathscr{K}_{a,k}, U_{a,k})$ for which $\mathscr{K}_{a,k}$ is the full subcategory of $\mathscr{K}_a$ determined by objects $a^0, \dots, a^{k-1}$, and $U_{a,k}$ is the domain restriction of U_a to $\mathscr{K}_{a,k}$ is denoted $\mathrm{Clo}_k(a)$ and called the *k-segment* of $\mathrm{Clo}(a)$.

The reason why the object a^0 was not included in the definition of a clone of a topological space X given in the introduction is simply that, being just the singleton space X^0, it has no significance for the results mentioned there. This phenomenon is common to concrete categories $(\mathscr{K}, U)$ with concrete finite products and with all constants, for a^0 is a terminal object in any of these categories.

Let $a, b \in \operatorname{obj}\mathscr{K}$, and $\mathrm{Clo}(a) = (\mathscr{K}_a, U_a)$, $\mathrm{Clo}(b) = (\mathscr{K}_b, U_b)$. For $k \geq 1$, we write

$$\mathrm{Clo}_k(a) \cong \mathrm{Clo}_k(b)$$

if there is an isofunctor Φ of $\mathscr{K}_{a,k}$ onto $\mathscr{K}_{b,k}$. If Φ also satisfies $U_{b,k} \circ \Phi = U_{a,k}$ we write

$$\mathrm{Clo}_k(a) = \mathrm{Clo}_k(b).$$

In universal algebra, clones on sets are sometimes considered distinct only when there is no categorical isomorphism between them. This view is reflected by the notion of abstract clone, see [5], closely corresponding to the notion of an algebraic theory in the sense of [3]. An *algebraic theory* with a *base object* a is then any category $\mathscr{T}$ with

$$\operatorname{obj}\mathscr{T} = \{a^n \mid n = 0, 1, \dots\}$$

and, for every $n \in \{0, 1, \dots\} = \omega$, a specified n-tuple of morphisms

$$\pi_i^{(n)} : a^n \to a \text{ with } i \in n$$

forming a categorical product of n copies of a, meaning that for every n-tuple $f_0, \dots, f_{n-1} : a^m \to a$ of $\mathscr{T}$-morphisms there exists a unique $\mathscr{T}$-morphism $f : a^m \to a^n$ satisfying $\pi_i^{(n)} \circ f = f_i$ for all $i \in n$; we then denote $f = f_0 \dot{\times} \cdots \dot{\times} f_{n-1}$.

Let $\mathscr{T}$ and $\mathscr{T}'$ be algebraic theories whose respective base objects are a and a'. A functor $H : \mathscr{T} \to \mathscr{T}'$ is called a *homomorphism* of these theories if $H(a^n) = (a')^n$ and $H(\pi_i^{(n)}) = \pi_i'^{(n)}$ for every integer $n \geq 0$ and for all $i \in n$. Let $\mathscr{K}$ be a category with finite products, and let $\Psi : \mathscr{T} \to \mathscr{K}$ be a full and faithful functor. We say that Ψ is a *representation* of a theory $\mathscr{T}$ in $\mathscr{K}$ if it preserves the finite powers of a, that is, the object $\Psi(a^n)$ together with $\{\Psi(\pi_i^{(n)}) \mid i \in n\}$ is the n-th power of $\Psi(a)$ in $\mathscr{K}$ for every $n \in \omega$. In particular, the object $\Psi(a^0)$ is a terminal object of $\mathscr{K}$. Since any finite product is determined uniquely up to an isomorphism, any representation $\Psi : \mathscr{T} \to \mathscr{K}$ is determined by the object $\Psi(a)$ uniquely up to a natural equivalence.

Let $(\mathscr{K}, U)$ be a concrete category with concrete finite products and the transfer property. If, for its terminal object t (which is a product of an empty collection of its objects), the hom-functor $\mathscr{K}(t, -) : \mathscr{K} \to \mathrm{Set}$ and the faithful functor $U : \mathscr{K} \to \mathrm{Set}$ are naturally equivalent, we say that $(\mathscr{K}, U)$ is *well-pointed.* (This is the case for the category Top and its natural forgetful functor.) If $\Psi : \mathscr{T} \to \mathscr{K}$ is any representation of an algebraic theory $\mathcal{T}$ with a base object a in the category $\mathcal{K}$, then $\Psi(a^0)$ must be a terminal object t of $\mathcal{K}$, and the set $\mathcal{T}(a^0, a)$ must be sent bijectively onto the hom-set $\mathcal{K}(t, \Psi(a))$ which, in turn, is bijective to the set $U\Psi(a)$. Therefore finding a representation of $\mathcal{T}$ in Top amounts to finding a topology τ on the set $P = \mathcal{T}(a^0, a)$ for which the clone of the space $X = (P, \tau)$ in Top is isomorphic to $\mathcal{T}$. Any representation $\Psi : \mathcal{T} \to$ Top thus determines which actual maps between finite powers of P are continuous, and this fact substantially restricts the extent of algebraic theories that can be represented in Top.

Next we define algebraic theories we shall be concerned with.

Let $\Sigma = \bigcup_{n=0}^{\infty} \Sigma_n$ be a finitary type of universal algebras in which Σ_n denotes the set of all n-ary operation symbols. For any $\sigma \in \Sigma$, we write $\operatorname{ar}\sigma = n$ to mean $\sigma \in \Sigma_n$. Throughout the paper, Σ_0 will be infinite and $\Sigma \setminus \Sigma_0 \neq \emptyset$.

Let $\mathcal{T}(\Sigma)$ denote the algebraic theory freely generated by Σ. Explicitly, $\mathcal{T}(\Sigma)$ is an algebraic theory such that every $\sigma \in \Sigma_n$ is a $\mathcal{T}(\Sigma)$-morphism $a^n \to a$ (where a denotes the base object of $\mathcal{T}(\Sigma)$), and $\mathcal{T}(\Sigma)$ has the usual universal property: for any algebraic theory $\mathcal{T}'$ with a base object a', any mapping $G : \Sigma \to \mathcal{T}'$ with $G(\sigma) \in \mathcal{T}'((a')^n, a')$ for every $\sigma \in \Sigma_n$ has a unique extension to a homomorphism $H : \mathcal{T}(\Sigma) \to \mathcal{T}'$.

For any set $\Omega \subseteq \Sigma \setminus (\Sigma_0 \cup \Sigma_1)$, let $\mathcal{T}(\Sigma, \Omega)$ denote the largest subtheory of $\mathcal{T}(\Sigma)$ such that

(i) if $\sigma \in \Omega$, then σ does not belong to $\mathcal{T}(\Sigma, \Omega)$, and

(ii) if $\sigma \in \Sigma$ and α are $\mathcal{T}(\Sigma)$-morphisms whose composite $a^m \xrightarrow{\alpha} a^n \xrightarrow{\sigma} a$ belongs to $\mathcal{T}(\Sigma, \Omega)$, then α belongs to $\mathcal{T}(\Sigma, \Omega)$ as well.

In [12], it is shown that the free theory $\mathcal{T} = \mathcal{T}(\Sigma)$ with $\operatorname{card}\Sigma > 1$ and $\Sigma_0 \neq \emptyset$, and also any of its subtheories $\mathcal{T}^{\Omega} = \mathcal{T}(\Sigma, \Omega)$ with $\Omega \subseteq \Sigma \setminus (\Sigma_0 \cup \Sigma_1)$ are well-pointed and satisfy

$$\mathcal{T}^{\Omega}(a^0, a) = \mathcal{T}(a^0, a).$$

Since Σ_0 will always be infinite, all algebraic theories considered here will have these two properties. The result of [12] describing their internal structure is recalled in the section below, and used extensively afterwards.

1. Representations of pairs of r-ultrametrics

DEFINITION 1.1. Let S be a set. A function $\varphi : S \times S \to \{2, 3, \ldots, \infty\}$ is called a *reciprocal ultrametric,* or an *r-ultrametric* on S if, for all $s, t, u \in S$,

(1) $\varphi(s, s) = \infty$,

(2) $\varphi(s, t) = \varphi(t, s)$, and

(3) $\varphi(s, u) \geq \min\{\varphi(s, t), \varphi(t, u)\}$.

Let $\mathbb{S} = \{a_s \mid s \in S\}$ be a set of objects of a concrete category $(\mathscr{K}, U)$ with concrete finite products and the transfer property, such that

$$U(a_s) = U(a_t)$$

and

$$\{U(\alpha) \mid \alpha \in \mathscr{K}(a_s, a_s)\} = \{U(\beta) \mid \beta \in \mathscr{K}(a_t, a_t)\} \text{ for all } s, t \in S,$$

that is, all members of $\mathbb{S}$ have the same underlying set and the same endomorphism monoid. Any such set $\mathbb{S}$ determines a pair of r-ultrametrics on S as follows:

$$\varphi(s, t) = \sup\{k \mid \mathrm{Clo}_k(a_s) = \mathrm{Clo}_k(a_t)\} \text{ and}$$
$$\widetilde{\varphi}(s, t) = \sup\{k \mid \mathrm{Clo}_k(a_s) \cong \mathrm{Clo}_k(a_t)\}.$$

It is clear that these r-ultrametrics satisfy $\varphi(s, t) \leq \widetilde{\varphi}(s, t)$ for all $s, t \in S$. In this section we aim to show that for certain concrete categories $(\mathscr{K}, U)$, these necessary conditions are also sufficient, in the sense that every pair $\varphi \leq \widetilde{\varphi}$ of r-ultrametrics on a set S can be represented by a system $\mathbb{S} = \{a_s \mid s \in S\}$ of its objects, see Theorem 1.3 below.

DEFINITION 1.2. A concrete category $(\mathscr{K}, U)$ is *comprehensive* if there exists a cardinal number α such that every algebraic theory $\mathscr{T}(\Sigma, \Omega)$ with

$$\operatorname{card} \Sigma_0 \geq \alpha + \operatorname{card}(\Sigma \setminus \Sigma_0)$$

is representable in $\mathscr{K}$. When it is necessary to indicate the cardinal α, we say that $(\mathscr{K}, U)$ is *α-comprehensive.*

In this section we prove the following result.

THEOREM 1.3. *Let $(\mathscr{K}, U)$ be a well-pointed comprehensive concrete category. Then any two r-ultrametrics $\varphi \leq \widetilde{\varphi}$ on a set S can be represented by a system $\{a_s \mid s \in S\} \subseteq \operatorname{obj} \mathscr{K}$ in the sense that, for all $s, t \in S$,*

$$U(a_s) = U(a_t) \text{ and } \{U(\alpha) \mid \alpha \in \mathscr{K}(a_s, a_s)\} = \{U(\beta) \mid \beta \in \mathscr{K}(a_t, a_t)\},$$
$$\varphi(s, t) = \sup\{k \mid \mathrm{Clo}_k(a_s) = \mathrm{Clo}_k(a_t)\},$$
$$\widetilde{\varphi}(s, t) = \sup\{k \mid \mathrm{Clo}_k(a_s) \cong \mathrm{Clo}_k(a_t)\}.$$

REMARK. Theorem 1.3 can be applied to the category MTop of metrizable topological spaces and their continuous maps because, clearly, MTop is well-pointed and, as shown implicitly in [6] and stated explicitly in [12], it is also comprehensive. Hence any two r-ultrametrics $\varphi \leq \widetilde{\varphi}$ can be represented by the equality and the isomorphism of clone segments of metrizable topological spaces.

Now we turn to the proof of Theorem 1.3.

LEMMA 1.4. *Let φ and $\widetilde{\varphi}$ be r-ultrametrics on S such that*

$$\varphi(s,t) \leq \widetilde{\varphi}(s,t) \quad \text{for all} \quad s,t \in S.$$

Then, for every $s \in S$, there is a sequence $\{M_n(s) \mid n = 2,3,\dots\}$ of sets such that the collection

$$\mathbf{M} = \{M_n(s) \mid n = 2,3,\dots \text{ and } s \in S\}$$

satisfies these two conditions:

(1) *if $n < \varphi(s,t)$ then $M_n(s) = M_n(t)$,*
if $n = \varphi(s,t)$ then $M_n(s) \cap M_n(t) = \emptyset$,

(2) *if $n < \widetilde{\varphi}(s,t)$ then $\operatorname{card} M_n(s) = \operatorname{card} M_n(t)$,*
if $n = \widetilde{\varphi}(s,t)$ then $\operatorname{card} M_n(s) \neq \operatorname{card} M_n(t)$.

Furthermore, if S is countable, then all sets $M_n(s)$ may be chosen to be finite; if S is finite, then all but finitely many sets $M_n(s)$ may be chosen to be empty.

PROOF. Since $\varphi \leq \widetilde{\varphi}$ are r-ultrametrics on S, the sets $e_n = \{(s,t) \in S^2 \mid n < \varphi(s,t)\}$ and $\tilde{e}_n = \{(s,t) \in S^2 \mid n < \widetilde{\varphi}(s,t)\}$ are equivalences on S, and $e_n \subseteq \tilde{e}_n$ for all $n \geq 1$. It is also clear that $e_{n+1} \subseteq e_n$ and $\tilde{e}_{n+1} \subseteq \tilde{e}_n$ for every $n \geq 1$, and that $e_1 = \tilde{e}_1 = S^2$. Furthermore, $(s,t) \in e_n \setminus e_{n+1}$ exactly when $\varphi(s,t) = n+1$, and similarly for $\widetilde{\varphi}$.

In what follows, we regard a cardinal number as the set of all smaller ordinals.

To produce the required sequences, we set $M_n(s) = \emptyset$ for every $s \in S$ whenever $n < \varphi(s,t)$ for all $s,t \in S$, or whenever $\widetilde{\varphi}(s,t) < n$ for all $s,t \in S$. Next, we select and fix a bijection $\beta : \operatorname{card} S \to S$, and for all other $n \geq 2$ set $\kappa_n(s) = \min\{\kappa \in \operatorname{card} S \mid (\beta(\kappa), s) \in e_n\}$. Let $\lambda_n : S \to \mathrm{Card}^+$ be a mapping into the class Card^+ of all nonzero cardinals such that $\lambda_n(s) = \lambda_n(t)$ exactly when $(s,t) \in \tilde{e}_n$. Write $M_n(s) = \{\kappa_n(s)\} \times \lambda_n(s)$. The system $\mathbf{M} = \{M_n(s) \mid n = 2,3,\dots \text{ and } s \in S\}$ then satisfies (1) and (2). If S is countable, then the values of each λ_n may be chosen to be finite. If S is finite, then the system $\mathbf{M}$ will have finitely many nonempty (and finite) members. $\square$

Next we describe the free theory $\mathscr{T} = \mathscr{T}(\Sigma)$ and its subtheories $\mathscr{T}^\Omega = \mathscr{T}(\Sigma, \Omega)$ for $\Omega \subseteq \Sigma \setminus (\Sigma_0 \cup \Sigma_1)$ in structural terms.

We begin with $\mathscr{T} = \mathscr{T}(\Sigma)$. It is well-known that the set $M^{(m)}$ of all $\mathscr{T}$-morphisms $a^m \to a$ is the underlying set of a free Σ-algebra over m generators. This set has the form $M^{(m)} = \bigcup_{k=0}^{\infty} M_k^{(m)}$, where the subsets $M_k^{(m)}$ are defined inductively as follows. First,

$$M_0^{(m)} = \{\pi_0^{(m)}, \dots, \pi_{m-1}^{(m)}\} \cup \{\sigma \cdot \tau^{(m)} \mid \sigma \in \Sigma_0\},$$

where the product projections $\pi_j^{(m)}$ play the role of generators, the expressions $\sigma \cdot \tau^{(m)}$ with $\sigma \in \Sigma_0$ are constant terms, and $\tau^{(m)} : a^m \to a^0$ denotes the unique morphism of a^m into the terminal object a^0 in $\mathscr{T}$. The remaining sets $M_k^{(m)}$ are defined successively by the inductive formula

$$M_{k+1}^{(m)} = M_k^{(m)} \cup \bigcup_{\sigma \in \Sigma \setminus \Sigma_0} \{\sigma(f_0, \dots, f_{\operatorname{ar}\sigma - 1}) \mid f_i \in M_k^{(m)} \text{ for all } i \in \operatorname{ar}\sigma\},$$

where $\sigma(f_0, \dots, f_{\operatorname{ar}\sigma-1})$ is a term formally created from σ and already existing terms $f_0, \dots, f_{\operatorname{ar}\sigma-1}$.

If $m = 0$, then $\tau^{(0)} : a^0 \to a^0$ is nothing but 1_{a^0}, so that instead of $\sigma \cdot \tau^{(0)}$ with $\sigma \in \Sigma_0$ we shall write σ. Hence $M_0^{(0)} = \Sigma_0$ and the formula for $M_{k+1}^{(0)}$ remains unchanged. For $m = 1$, we have only one product projection $\pi_0^{(1)}$, namely 1_a. Hence $M_0^{(1)} = \{1_a\} \cup \{\sigma \cdot \tau^{(1)} \mid \sigma \in \Sigma_0\}$ and the formula for $M_{k+1}^{(1)}$ also remains unchanged.

For $n > 1$, the set $\mathscr{T}(a^m, a^n)$ is defined to be the set of all n-tuples $(f_0, \dots, f_{n-1})$ of elements of $\mathscr{T}(a^m, a)$. The unique element of $\mathscr{T}(a^m, a^0)$ has been already denoted by $\tau^{(m)}$. The inductive definition of composition as well as the proof that $\mathscr{T}(\Sigma)$ is really an algebraic theory (meaning that $\pi_i^{(n)} : a^n \to a$ with $i \in n$ form a categorical product of n copies of a) is given in [12].

Also in [12], the structure of the subtheory $\mathscr{T}^{\Omega} = \mathscr{T}(\Sigma, \Omega)$ of $\mathscr{T} = \mathscr{T}(\Sigma)$ is described as follows. For any $m \geq 0$, the hom-set $T^{(m)} = \mathscr{T}^{\Omega}(a^m, a)$ has the form $T^{(m)} = \bigcup_{k=0}^{\infty} T_k^{(m)}$, where

$$\begin{aligned}
T_0^{(m)} &= M_0^{(m)} = \left\{\pi_0^{(m)}, \dots, \pi_{m-1}^{(m)}\right\} \cup \left\{\sigma_0 \cdot \tau^{(m)} \mid \sigma_0 \in \Sigma_0\right\}, \\
T_1^{(m)} &= T_0^{(m)} \cup \bigcup_{\sigma \in \Sigma \setminus (\Sigma_0 \cup \Omega)} \left\{p_\sigma(f_0, \dots, f_{\operatorname{ar}\sigma-1}) \mid f_i \in T_0^{(n)} \text{ for all } i \in \operatorname{ar}\sigma\right\} \\
&\quad \cup \bigcup_{\sigma \in \Omega} \Big\{p_\sigma(f_0, \dots, f_{\operatorname{ar}\sigma-1}) \mid f_i \in T_0^{(m)} \text{ for all } i \in \operatorname{ar}\sigma \\
&\qquad\qquad \text{and either } f_i = f_j \text{ for some } i, j \in \operatorname{ar}\sigma \text{ with } i \neq j \\
&\qquad\qquad \text{or } f_i = \sigma_0 \cdot \tau^{(m)} \text{ for some } i \in \operatorname{ar}\sigma \text{ and } \sigma_0 \in \Sigma_0\Big\}, \\
T_{k+1}^{(m)} &= T_k^{(m)} \cup \bigcup_{\sigma \in \Sigma \setminus \Sigma_0} \Big\{p_\sigma(f_0, \dots, f_{\operatorname{ar}\sigma-1}) \mid f_i \in T_k^{(m)} \text{ for all } i \in \operatorname{ar}\sigma \\
&\qquad\qquad \text{and } f_j \notin T_{k-1}^{(m)} \text{ for some } j \in \operatorname{ar}\sigma\Big\}, \text{ for any } k \geq 1.
\end{aligned}$$

Less formally, this means that $T^{(m)}$ is the set of all those terms in $M^{(m)}$ which do not contain any subterm of the form $\sigma(\pi_{\psi(0)}^{(m)}, \dots, \pi_{\psi(n-1)}^{(m)})$ with $\sigma \in \Omega \cap \Sigma_n$ and a one-to-one map $\psi : n \to m$.

If $n > 1$, then $\mathscr{T}^{\Omega}(a^m, a^n)$ is the set of all n-tuples $(f_0, \dots, f_{n-1})$ of $\mathscr{T}^{\Omega}(a^m, a)$, while the morphism $\tau^{(m)}$ is the unique member of $\mathscr{T}^{\Omega}(a^m, a^0)$, just as in $\mathscr{T} = \mathscr{T}(\Sigma)$.

Let $\widetilde{\varphi} \geq \varphi$ be two r-ultrametrics on the set S, and let α be an infinite cardinal for which $(\mathscr{K}, U)$ is α-comprehensive. To any $s \in S$ we now assign an algebraic theory $\mathscr{T}(\Sigma, \Omega_s)$ as follows.

Let $\mathbf{M} = \{M_n(s) \mid n = 2, 3, \ldots \text{ and } s \in S\}$ be the system of sequences from Lemma 1.4. First we define a type Σ by $\Sigma_1 = \emptyset$ and $\Sigma_n = \bigcup\{M_n(s) \mid s \in S\}$ for each $n \geq 2$, select a set Σ_0 with card $\Sigma_0 \geq \alpha + \operatorname{card}\bigcup\{\Sigma_n \mid n \geq 1\}$, and let Σ stand for the disjoint union of all sets Σ_n with $n \in \omega$.

For every $s \in S$ and $n \geq 2$, we define

$$\Omega_{n,s} = \Sigma_n \setminus M_n(s),$$

and then set $\Omega_s = \bigcup\left\{\Omega_{n,s} \mid n \geq 2\right\}$.

For every $s \in S$ and each $n \geq 1$, we let $\mathscr{T}(\Sigma, \Omega_s)_n$ denote the n-segment of the algebraic theory $\mathscr{T}(\Sigma, \Omega_s)$, that is, its full subcategory determined by the first n powers $a^0, a, \ldots, a^{n-1}$ of its base object a. Since $\mathscr{T}(\Sigma)$ and all $\mathscr{T}(\Sigma, \Omega_s)$ are well-pointed, and because $\Omega \subseteq \Sigma \setminus (\Sigma_0 \cup \Sigma_1)$, from the definition of $\mathscr{T}(\Sigma, \Omega)$ it immediately follows that $\mathscr{T}(\Sigma)_2 = \mathscr{T}(\Sigma, \Omega_s)_2$ for every $s \in S$.

LEMMA 1.5. *Let $\varphi \leq \widetilde{\varphi}$ be r-ultrametrics on S. For $s \in S$, let $\mathscr{T}(\Sigma, \Omega_s)$ be the algebraic theory just defined. Then, for any $s, t \in S$,*

(a) *$\mathscr{T}(\Sigma, \Omega_s)_n = \mathscr{T}(\Sigma, \Omega_t)_n$ if and only if $n < \varphi(s, t)$, and*
(b) *$\mathscr{T}(\Sigma, \Omega_s)_n \cong \mathscr{T}(\Sigma, \Omega_t)_n$ if and only if $n < \widetilde{\varphi}(s, t)$.*

PROOF. First we show that $\mathscr{T}(\Sigma, \Omega_s)_n = \mathscr{T}(\Sigma, \Omega_t)_n$ if and only if $n < \varphi(s, t)$. Suppose first that $n < \varphi(s, t)$. In view of the paragraph preceding this lemma, we may assume that $n \geq 2$. Let $f : a^n \to a$ belong to $\mathscr{T}(\Sigma) \setminus \mathscr{T}(\Sigma, \Omega_s)$. In the algebraic interpretation, this means that the term f has a subterm $g = \sigma(x_{\psi(0)}, \ldots, x_{\psi(\operatorname{ar}\sigma - 1)})$ with $\sigma \in \Omega_s$ and a map $\psi : \operatorname{ar}\sigma \to n$ that is one-to-one. Hence $\operatorname{ar}\sigma \leq n < \varphi(s, t)$, so that $M_{\operatorname{ar}\sigma}(s) = M_{\operatorname{ar}\sigma}(t)$ by Lemma 1.4(1), and therefore $\sigma \in \Omega_t$. But then $f \notin \mathscr{T}(\Sigma, \Omega_t)$. Together with a symmetric argument, this proves that $\mathscr{T}(\Sigma, \Omega_s)_n = \mathscr{T}(\Sigma, \Omega_t)_n$ whenever $n < \varphi(s, t)$. Conversely, if $n \geq \varphi(s, t)$, then $M_{\varphi(s,t)}(s) \neq M_{\varphi(s,t)}(t)$, see Lemma 1.4(1). If $\sigma \in M_{\varphi(s,t)}(s) \setminus M_{\varphi(s,t)}(t)$, then σ – interpreted as a term $\sigma(x_0, \ldots, x_{\varphi(s,t)-1})$ – belongs to $\mathscr{T}(\Sigma, \Omega_s)_n$ but not to $\mathscr{T}(\Sigma, \Omega_t)_n$. Altogether, $\mathscr{T}(\Sigma, \Omega_s)_n = \mathscr{T}(\Sigma, \Omega_t)_n$ exactly when $n < \varphi(s, t)$.

Next we show that $\mathscr{T}(\Sigma, \Omega_s)_n \cong \mathscr{T}(\Sigma, \Omega_t)_n$ if and only if $n < \widetilde{\varphi}(s, t)$. Assume first that $\mathscr{T}(\Sigma, \Omega_s)_n \cong \mathscr{T}(\Sigma, \Omega_t)_n$. In [12, Section VII] it was shown that, for any two subsets Ω_s and Ω_t of $\Sigma \setminus \Sigma_0$, the n-segment $\mathscr{T}(\Sigma, \Omega_s)_n$ is isomorphic to $\mathscr{T}(\Sigma, \Omega_t)_n$ only when $\operatorname{card}(\Sigma_k \setminus \Omega_s) = \operatorname{card}(\Sigma_k \setminus \Omega_t)$ for every $k \in n$. Since $\Sigma_k \setminus \Omega_s = M_k(s)$ and $\Sigma_k \setminus \Omega_t = M_k(t)$ in our case, from Lemma 1.4(2) it follows that $n < \widetilde{\varphi}(s, t)$. For the converse, assume that $n < \widetilde{\varphi}(s, t)$. If $\varphi(s, t) \leq k < n$, then the sets $M_k(s), M_k(t) \subseteq \Sigma_k$ have the same cardinality. Hence there is an involution $\beta_k : \Sigma_k \to \Sigma_k$ which maps $M_k(s)$ onto $M_k(t)$ and vice versa. We now define a mapping $h : \Sigma \to \Sigma$ by

$$h(\sigma) = \begin{cases} \beta_k(\sigma) & \text{if } \varphi(s,t) \leq k < n \text{ and } \sigma \in \Sigma_k \\ \sigma & \text{in all other cases.} \end{cases}$$

Then h^2 is the identity on Σ and, because the algebraic theory $\mathscr{T}(\Sigma)$ is free over Σ, there is an involutory endomorphism Ψ of $\mathscr{T}(\Sigma)$ extending h. Since Ψ maps the n-segment $\mathscr{T}(\Sigma, \Omega_s)_n$ into $\mathscr{T}(\Sigma, \Omega_t)_n$ and vice versa, its restriction Ψ_n to $\mathscr{T}(\Sigma, \Omega_s)_n$ is an isomorphism of $\mathscr{T}(\Sigma, \Omega_s)_n$ onto the n-segment $\mathscr{T}(\Sigma, \Omega_t)_n$. □

PROOF OF THEOREM 1.3 CONCLUDED. Since $\mathscr{K}$ is comprehensive, for every $s \in S$ there exists a representation Φ_s of the theory $\mathscr{T}^{(s)} = \mathscr{T}(\Sigma, \Omega_s)$ in $\mathscr{K}$. Write $a_s = \Phi_s(a)$ for every $s \in S$. Then $a_s^0 = \Phi_s(a^0)$ is a terminal object of $\mathscr{K}$, and hence we may choose the representations Φ_s so that $\Phi_s(a^0) = z$ for all $s \in S$. Recall that $\mathscr{T}^{(s)}(a^0, a)$ is bijective to the underlying set P of the Σ-algebra freely generated by the empty set, for every $s \in S$. From the fullness of Φ_s and the fact that U is naturally equivalent to $\mathscr{K}(z, -)$ it follows that $U(a_s)$ is bijective to P for every $s \in S$. Since $(\mathscr{K}, U)$ has the transfer property, we may thus also assume that $U(a_s) = P$ for every $s \in S$. When combined with the transfer property, claims (a) and (b) of Lemma 1.5 complete the proof of Theorem 1.3.

2. Representations of pairs of grounded quadruples

DEFINITION 2.1. A quadruple $R = (r, s, t_1, t_2)$ of elements of the set $\{2, 3, \ldots, \infty\}$ is *grounded* if none of its four entries is strictly smaller than the remaining three.

Let $(\mathscr{K}, U)$ and $(\mathscr{K}', U')$ be well-pointed concrete categories, and let $F : \mathscr{K} \to \mathscr{K}'$ be a functor such that $U' \circ F = U$. Suppose that four objects $a_1, a_2 \in \operatorname{obj} \mathscr{K}$ and $F(a_1), F(a_2) \in \operatorname{obj} \mathscr{K}'$ have the same underlying set and the same endomorphism monoid. In other words, let

$$U(a_1) = U(a_2)(= U'F(a_1) = U'F(a_2)) \text{ and } M(a_i) = M(F(a_j)) \text{ for } i, j = 1, 2,$$

where

$$M(a_i) = \{U(\alpha) \mid \alpha \in \mathscr{K}(a_i, a_i)\} \text{ and } M(F(a_j)) = \{U'(\beta) \mid \beta \in \mathscr{K}'(F(a_j), F(a_j))\}.$$

This section discusses the equality and the isomorphism of the clone segments of the four objects $a_1, a_2 \in \operatorname{obj} \mathscr{K}$ and $F(a_1), F(a_2) \in \operatorname{obj} \mathscr{K}'$. We define

$$r = \sup\{k \mid \mathrm{Clo}_k(a_1) = \mathrm{Clo}_k(a_2)\},$$

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$$s = \sup\{k \mid \mathrm{Clo}_k(F(a_1)) = \mathrm{Clo}_k(F(a_2))\}, \quad \text{and}$$
$$t_i = \sup\{k \mid \mathrm{Clo}_k(a_i) = \mathrm{Clo}_k(F(a_i))\} \quad \text{for } i = 1, 2.$$

Replacing $\mathrm{Clo}_k(\dots) = \mathrm{Clo}_k(\dots)$ in these four expressions by $\mathrm{Clo}_k(\dots) \cong \mathrm{Clo}_k(\dots)$, we obtain four more parameters $\tilde{r}$, $\tilde{s}$, $\tilde{t}_1$ and $\tilde{t}_2$ reflecting isomorphism properties of the respective clone segments. It is not difficult to see that $R = (r, s, t_1, t_2)$ and $\widetilde{R} = (\tilde{r}, \tilde{s}, \tilde{t}_1, \tilde{t}_2)$ are grounded quadruples – this is because $=$ and $\cong$ are transitive relations, and that $r \le \tilde{r}$, $s \le \tilde{s}$, and $t_i \le \tilde{t}_i$ for $i = 1, 2$. We abbreviate the latter four relations by writing $R \le \widetilde{R}$.

Recall that for any finitary type Σ and any two sets $\Omega' \subseteq \Omega \subseteq \Sigma \setminus (\Sigma_0 \cup \Sigma_1)$, the algebraic theory $\mathscr{T}(\Sigma, \Omega)$ is a subtheory of $\mathscr{T}(\Sigma, \Omega')$. Let

$$E : \mathscr{T}(\Sigma, \Omega) \to \mathscr{T}(\Sigma, \Omega')$$

denote the resulting inclusion functor.

Definition 2.2. Let $(\mathscr{K}, U)$ and $(\mathscr{K}', U')$ be well-pointed concrete categories, and let $F : \mathscr{K} \to \mathscr{K}'$ be a functor satisfying $U' \circ F = U$. We say that such a functor F is *comprehensive* if there exists a cardinal number α such that, for every finitary type Σ with

$$\mathrm{card}\,\Sigma_0 \ge \alpha + \mathrm{card}(\Sigma \setminus \Sigma_0)$$

and for any choice of

$$\Omega' \subseteq \Omega \subseteq \Sigma \setminus (\Sigma_0 \cup \Sigma_1),$$

there exist representations $\Phi : \mathscr{T}(\Sigma, \Omega) \to \mathscr{K}$ and $\Phi' : \mathscr{T}(\Sigma, \Omega') \to \mathscr{K}'$ such that the diagram

$$\begin{array}{ccc} \mathscr{T}(\Sigma, \Omega) & \xrightarrow{E} & \mathscr{T}(\Sigma, \Omega') \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\Phi'} \\ \mathscr{K} & \xrightarrow{F} & \mathscr{K}' \end{array}$$

commutes. We say that F is *α-comprehensive* if the cardinal α needs to be mentioned explicitly.

Theorem 2.3. *Let $(\mathscr{K}, U)$ and $(\mathscr{K}', U')$ be well-pointed concrete categories and let $F : \mathscr{K} \to \mathscr{K}'$ be a comprehensive functor such that $U' \circ F = U$. Then for any two grounded quadruples $R = (r, s, t_1, t_2)$, $\widetilde{R} = (\tilde{r}, \tilde{s}, \tilde{t}_1, \tilde{t}_2)$ with $R \le \widetilde{R}$ there exist $a_1, a_2 \in \mathrm{obj}\,\mathscr{K}$ and $F(a_1), F(a_2) \in \mathrm{obj}\,\mathscr{K}'$ such that*

$$U(a_1) = U(a_2) (= U'F(a_1) = U'F(a_2)) \text{ and } M(a_i) = M(F(a_j)) \text{ for } i, j = 1, 2$$

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and, for $i = 1, 2$, their clones satisfy

$$\begin{aligned} r &= \sup\{k \mid \mathrm{Clo}_k(a_1) = \mathrm{Clo}_k(a_2)\}, & \tilde{r} &= \sup\{k \mid \mathrm{Clo}_k(a_1) \cong \mathrm{Clo}_k(a_2)\}, \\ s &= \sup\{k \mid \mathrm{Clo}_k(F(a_1)) = \mathrm{Clo}_k(F(a_2))\}, & \tilde{s} &= \sup\{k \mid \mathrm{Clo}_k(F(a_1)) \cong \mathrm{Clo}_k(F(a_2))\}, \\ t_i &= \sup\{k \mid \mathrm{Clo}_k(a_i) = \mathrm{Clo}_k(F(a_i))\}, & \tilde{t}_i &= \sup\{k \mid \mathrm{Clo}_k(a_i) \cong \mathrm{Clo}_k(F(a_i))\}. \end{aligned}$$

The proof of Theorem 2.3 involves a combinatorial discussion, and will be the subject of the remainder of this section. In Section 3, we apply Theorem 2.3 to the case when $(\mathscr{K}, U) = (\mathscr{K}', U') = \mathbf{Top}$ and $F : \mathbf{Top} \to \mathbf{Top}$ is a topological modification satisfying certain natural requirements, stated in Theorem 3.3 for lower modifications and in Theorem 3.8 for upper modifications. The requirements are mild enough to allow these theorems to be applied to the compactly generated modification, the sequential modification, the completely regular (= Tychonoff) modification, and to some other modifications. Proofs of topological results described in Section 3, namely that all these modifications are $2^{\aleph_0}$-comprehensive, are presented in the Section 4 and 5. These three sections thus form the 'topological' part of the paper. Moreover, the spaces $F(a_1)$ and $F(a_2)$ constructed in Sections 4 and 5 are metrizable, and this provides a joint strengthening of those results of [8, 9] and [6] which concern clone segment equality and isomorphism.

We begin to prove Theorem 2.3 as follows.

DEFINITION 2.4. Let $R = (r, s, t_1, t_2)$ be a grounded quadruple. On a four-element set $D = \{a_1, a_2, \bar{a}_1, \bar{a}_2\}$ define a binary symmetric function $\varphi : D \times D \to \{2, 3, \dots, \infty\}$ by setting $\varphi(d, d) = \infty$ for all $d \in D$, by $\varphi(a_1, a_2) = r$, $\varphi(\bar{a}_1, \bar{a}_2) = s$, by $\varphi(a_i, \bar{a}_i) = t_i$ for $i = 1, 2$, as indicated in the diagram

$$\begin{array}{ccccc} a_1 & \longleftarrow & (t_1) & \longrightarrow & \bar{a}_1 \\ \uparrow & & & & \uparrow \\ (r) & & & & (s) \\ \downarrow & & & & \downarrow \\ a_2 & \longleftarrow & (t_2) & \longrightarrow & \bar{a}_2 \end{array}$$

and by

$$\varphi(a_i, \bar{a}_{3-i}) = \max\{\min\{r, t_{3-i}\}, \min\{s, t_i\}\} \quad \text{for} \quad i = 1, 2.$$

LEMMA 2.5. *The function φ is an r-ultrametric on the set $D = \{a_1, a_2, \bar{a}_1, \bar{a}_2\}$. Furthermore, for any $n \geq 1$, the relation*

$$e_n = \{(d, d') \in D \times D \mid n < \varphi(d, d')\}$$

is an equivalence satisfying

$$(a_1, \bar{a}_2), (\bar{a}_1, a_2) \in e_n \quad \Rightarrow \quad e_n = D \times D.$$

PROOF. Since the quadruple $R = (r, s, t_1, t_2)$ is grounded, at least two of its entries equal $m = \min\{r, s, t_1, t_2\}$. If $r = s = m$ or $t_1 = t_2 = m$ then $\varphi(a_1, \bar{a}_2) = \varphi(\bar{a}_1, a_2) = m$ by the definition of φ, and so it follows that $\varphi(x, y) \geq \min\{\varphi(x, z), \varphi(y, z)\}$ for all $x, y, z \in D$. Otherwise $r \neq s$ and $t_1 \neq t_2$. With no loss of generality, assume that $r < s$, and let $t_i < t_{3-i}$. But then $r = t_i = m$ because (r, s, t_1, t_2) is grounded, and hence $\varphi(a_i, \bar{a}_{3-i}) = m$ and $\varphi(\bar{a}_i, a_{3-i}) = \min\{s, t_{3-i}\}$. Again, $\varphi(x, y) \geq \min\{\varphi(x, z), \varphi(y, z)\}$ for all $x, y, z \in D$ is easily verified. Therefore φ is an r-ultrametric on D.

The relation e_n is clearly an equivalence. If $(a_i, \bar{a}_{3-i}) \in e_n$ for $i = 1, 2$, then n is smaller than at least three entries of the grounded quadruple R. Since e_n is transitive and D has only four elements, it follows that $e_n = D \times D$. □

Let $R = (r, s, t_1, t_2)$ and $\widetilde{R} = (\tilde{r}, \tilde{s}, \tilde{t}_1, \tilde{t}_2)$ be grounded quadruples satisfying $R \leq \tilde{R}$. Let φ be the r-ultrametric on $D = \{a, \bar{a}, b, \bar{b}\}$ defined in 2.4, and let e_n be its associated equivalence from Lemma 2.5. Replacing R by $\widetilde{R}$, we similarly define an r-ultrametric $\widetilde{\varphi}$ on D and its associated equivalence $\tilde{e}_n$. Clearly $\varphi \leq \widetilde{\varphi}$. Set $N = (1 + \sup \widetilde{\varphi}) \setminus \{0, 1\}$. It is clear that $e_{n+1} \subseteq e_n \subseteq \tilde{e}_n$ and $\tilde{e}_{n+1} \subseteq \tilde{e}_n$ for all $n \in N$, and that Lemma 2.5 holds for $\widetilde{\varphi}$ and $\tilde{e}_n$ as well as for φ and e_n.

COROLLARY 2.6. *For $n \in N$, the equivalence pairs $e_n \subseteq \tilde{e}_n$ associated with r-ultrametrics $\varphi \leq \widetilde{\varphi}$ by 2.5 are exactly those in which neither e_n nor $\tilde{e}_n$ partition D into classes $\{a_1, \bar{a}_2\}$, $\{\bar{a}_1, a_2\}$. Such pairs are called admissible.*

LEMMA 2.7. *For $D = \{a_1, a_2, \bar{a}_1, \bar{a}_2\}$ and any admissible pair $e_n \subseteq \tilde{e}_n$ of equivalences on D, there exists a collection $\{M_n(d) \mid d \in D,\ n \in N\}$ of sets such that*

(a) $M_n(a_i) \subseteq M_n(\bar{a}_i)$ *for* $i = 1, 2$,
(b) $M_n(d) = M_n(d')$ *if and only if* $(d, d') \in e_n$,
(c) $\operatorname{card} M_n(d) = \operatorname{card} M_n(d')$ *if and only if* $(d, d') \in \tilde{e}_n$.

PROOF. For each admissible pair $e_n \subseteq \tilde{e}_n$ of equivalences and $i = 1, 2$, we shall construct four sets $A_i = M_n(a_i)$ and $M_n(\bar{a}_i) = A_i \cup A'_i$ – where the latter union will always be disjoint, and the set A'_i will be infinite when $(a_i, \bar{a}_i) \in \tilde{e}_n \setminus e_n$. Nonextremal equivalences will be identified by their nonsingleton classes. The obvious choice of cardinalities in cases when $\tilde{e}_n$ is the total equivalence will be left out in what follows. Case 0: Let e_n be the diagonal. To satisfy (a) and (b), we choose nonvoid sets $A_1 \neq A_2$ and $A'_1 \neq A'_2$. In order to also satisfy (c), we select their cardinalities as follows:

for $\tilde{e}_n = e_n$ select $\operatorname{card} A_1 < \operatorname{card} A_2 < \operatorname{card} A'_1 < \operatorname{card} A'_2$,

for $\tilde{e}_n \sim \{\bar{a}_1, \bar{a}_2\}$ select $\operatorname{card} A_1 < \operatorname{card} A_2 < \operatorname{card} A'_1 = \operatorname{card} A'_2$,
for $\tilde{e}_n \sim \{a_1, a_2\}$ select $\operatorname{card} A_1 = \operatorname{card} A_2 < \operatorname{card} A'_1 < \operatorname{card} A'_2$,
for $\tilde{e}_n \sim \{a_i, \bar{a}_i\}$ select $\operatorname{card} A_i = \operatorname{card} A'_i < \operatorname{card} A_{3-i} < \operatorname{card} A'_{3-i}$,
for $\tilde{e}_n \sim \{a_{3-i}, \bar{a}_i\}$ select $\operatorname{card} A_i < \operatorname{card} A'_i = \operatorname{card} A_{3-i} < \operatorname{card} A'_{3-i}$,
for $\tilde{e}_n \sim \{\bar{a}_i, a_{3-i}, \bar{a}_{3-i}\}$ select $\operatorname{card} A_i < \operatorname{card} A_{3-i} = \operatorname{card} A'_i = \operatorname{card} A'_{3-i}$,
for $\tilde{e}_n \sim \{a_i, a_{3-i}, \bar{a}_{3-i}\}$ select $\operatorname{card} A_i = \operatorname{card} A_{3-i} = \operatorname{card} A'_{3-i} < \operatorname{card} A'_i$,
for $\tilde{e}_n \sim \{a_1, a_2\}, \{\bar{a}_1, \bar{a}_2\}$ select $\operatorname{card} A_1 = \operatorname{card} A_2 < \operatorname{card} A'_1 = \operatorname{card} A'_2$,
for $\tilde{e}_n \sim \{a_1, \bar{a}_1\}, \{a_2, \bar{a}_2\}$ select $\operatorname{card} A_1 = \operatorname{card} A'_1 < \operatorname{card} A_2 = \operatorname{card} A'_2$.

Case 1: $e_n \sim \{\bar{a}_1, \bar{a}_2\}$. To satisfy (a) and (b), we choose pairwise disjoint sets C and $A_i \neq \emptyset$ with $i = 1, 2$, and set $A'_{3-i} = A_i \cup C$ for $i = 1, 2$. *For the equivalences* $\tilde{e}_n \supseteq e_n$ we satisfy (c) through the following choices of cardinalities:

for $\tilde{e}_n = e_n$ select $\operatorname{card} A_1 < \operatorname{card} A_2 < \operatorname{card} C$,
for $\tilde{e}_n \sim \{\bar{a}_i, a_{3-i}, \bar{a}_{3-i}\}$ select $\operatorname{card} A_i < \operatorname{card} A_{3-i} = \operatorname{card} C$,
for $\tilde{e}_n \sim \{a_1, a_2\}, \{\bar{a}_1, \bar{a}_2\}$ select $\operatorname{card} A_1 = \operatorname{card} A_2 < \operatorname{card} C$.

Case 2: $e_n \sim \{a_1, a_2\}$. Here we set $A_1 = A_2 = A$ and choose nonvoid $A'_1 \neq A'_2$. Then (a) and (b) hold. To satisfy (c), we make these choices of cardinalities:

for $\tilde{e}_n = e_n$ select $\operatorname{card} A < \operatorname{card} A'_2 < \operatorname{card} A'_1$,
for $\tilde{e}_n \sim \{a_i, a_{3-i}, \bar{a}_{3-i}\}$ select $\operatorname{card} A = \operatorname{card} A'_{3-i} < \operatorname{card} A'_i$,
for $\tilde{e}_n \sim \{a_1, a_2\}, \{\bar{a}_1, \bar{a}_2\}$ select $\operatorname{card} A < \operatorname{card} A'_2 = \operatorname{card} A'_1$.

Case 3: $e_n \sim \{a_i, \bar{a}_i\}$. We set $A'_i = \emptyset$ and choose nonvoid sets $A_1 \neq A_2$. Then (a) and (b) hold with any $A'_{3-i} \neq \emptyset$, and we choose the cardinalities as follows:

for $\tilde{e}_n = e_n$ select $\operatorname{card} A_{3-i} < \operatorname{card} A_i < \operatorname{card} A'_{3-i}$,
for $\tilde{e}_n \sim \{a_i, \bar{a}_i, \bar{a}_{3-i}\}$ select $\operatorname{card} A_{3-i} < \operatorname{card} A_i = \operatorname{card} A'_{3-i}$,
for $\tilde{e}_n \sim \{a_i, a_{3-i}, \bar{a}_{3-i}\}$ select $\operatorname{card} A_{3-i} = \operatorname{card} A_i < \operatorname{card} A'_{3-i}$,
for $\tilde{e}_n \sim \{a_1, \bar{a}_1\}, \{a_2, \bar{a}_2\}$ select $\operatorname{card} A_i < \operatorname{card} A_{3-i} = \operatorname{card} A'_{3-i}$.

Case 4: $e_n \sim \{a_{3-i}, \bar{a}_i\}$. Here we choose nonvoid disjoint sets A'_1 and A'_2, and set $A_{3-i} = A_i \cup A'_i$. It is clear that (a) and (b) hold with any such choice. To satisfy (c), we choose cardinalities of these sets as follows:

for $\tilde{e}_n = e_n$ select $\operatorname{card} A_i < \operatorname{card} A'_i < \operatorname{card} A'_{3-i}$,
for $\tilde{e}_n \sim \{a_{3-i}, \bar{a}_i, \bar{a}_{3-i}\}$ select $\operatorname{card} A_i < \operatorname{card} A'_i = \operatorname{card} A'_{3-i}$,
for $\tilde{e}_n \sim \{a_i, a_{3-i}, \bar{a}_i\}$ select $\operatorname{card} A_i = \operatorname{card} A'_i < \operatorname{card} A'_{3-i}$.

In view of Corollary 2.6, this completes the proof for all pairs $e_n \subseteq \tilde{e}_n$ in which e_n has more than two classes. The remainder, concerning pairs in which $e_n \not\sim \{a, \bar{b}\}, \{\bar{a}, b\}$ and e_n has at most two classes, is left to the reader. □

PROOF OF THEOREM 2.3 CONCLUDED. Let $D = \{a_1, a_2, \bar{a}_1, \bar{a}_2\}$, and let $\widetilde{\varphi} \geq \varphi$ be r-ultrametrics on D. Given their associated system $\{M_n(d) \mid d \in D,\ n \in N\}$ constructed in Lemma 2.7, for any $n \in N$ we set $\Sigma_n = \bigcup\{M_n(d) \mid d \in D\}$, and then

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choose Σ_0 so that card $\Sigma_0 \geq \alpha + \operatorname{card} \bigcup \{\Sigma_n \mid n \in N\}$, where α is an infinite cardinal for which $F : \mathscr{K} \to \mathscr{K}'$ is comprehensive. We set $\Omega_{n,d} = \Sigma_n \setminus M_n(d)$ for all $n \in N$ and $d \in D$, and then write $\Omega_d = \bigcup \{\Omega_{n,d} \mid n \in N\}$.

By Lemma 1.5 and Lemma 2.7, the algebraic theories $\mathscr{T}(\Sigma, \Omega_d)$ with $d \in D$ are such that their n-segments are isomorphic exactly when $n < \widetilde{\varphi}(d, d')$ and coincide exactly when $n < \varphi(d, d')$. Since $M_n(a_i) \subseteq M_n(\bar{a}_i)$ for $i = 1, 2$ and all $n \in N$, there exist inclusion homomorphisms

$$E_i : \mathscr{T}(\Sigma, \Omega_{a_i}) \to \mathscr{T}(\Sigma, \Omega_{\bar{a}_i}) \quad \text{for} \quad i = 1, 2.$$

Since the functor $F : \mathscr{K} \to \mathscr{K}'$ is comprehensive for the infinite cardinal α, we have $U' \circ F = U$ for the respective forgetful functors U and U', and there exist representations

$$\Phi_i : \mathscr{T}(\Sigma, \Omega_{a_i}) \to \mathscr{K} \quad \text{and} \quad \Phi_i' : \mathscr{T}(\Sigma, \Omega_{\bar{a}_i}) \to \mathscr{K}'$$

satisfying $\Phi_i' \circ E_i = F \circ \Phi_i$ for $i = 1, 2$. Since $\mathscr{T}^{\Omega}(a^0, a) = \mathscr{T}(a^0, a)$ is the underlying set of the images $\Phi_i(a)$ and $\Phi_i'(a)$ of the base object a of any of these theories, as in Section 1 we conclude that $U\Phi_1(a) = U\Phi_2(a)$ and $U'\Phi_1'(a) = U'\Phi_2'(a)$. From $U' \circ U = F$ it then follows that the four objects $\Phi_i(a) \in \operatorname{obj} \mathscr{K}$ and $\Phi_i'(a) \in \mathscr{K}'$ have the same underlying set. But then the grounded quadruples $R \leq \widetilde{R}$ are represented by these four objects, exactly as claimed in Theorem 2.3.

3. Topological modifications

DEFINITION 3.1. Recall that a functor $m : \mathrm{Top} \to \mathrm{Top}$ is called a *modification* if $m \circ m = m$, and $U \circ m = U$ for the forgetful functor $U : \mathrm{Top} \to \mathrm{Set}$. If this is the case, for any space $X = (P, t)$ we write $mX = (P, mt)$. If $t \leq mt$ for all spaces $X = (P, t)$ (which means that the identity map $(P, t) \to (P, mt)$ is continuous, that is, the topology t is finer than mt), we say that m is an *upper modification* and write $X \leq mX$. If $mt \leq t$ for all $X = (P, t)$, then m is a *lower modification*, and we write $mX \leq X$. *We also note that [2] uses these terms in a somewhat wider sense.*

To apply Theorem 2.3 in case when $F = m$ is an upper or lower topological modification, we need only show that m is comprehensive. Theorems 3.3 and 3.8 below give the respective sufficient conditions for comprehensivity.

OBSERVATION 3.2. We note that the lower modifications $m : \mathrm{Top} \to \mathrm{Top}$ are exactly all full coreflections $c : \mathrm{Top} \to \mathrm{Top}$ other than the constant one to the empty space. Indeed, if m is a lower modification, then the identity map 1_P carries the coreflecting map $c_X : mX \to X$ for every space $X = (P, t)$, and the resulting coreflection c maps

Top onto the full subcategory $\mathscr{K}_m$ determined in Top by all spaces mX with $X \in$ Top. Conversely, if c : Top $\to$ Top is a coreflection onto a full subcategory $\mathscr{K}$ of Top containing a nonvoid space then, because of the continuity of constant maps, every coreflecting map $c_X : cX \to X$ is carried by the identity map, and hence $cX \leq X$. Therefore $U \circ c = U$, and $c \circ c = c$ follows.

THEOREM 3.3. *A lower modification* m : Top $\to$ Top *is comprehensive whenever*

(a) *$mX = X$ for every metrizable space X,*
(b) *mY is a closed subspace of mX whenever Y is a closed subspace of X, and*
(c) *there exists a Hausdorff totally disconnected space X_0 for which $mX_0 \neq X_0$ and mX_0 is metrizable.*

Theorem 3.3 can be applied to many coreflections. From [2], we recall a well-known fact that every class $\mathscr{C}$ of topological spaces determines a lower modification $m_{\mathscr{C}}$ by the following rule:

> for every space X, a set O is open in $m_{\mathscr{C}}X$ exactly when $f^{-1}(O)$ is open in Y for every continuous $f : Y \to X$ with $Y \in \mathscr{C}$.

For any given class $\mathscr{C} \subseteq$ Top, the full subcategory $\widetilde{\mathscr{C}}$ of Top determined by all spaces X with $m_{\mathscr{C}}X = X$ is the coreflective hull of $\mathscr{C}$, and it is clear that $m_{\mathscr{C}} = m_{\widetilde{\mathscr{C}}}$ is its corresponding coreflection.

OBSERVATION 3.4. Let $\mathscr{S} \subseteq$ Top consist of all finite T_1-spaces and the subspace $X_0 = \{0\} \cup \{1/n \mid n = 1, 2, \dots\}$ of reals (that is, a convergent sequence). Then the functor $m_{\mathscr{S}}$ is the well-known *sequential modification.* It is also well-known and easy to verify that

(i) if $\mathscr{C} \supseteq \mathscr{S}$ then $m_{\mathscr{C}}X = X$ for any metrizable X, and hence $m_{\mathscr{C}}$ satisfies (a) in Theorem 3.3,
(ii) if $\mathscr{C}$ is closed under continuous images then $m_{\mathscr{C}}Y$ is a closed subspace of $m_{\mathscr{C}}X$ whenever Y is a closed subspace of X, and hence $m_{\mathscr{C}}$ satisfies (b) in Theorem 3.3.

CLAIM 3.5. *Theorem* 3.3 *can be applied when*

> *$\mathscr{C} = \mathscr{S}$, that is, when $m_{\mathscr{C}}$ is the sequential modification, or*
> *$\mathscr{C}$ consists of all compact spaces, that is, when $m_{\mathscr{C}} = CG$ is the compactly generated modification, or*
> *$\mathscr{C}$ consists of all spaces X with* card $X \leq \gamma$ *for some infinite cardinal γ, that is, $m_{\mathscr{C}}$ is the coreflection onto the full subcategory of all spaces with tightness $\leq \gamma$, or*
> *$\mathscr{C}$ consists of all compact spaces of cardinality $\leq \gamma$.*

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Indeed, for these four types of modifications the class $\mathscr{C}$ contains $\mathscr{S}$, and is closed under the formation of continuous images. Thus, by Observation 3.4, the hypotheses (a) and (b) of Theorem 3.3 are satisfied. Given an infinite cardinal γ, in all cases the space $X_0 = (A, u)$ can be chosen so that $\operatorname{card} A > \gamma$, all points of X_0 except $o \in A$ are isolated, and o has a local open basis

$$\{\{o\} \cup (A \setminus D) \mid D \subset A \text{ and } \operatorname{card} D \leq \gamma\}.$$

Thus, by Theorem 3.3, these four types of topological modification are comprehensive.

The somewhat involved proof of Theorem 3.3 will be presented in Sections 4 and 5. Here we prove only two easy claims about lower modifications needed in these sections. Since these claims generally do not hold for upper modifications, the hypotheses of Theorem 3.8 below are more restrictive than those of Theorem 3.3.

Being a coreflection, any lower modification m satisfies

$$m(X_1 \times X_2) = m(mX_1 \times mX_2),$$

and hence the claim below is immediate.

CLAIM 3.6. *If m satisfies* (a) *of Theorem* 3.3, *and if X is a space for which mX is metrizable, then, for every integer $n \geq 1$,*

$$m(X^n) = (mX)^n.$$

For the next lemma, we need to introduce some notation. Suppose that ϱ is a metric on P with $\operatorname{diam}(P, \varrho) = 1$ and such that $\varrho(c, c') = 1$ whenever $c, c' \in C \subseteq P$ are distinct, we then say that ϱ is 1-*discrete* on C. For any topology w on C, we use the symbol $\varrho * w$ to denote the topology on P determined by the following two requirements:

(i) for any $x \in P \setminus C$, the system of all $(\varrho * w)$-neighborhoods of x and the system of all its ϱ-neighborhoods coincide,

(ii) for any $x \in C$, the system

$$\{U_{\varrho,\varepsilon} \mid U \text{ is a } w\text{-open neighborhood of } x \text{ and } \varepsilon > 0\},$$

where $U_{\varrho,\varepsilon} = \{y \in P \mid \varrho(y, U) < \varepsilon\}$, forms a local open basis of x in $(P, \varrho * w)$.

LEMMA 3.7. *Let m be a lower modification satisfying* (a) *and* (b) *of Theorem* 3.3. *Let ϱ be a metric on P which is* 1-*discrete on $C \subseteq P$. Let w be a topology on C such that (C, mw) is metrizable. Then*

$$m(\varrho * w) = \varrho * mw.$$

PROOF. Since $mw \le w$ and mw is metrizable, the topology $t = \varrho * mw$ is also metrizable, and hence $mt = t$. Clearly $t \le \varrho * w$. Since $t_1 = m(\varrho * w)$ is the coarsest topology with $mt_1 = t_1$ that is finer than $\varrho * w$, it follows that $\varrho * mw = t \le t_1 = m(\varrho * w)$.

To prove the reverse inequality $t_1 \le t$, we show that for any $M \subseteq P$, every x from the t_1-closure $t_1 M$ of M belongs also to the t-closure tM of M. This is clear for any $x \in P \setminus C$. So let $x \in C$ and suppose that $x \in t_1 M \setminus tM$. Since $x \notin tM$, there exist an mw-neighborhood O of x and an $\varepsilon > 0$ such that $O_{\varrho,\varepsilon} \cap M = \emptyset$. We may suppose that $\varepsilon < 1$. Since C is closed in $(P, \varrho * w)$ and because ϱ is 1-discrete on C, we have $t_1 \restriction C = m((\varrho * w) \restriction C) = mw$. Thus O is also a neighborhood of x in $(C, t_1 \restriction C)$. For every $c \in C$ set $K_c = \{z \in P \mid \varrho(z, c) \le \varepsilon/2\}$. Then $R = \{K_c \mid c \in C\}$ is closed in $(P, \varrho * w)$. Since $S = \{z \in P \mid \varrho(z, C) \ge \varepsilon/3\}$ is also closed in $(P, \varrho * w)$, and because $\varrho * w$ is metrizable on S, the three topologies $\varrho * w$, t_1 and t coincide on S, so we may assume that $M \subseteq R$. The map $f : (R, (\varrho * w) \restriction R) \to (C, w)$ given by $f^{-1}(c) = K_c$ for every $c \in C$ is obviously continuous, and hence it is also continuous as a map from $m((\varrho * w) \restriction R) = (m(\varrho * w)) \restriction R = t_1 \restriction R$ to $mw = m((\varrho * w) \restriction C) = (m(\varrho * w)) \restriction C = t_1 \restriction C$. Whence $f(x) \in (t_1 \restriction C)(f(M))$ follows from $x \in t_1 M$. But $x = f(x) \in C$ has the $(t_1 \restriction C)$-neighborhood O for which $O_{\varrho,\varepsilon} \cap M = \emptyset$ and, because $M \subseteq R$, this is possible only when $f(M) \subseteq C \setminus O$, a contradiction. □

Now we formulate a sufficient condition for comprehensiveness of an upper modification.

THEOREM 3.8. *An upper modification* $m : \mathrm{Top} \to \mathrm{Top}$ *is comprehensive whenever*

(a) *$mX = X$ for any metrizable space X,*
(b) *if Y is a closed C^*-embedded subspace of X then mY is a closed C^*-embedded subspace of mX, and*
(c) *there exist a full subcategory $\mathscr{K}$ of* Top *and a space $X_0 \in \mathscr{K}$ such that*
    (c1) *$\mathscr{K}$ is closed under finite products and the domain restriction $\mathscr{K} \to$ Top of m preserves them,*
    (c2) *$X_0 \in \mathscr{K}$ is a Hausdorff totally disconnected space for which $mX_0 \ne X_0$ and mX_0 is metrizable, and*
    (c3) *$\mathscr{K}$ contains all metrizable spaces, and also all spaces $(P, \varrho * u)$ such that ϱ is a 1-discrete metric on $C \subseteq P$ and $(C, u) \in \mathscr{K}$.*

Theorem 3.8 applies to the most important upper modification in Top, the completely regular modification CR (which is given by the requirement that the sets open in CRX are exactly the cozero sets in a space X). The fact that CR satisfies (a) and (b) is well-known. To see that CR satisfies also (c), we let $\mathscr{K}$ be determined by all

Hausdorff spaces X in which the closure of every neighborhood of any point contains its cozero neighborhood (see [8], where such spaces are said to have the R-property). Define a space X_0 on the set $\mathbf{Q} \cap [0, 1]$ of all rationals in the closed interval [0,1] whose Euclidean topology is changed at $0 \in X_0$ so that the system

$$\left\{\mathbf{Q} \cap \left[0, \frac{1}{n}\right) \setminus T \mid n = 1, 2, \ldots\right\}, \text{ where } T = \left\{\frac{1}{k} \mid k = 1, 2, \ldots\right\}$$

is its local open basis there. An easy proof that $\mathscr{K}$ and X_0 indeed satisfy (c) can be found in [8].

COROLLARY 3.9. *The completely regular modification* CR : Top $\to$ Top *is comprehensive.*

4. The basic construction

Let (A, u) be a totally disconnected Hausdorff space, let $\Sigma = \bigcup_{n=0}^{\infty} \Sigma_n$ be a finitary type such that

$$\operatorname{card} \Sigma_0 \geq 2^{\aleph_0} + \operatorname{card} A + \operatorname{card}(\Sigma \setminus \Sigma_0),$$

and let $\Omega \subseteq \Sigma \setminus (\Sigma_0 \cup \Sigma_1)$ be given. In this section, we construct a space $X = (P, t)$ satisfying the conditions (i)–(iii) below.

(i) X determines a representation of the algebraic theory $\mathscr{T} = \mathscr{T}(\Sigma, \Omega)$ in Top, that is,

$$\operatorname{Clo}(X) \simeq (\mathscr{T}, \mathscr{T}(a^0, -))$$

where a denotes the base object of $\mathscr{T}$.

Since Σ_0 is infinite, the algebraic theory $\mathscr{T}$ is well-pointed. According to Section 0, this implies that the hom-functor $\mathscr{T}(a^0, -) : \mathscr{T} \to$ Set is faithful.

(ii) (A, u) is a closed subspace of X.

Since the space X depends on (A, u), Σ and Ω, we write

$$X = \langle (A, u), (\Sigma, \Omega) \rangle.$$

This space will also be such that if m : Top $\to$ Top is either a lower modification satisfying (a), (b), (c) in Theorem 3.3 or an upper modification satisfying (a), (b), (c) in Theorem 3.8 and if (A, mu) is metrizable, then

(iii) $m(\langle (A, u), (\Sigma, \Omega) \rangle) = \langle (A, mu), (\Sigma, \Omega) \rangle.$

Since $X = (P, t)$ has to determine a representation of $\mathscr{T} = \mathscr{T}(\Sigma, \Omega)$ in Top, necessarily $P = \mathscr{T}(a^0, a)$, see Section 0. We use the fact, recalled already in Section 1, that $\mathscr{T}(a^0, a)$ is the underlying set of the initial Σ-algebra, that is, of the free Σ-algebra over the empty set of generators. We shall also need its operations, and hence we denote

$$\mathbb{P} = (P, \{p_\sigma \mid \sigma \in \Sigma\})$$

the initial Σ-algebra. Its underlying set P is bijective to the set $\bigcup_{k=0}^{\infty} M_k^{(0)}$ described in Section 1. Since we need to express points of P as values of operations

$$p_\sigma : P^{\operatorname{ar}\sigma} \to P \text{ with } \sigma \in \Sigma,$$

we use the following notation. For $\sigma \in \Sigma_0$, the nullary operation $p_\sigma : P^0 \to P$ is, as usual, identified with its value. Hence $p_\sigma \in P$ for every $\sigma \in \Sigma_0$, and we denote

$$G_0 = \{p_\sigma \mid \sigma \in \Sigma_0\}.$$

It is clear that G_0 is bijective to the set $M_0^{(0)} = \Sigma_0$ from Section 1.

If $\sigma \in \Sigma_n$ and $n \geq 1$, then $p_\sigma : P^n \to P$ is a one-to-one map. Let $B_\sigma = p_\sigma(P^n)$ denote its image. It is well-known that

$$\begin{aligned} B_\sigma \cap G_0 = \emptyset \quad &\text{for all} \quad \sigma \in \Sigma \setminus \Sigma_0, \\ B_\sigma \cap B_{\sigma'} = \emptyset \quad &\text{for all} \quad \sigma, \sigma' \in \Sigma \setminus \Sigma_0 \text{ with } \sigma \neq \sigma'. \end{aligned}$$

The inductive formula of Section 1 then takes on the following form. If we set

$$\begin{aligned} &B_{\sigma,1} = p_\sigma(G_0^{\operatorname{ar}\sigma}), && B_1 = \bigcup \left\{B_{\sigma,1} \mid \sigma \in \Sigma \setminus \Sigma_0\right\}, \\ &G_k = G_{k-1} \cup B_k && \text{(and this set is bijective to } M_k^{(0)}\text{)}, \\ &B_{\sigma,k+1} = p_\sigma(G_k^{\operatorname{ar}\sigma} \setminus G_{k-1}^{\operatorname{ar}\sigma}), && B_{k+1} = \bigcup \left\{B_{\sigma,k+1} \mid \sigma \in \Sigma \setminus \Sigma_0\right\}, \end{aligned}$$

then $P = \bigcup_{k=0}^{\infty} G_k$. Furthermore $B_\sigma = \bigcup_{k=1}^{\infty} B_{\sigma,k}$ and

$$\bigcup \{B_\sigma \mid \sigma \in \Sigma \setminus \Sigma_0\} = \bigcup_{k=1}^{\infty} B_k = P \setminus G_0.$$

Finally, we denote $B = P \setminus G_0$.

Since each space X constructed on the set P shall contain a given space (A, u), we continue by selecting a subset

$$A \subset G_0 \text{ with } \operatorname{card} A \leq \operatorname{card}(G_0 \setminus A).$$

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For another construction, we also select a one-to-one sequence $a_0, a_1, a_2, \dots$ in $G_0 \setminus A$, and let D stand for the set of its elements. With the set $B = P \setminus G_0$, we finally denote

$$C = A \cup B \cup D.$$

The basic construction will depend not only on the given (A, u) and (Σ, Ω), but also on a certain metric ϱ on P which is 1-discrete on C (see Lemma 3.7). The metric ϱ will be that from Claim 4.7 below, and *will remain fixed* throughout the remainder of the paper.

Let (P, ϱ) be a metric space whose metric ϱ is 1-discrete on the set $C = A \cup B \cup D$. For an integer $n \geq 1$, let ϱ^n denote the metric on P^n given by

$$\varrho^n(x, y) = \max\{\varrho(x_j, y_j) \mid j \in n\}$$

for all $x = (x_0, \dots, x_{n-1})$ and $y = (y_0, \dots, y_{n-1})$. Denote $a^{(n)} = (a_0, \dots, a_{n-1})$, where $\{a_0, \dots, a_{n-1}\} \subseteq D$.

Next we produce a metric μ on P^n inducing a topology that differs from that induced by ϱ^n in a closely specified way. First, for any $x \in P^n \setminus \{a^{(n)}\}$ we define

$$\lambda_n'(x) = \frac{n \prod_{j=0}^{n-1} \varrho(x_j, a_j)}{\sum_{j=0}^{n-1} (\varrho(x_j, a_j))^n} \quad \text{and} \quad \lambda_n''(x) = \max\left\{0, \frac{1}{2} - \varrho^n(x, a^{(n)})\right\}.$$

CLAIM 4.1. *The function λ_n defined by $\lambda_n(x) = \lambda_n'(x) \cdot \lambda_n''(x)$ for all $x \in P^n \setminus \{a^{(n)}\}$ and by $\lambda_n(a^{(n)}) = 0$ has these properties:*

(1) *λ_n is continuous on $(P^n, \varrho^n) \setminus \{a^{(n)}\}$,*
(2) *$0 \leq \lambda_n(x) \leq 1/2$ for all $x \in P^n$,*
(3) *$\lambda_n(x) = 0$ whenever $\varrho^n(x, a^{(n)}) \geq 1/2$, or $x_j = a_j$ for some $j \in n$, or $x_i = x_j$ for some distinct $i, j \in n$,*
(4) *if $M = \{x \in P^n \setminus \{a^{(n)}\} \mid \varrho(x_i, a_i) = \varrho(x_j, a_j)$ for all $i, j \in n\}$, then*

$$\lim_{x \in M,\, x \to a^{(n)}} \lambda_n(x) = \frac{1}{2}.$$

PROOF. The property (1) is obvious. For (2), we recall that $\operatorname{diam}(P, \varrho) = 1$, and hence $0 \leq \lambda_n'(x) \leq 1$ and, obviously, $0 \leq \lambda_n''(x) \leq 1/2$ for all $x \in P^n \setminus \{a^{(n)}\}$. To prove (3), we first note that $\lambda_n'(x) = 0$ when $x_j = a_j$ for some $j \in n$, and $\lambda_n''(x) = 0$ for all $x \in P^n$ with $\varrho^n(x, a^{(n)}) \geq 1/2$. If $x_i = x_j$ for some distinct $i, j \in n$, then $1 = \varrho(a_i, a_j) \leq \varrho(a_i, x_i) + \varrho(a_j, x_j) \leq 2\varrho^n(x, a^{(n)})$, and hence $\lambda_n''(x) = 0$. Finally, for any $x \in M$ we have $\lambda_n'(x) = 1$, and (4) follows because $\lim_{x \to a^{(n)}} \lambda_n''(x) = 1/2$. □

Now we are in a position to define a new metric μ on P^n by

$$\mu(x, y) = \min\{1, \varrho^n(x, y) + |\lambda_n(x) - \lambda_n(y)|\}.$$

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If $x = (x_0, \dots, x_{n-1})$ and $y = (y_0, \dots, y_{n-1})$ are such that $x_i = y_i = a_i$ for some $i \in n$ then $\mu(x, y) = \varrho^n(x, y)$. Whence for every $i \in n$ we have

$$\mu \restriction (\pi_i^{(n)})^{-1}\{a_i\} = \varrho^n \restriction (\pi_i^{(n)})^{-1}\{a_i\}$$

(where $\pi_i^{(n)} : P^n \to P$ denotes the i-th projection). Furthermore, the ϱ^n-neighborhoods and the μ-neighborhoods of any $x \in P^n \setminus \{a^{(n)}\}$ are equivalent in the sense that each ϱ^n-neighborhood of x contains a μ-neighborhood of x and vice versa. This is no longer true for the point $a^{(n)}$: no μ-neighborhood

$$K_\varepsilon = \{y \in P^n \mid \mu(y, a^{(n)}) < \varepsilon\}$$

with $0 < \varepsilon < 1/2$ contains any ϱ^n-neighborhood of $a^{(n)}$ because $K_\varepsilon \cap M = \emptyset$. Indeed, any $y \in K_\varepsilon \cap M$ would satisfy $\mu(y, a^{(n)}) = \varrho^n(y, a^{(n)}) + \lambda_n(y)$ and $\lambda_n(y) = 1/2 - \varrho^n(y, a^{(n)})$, which is impossible. On the other hand, it is clear that each ϱ^n-neighborhood of $a^{(n)}$ contains a μ-neighborhood of $a^{(n)}$. For a future use, we denote

$$\mathscr{M}^{(n)} = \left\{K_\varepsilon \mid 0 < \varepsilon < \frac{1}{2}\right\}.$$

We also observe that, for any $i \in n$, if restricted to the set $(\pi_i^{(n)})^{-1}\{a_i\}$, the μ-neighborhoods and the ϱ^n-neighborhoods of $a^{(n)}$ coincide.

Depending on the above data, and in the notation developed so far, we now use transfinite induction to define increasing chains of topologies

$$u_0 \leq u_1 \leq u_2 \leq \cdots \quad \text{on } C, \quad \text{and} \quad t_0 \leq t_1 \leq t_2 \leq \cdots \quad \text{on } P$$

as follows.

CONSTRUCTION 4.2. Let u_0 denote the topology on C which coincides with u on A, is discrete on the complement $B \cup D$ of A in C, and such that A is clopen (= closed-and-open) in (C, u_0).

Set $t_0 = \varrho * u_0$.

Suppose that (P, t_β) and (C, u_β) have already been defined for all $\beta < \alpha$.

For $\alpha = \beta + 1$, we let u_α be the topology on C whose restriction $u_\alpha \restriction A$ coincides with the original topology u on A, it is discrete on D, and the sets A, D and all the sets B_σ with $\sigma \in \Sigma \setminus \Sigma_0$ are clopen in (C, u_α); finally, for each $\sigma \in \Sigma_n$ with $n \geq 1$, the restriction $u_\alpha \restriction B_\sigma$ is that topology for which

the bijection $p_\sigma : P^n \to B_\sigma$ is a homeomorphism of
$(P, t_\beta)^n$ onto $(B_\sigma, u_\alpha \restriction B_\sigma)$ whenever $\sigma \in \Sigma_n \setminus \Omega$,
$(P^n, z_{\beta,\sigma})$ onto $(B_\sigma, u_\alpha \restriction B_\sigma)$ whenever $\sigma \in \Sigma_n \cap \Omega$,

where $z_{\beta,\sigma}$ is that topology for which $\mathscr{M}^{(n)}$ is a local open basis of $a^{(n)}$, while a local open basis of any $x \in P^n \setminus \{a^{(n)}\}$ is the system of all its open t^n_β-neighbourhoods.

These requirements determine u_α on $C = A \cup D \cup B$ uniquely. We set $t_\alpha = \varrho * u_\alpha$.

For a limit ordinal α, we define $u_\alpha = \sup\{u_\beta \mid \beta < \alpha\}$ – that is, we let u_α be the finest topology which is coarser than every u_β with $\beta < \alpha$, and then set $t_\alpha = \varrho * u_\alpha$ again.

It is readily seen that these topologies form increasing chains $u_0 \leq u_1 \leq \cdots$ and $t_0 \leq t_1 \leq \cdots$. The claim below is immediate.

CLAIM 4.3. *There exists an ordinal α_0 such that $u_\gamma = u_{\alpha_0}$ and $t_\gamma = t_{\alpha_0}$ for every $\gamma \geq \alpha_0$. We set $t = t_{\alpha_0}$.*

The manner in which $X = (P, t)$ was constructed implies that C is closed in (P, t), and that D with the discrete topology d and (A, u) are clopen subspaces of $(C, t \restriction C)$. Furthermore, the set $B_\sigma \subseteq C$ is clopen in $(C, t \restriction C)$ and p_σ is a homeomorphism of $(P, t)^{\mathrm{ar}\,\sigma}$ onto $(B_\sigma, t \restriction B_\sigma)$ for every $\sigma \in \Sigma \setminus (\Sigma_0 \cup \Omega)$; for $\sigma \in \Omega$, the map p_σ is a homeomorphism of $(P^{\mathrm{ar}\,\sigma}, z_\sigma)$ onto $(B_\sigma, t \restriction B_\sigma)$ where $z_\sigma < t^{\mathrm{ar}\,\sigma}$, and the reason why $z_\sigma \neq t^{\mathrm{ar}\,\sigma}$ is that in z_σ the system $\mathscr{M}^{(\mathrm{ar}\,\sigma)}$ replaces the $t^{\mathrm{ar}\,\sigma}$-neighborhood system of $a^{(\mathrm{ar}\,\sigma)} \in D^{\mathrm{ar}\,\sigma}$.

The claim below is an easy generalization of [8, Lemma II.6].

LEMMA 4.4. *For any $k \geq 0$, the set G_k is open in $X = (P, t)$ and $t_k \restriction G_k = t \restriction G_k$. Moreover, every $x \in G_k$ has a neighborhood in G_k whose t-closure is still contained in G_k.*

PROOF. To prove the first claim, we first note that G_0 is open in X, evidently. On $G_0 \setminus C$, the topology t is induced by the metric ϱ, while (A, u) and (D, d) are clopen subspaces of $(C, t \restriction C)$. Since $A \cup D = G_0 \cap C$, the restriction of $t_0 = \varrho * u$ to G_0 coincides with $t \restriction G_0$. This proves the first claim for $k = 0$, and the remainder follows by an induction on k.

Now we turn to the second claim, beginning with G_0. Since G_0 is open, we have $\varrho(x, B) = \varepsilon > 0$ for every $x \in G_0$. Clearly, the closure $\overline{U}$ of $U = \{z \mid \varrho(x, z) < \varepsilon/2\}$ is contained in G_0 and hence the second claim holds for $k = 0$. Proceeding by induction in k, we now assume that $x \in G_{k+1} \setminus G_k = \bigcup\{B_{\sigma,k+1} \mid \sigma \in \Sigma \setminus \Sigma_0\}$. Since the latter union is disjoint, there is a unique $\tau \in \Sigma \setminus \Sigma_0$ for which $x \in B_{\tau,k+1} = p_\tau(G_k^{\mathrm{ar}\,\tau} \setminus G_{k-1}^{\mathrm{ar}\,\tau})$. Write $n = \mathrm{ar}\,\tau$ and denote $(x_0, \ldots, x_{n-1}) = p_\tau^{-1}(x)$. Then $x_i \in G_{k_i}$ with $k_i \leq k$ for all $i \in n$, and hence the induction hypothesis provides t-neighbourhoods U_i of x_i such that $\overline{U}_i \subseteq G_{k_i}$ for all $i \in n$. But then $U = U_0 \times \cdots \times U_{n-1}$ is a neighborhood of $p_\tau^{-1}(x)$ in $X^n = (P^n, t^n)$ such that $\overline{U} \subseteq G_k^n$. If $\tau \notin \Omega$, then p_τ is a homeomorphism of X^n onto the clopen subset B_τ of X, and the image $p_\tau(\overline{U})$

is a closed neighborhood of x contained in $p_\tau(G_k^n) \subseteq G_{k+1}$. It follows that any $V = (p_\tau(U))_{\varrho,\varepsilon}$ with $0 < \varepsilon < 1/2$ is a neighborhood of x in $X = (P, t)$ for which $\overline{V} \subseteq G_{k+1}$. If $\tau \in \Omega$, we proceed analogously whenever $p_\tau^{-1}(x) \neq a^{(n)}$. For $a^{(n)}$, we work with the metric μ described earlier, and find an open δ-neighborhood of $a^{(n)}$ contained in $U_0 \times \cdots \times U_{n-1}$. Then p_σ sends its closure to a closed neighborhood of x in B_τ as well, and we then proceed as in the previous case. □

COROLLARY 4.5. *Since $P = \bigcup\{G_k \mid k = 0, 1, \dots\}$ and $t = \sup\{t_k \mid k = 0, 1, \dots\}$, the transfinite induction defining t is, in fact, countable.*

LEMMA 4.6. *$X = (P, t)$ is a Hausdorff space.*

PROOF. Let $x, y \in P$ be distinct. We need to separate these elements by disjoint t-neighborhoods. Since (A, u) is a Hausdorff space and ϱ is 1-discrete on C, such neighborhoods exist when $x, y \in G_0$. Proceeding by induction, we now suppose that the conclusion holds in all G_l with $l \leq k$. Several cases need to be discussed. We select the case when $x, y \in G_{k+1} \setminus G_k$; the other cases are simpler.

Under this assumption, there are uniquely determined $\sigma, \tau \in \Sigma \setminus \Sigma_0$ for which $x \in B_{\sigma,k+1}$ and $y \in B_{\tau,k+1}$. If $\sigma \neq \tau$ then $(B_{\sigma,k+1})_{\varrho,1/3}$ and $(B_{\tau,k+1})_{\varrho,1/3}$ are respective disjoint t-neighborhoods of these elements. When $\sigma = \tau$, we denote $n = \operatorname{ar}\sigma$, $(x_0, \dots, x_{n-1}) = p_\sigma^{-1}(x)$ and $(y_0, \dots, y_{n-1}) = p_\tau^{-1}(y)$. Then $x_i, y_i \in G_k$ for all $i \in n$, and $x_j \neq y_j$ for some $j \in n$. By the induction hypothesis, there exist disjoint neighborhoods U_j of x_j and V_j of y_j. Set $U = p_\sigma(\pi_j^{-1}(U_j))$ and $V = p_\sigma(\pi_j^{-1}(V_j))$. Then U, V are disjoint neighborhoods of x, y in $(B_\sigma, t \restriction B_\sigma)$ and, consequently, $U_{\varrho,1/3}$ and $V_{\varrho,1/3}$ are disjoint neighborhoods of x and y in $(P, t) = X$. □

PROPOSITION 4.7. *If $(P, t) = \langle (A, u), (\Sigma, \Omega) \rangle$ and a modification $m : \mathrm{Top} \to \mathrm{Top}$ is either a lower modification satisfying* (a), (b) *in* 3.3 *or an upper modification satisfying* (a), (b), (c) *in* 3.8 *and if (A, mu) is metrizable, then $(P, mt) = \langle (A, mu), (\Sigma, \Omega) \rangle$.*

PROOF. Let

$$u_0 \leq u_1 \leq u_2 \leq \cdots \quad \text{and} \quad t_0 \leq t_1 \leq t_2 \leq \cdots$$

be the respective chains of topologies on C and on P obtained in the construction of (P, t), that is, those beginning with the topology u on A, and let

$$\tilde{u}_0 \leq \tilde{u}_1 \leq \tilde{u}_2 \leq \cdots \quad \text{and} \quad \tilde{t}_0 \leq \tilde{t}_1 \leq \tilde{t}_2 \leq \cdots$$

be the corresponding chains of topologies on C and on P obtained by the same construction employing the same metric ϱ on P, but beginning with the topology mu on A. By Lemma 4.4, it suffices to prove that $\tilde{u}_k = mu_k$ for all integers $k \geq 0$.

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This can be shown by an easy induction in k: for a lower modification it follows from Claim 3.6 and Lemma 3.7, and for an upper modification from (a), (b), (c) in Theorem 3.8. $\square$

To prove that $(P, t) = \langle (A, u), (\Sigma, \Omega) \rangle$ determines a representation of $\mathcal{T}(\Sigma, \Omega)$ in Top, we need certain specific properties of the metric ϱ on P. The proof follows, in part, the reasoning used in [6] to deal with metric spaces. Here, however, the insertion of a space (A, u) into the space (P, t) determining the representation of $\mathcal{T}(\Sigma, \Omega)$ need not conserve metrizability, and this is why a separate proof is necessary. To proceed, we need to recall some notions and facts presented elsewhere.

(a) In [9], a topological space $X = (P, t)$ was called *C-semirigid* (for a set $C \subseteq P$) if $f(P) \subseteq C$ for any continuous selfmap $f : X \to X$ other than a constant or the identity. It is clear that any C-semirigid space X with $P \setminus C \neq \emptyset$ must be connected. If X is a C-semirigid space with card$(P \setminus C) \geq 3$ and X^k its k-th power, then $f(P^k) \subseteq C$ for any continuous map $f : X^k \to X$ other than a constant or a projection, see [9].

(b) In [8], a metric ϱ on P was called *extremally C-semirigid* (for a set $C \subseteq P$) if

(1) $\operatorname{diam}(P, \varrho) = 1$, and $\varrho(x, y) = 1$ whenever $x, y \in C$ are distinct,

(2) $(P, \varrho * w)$ is C-semirigid for any Hausdorff topology w on P.

(c) In [9], it was shown that an extremally C-semirigid metric ϱ does exist on a set $P \supseteq C$ whenever $\operatorname{card}(P \setminus C) = \operatorname{card} C \geq 2^{\aleph_0}$. Consequently, such a metric ϱ exists for the underlying set P of the initial Σ-algebra and for the earlier chosen set $C = A \cup D \cup B \subseteq P$.

We may thus assume

CLAIM 4.8. *The metric ϱ used in Construction 4.2 is extremally C-semirigid.*

Since $(P, t) = \langle (A, u), (\Sigma, \Omega) \rangle$ is Hausdorff, by Lemma 4.6, the space $(C, t \restriction C)$ is also Hausdorff. Therefore

CLAIM 4.9. *The space (P, t) is C-semirigid.*

This fact will be essential in the arguments that follow.

To prove that $X = (P, t)$ determines a representation of $\mathcal{T}^{\Omega} = \mathcal{T}(\Sigma, \Omega)$ in Top, we have to prove that, for every integer $m \geq 1$, the set $\mathcal{C}(X^m, X)$ of all continuous maps of the m-th power X^m of X into X is bijective to the set $\mathcal{T}^{\Omega}(a^m, a)$. Having recalled from Section 1 that $\mathcal{T}^{\Omega}(a^m, a) = \bigcup_{k=0}^{\infty} T_k^{(m)}$, we now analogously define a set $L^{(m)}$ of maps $P^m \to P$.

Let $\pi_i^{(m)} : P^m \to P$ denote the i-th projection for each $i \in m$, let $\gamma_g^{(m)} : P^m \to P$ denote the constant map with the value $g \in P$ and, for any n-tuple of maps $f_j : P^m \to P$, let $f_0 \dot{\times} \cdots \dot{\times} f_{n-1} : P^m \to P^n$ denote the map sending each $p \in P^m$ to $(f_0(p), \dots, f_{n-1}(p))$.

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DEFINITION 4.10. We set

$$
\begin{aligned}
L_0^{(m)} &= \{\pi_0^{(m)}, \dots, \pi_{m-1}^{(m)}\} \cup \{\gamma_g^{(m)} \mid g \in G_0\},\\
L_1^{(m)} &= L_0^{(m)} \cup \bigcup_{\sigma \in \Sigma \setminus (\Sigma_0 \cup \Omega)} \{p_\sigma \circ (f_0 \dot{\times} \cdots \dot{\times} f_{\operatorname{ar}\sigma - 1}) \mid f_i \in L_0^{(n)} \text{ for all } i \in \operatorname{ar}\sigma\}\\
&\quad \cup \bigcup_{\sigma \in \Omega} \{p_\sigma \circ (f_0 \dot{\times} \cdots \dot{\times} f_{\operatorname{ar}\sigma - 1}) \mid f_i \in L_0^{(m)} \text{ for all } i \in \operatorname{ar}\sigma\\
&\qquad \text{and either } f_i = f_j \text{ for some } i, j \in \operatorname{ar}\sigma \text{ with } i \neq j\\
&\qquad \text{or } f_i = \gamma_g^{(m)} \text{ for some } i \in \operatorname{ar}\sigma \text{ and } g \in G_0\},\\
L_{k+1}^{(m)} &= L_k^{(m)} \cup \bigcup_{\sigma \in \Sigma \setminus \Sigma_0} \{p_\sigma \circ (f_0 \dot{\times} \cdots \dot{\times} f_{\operatorname{ar}\sigma - 1}) \mid f_i \in L_k^{(m)} \text{ for all } i \in \operatorname{ar}\sigma\\
&\qquad \text{and } f_j \notin L_{k-1}^{(m)} \text{ for some } j \in \operatorname{ar}\sigma\} \text{ for any } k \geq 1.
\end{aligned}
$$

It is clear that $L^{(m)} = \bigcup_{k=0}^{\infty} L_k^{(m)}$ is bijective to $T^{(m)}$ of Section 1 for every m. A suitable bijection $\lambda : L^{(m)} \to T^{(m)}$ is given by

$$
\begin{aligned}
&\lambda(\pi_i^{(m)}) = \pi_i^{(m)}, \quad \lambda(\gamma_{p_\sigma}^{(m)}) = \sigma \cdot \tau^{(m)} \text{ for all } \sigma \in \Sigma_0,\\
&\lambda(p_\sigma \circ (f_0 \dot{\times} \cdots \dot{\times} f_{\operatorname{ar}\sigma - 1})) = \sigma(\lambda f_0, \dots, \lambda f_{\operatorname{ar}\sigma - 1}) \text{ for all other } \sigma,
\end{aligned}
$$

and hence it suffices to show that $L^{(m)}$ coincides with the set $\mathscr{C}(X^m, X)$ of all continuous maps $X^m \to X$.

First we show that $\mathscr{C}(X^m, X) \subseteq L^{(m)}$. We recall that $P = \bigcup_{i=0}^{\infty} G_i$ and, for any $f \in \mathscr{C}(X^m, X)$, we let $r(f)$ denote the least integer i for which $f(P^m) \cap G_i \neq \emptyset$, and then show that $f \in L^{(m)}$ inductively in the 'rank' $r(f)$ of f.

Let $r(f) = 0$, that is, let $f(P^m) \cap G_0 \neq \emptyset$, and assume that f is not a projection. Since X is C-semirigid, f must be a constant or $f(P^m) \subseteq C$. Thus, since $P \setminus C = G_0 \setminus (A \cup D)$, any f with $f(P^m) \setminus C \neq \emptyset$ is a constant whose value belongs to $G_0 \setminus (A \cup D)$, and hence $f \in L_0^{(m)}$. In the remaining case of $f(P^m) \subseteq C$, the image $f(P^m)$ must be connected (because X is C-semirigid and hence connected, and so are the spaces X^m and $f(X^m)$), and it must intersect the totally disconnected clopen subspace $A \cup D = G_0 \cap C$ of C. But this is possible only when f is a constant whose value belongs to $A \cup D$. Therefore $f \in L_{r(f)}^{(m)}$ for any $f \in \mathscr{C}(X^m, X)$ with $r(f) = 0$.

Continuing the induction, we let $r(f) > 0$. Then $f(P^m) \subseteq B$. Since $f(P^m)$ is connected and because the set B is a disjoint union of the clopen sets $B_\sigma = p_\sigma(P^{\operatorname{ar}\sigma})$ with $\sigma \in \Sigma \setminus \Sigma_0$, there must be a unique $\sigma \in \Sigma \setminus \Sigma_0$ such that $f(P^m) \subseteq p_\sigma(P^{\operatorname{ar}\sigma})$. For every $j \in \operatorname{ar}\sigma$, the map $f_j = \pi_j^{(\operatorname{ar}\sigma)} \circ p_\sigma^{-1} \circ f$ is continuous (the map p_σ^{-1} is always continuous, regardless of whether σ belongs to Ω or not, see the observation just following Claim 4.3), and satisfies $r(f_j) < r(f)$. Hence $f_j \in L_{k_j}^{(m)}$ for some $k_j < r(f)$ by the induction hypothesis. But then $f = p_\sigma \circ (f_0 \dot{\times} \cdots \dot{\times} f_{\operatorname{ar}\sigma - 1}) \in L_{r(f)}^{(m)}$. Therefore $\mathscr{C}(X^m, X) \subseteq L^{(m)}$.

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To prove the reverse inclusion $L^{(m)} \subseteq \mathscr{C}(X^m, X)$, here and also in Section 5 we need these subsets of P^m:

$$\begin{aligned} P^m[i,c] &= \{(x_0,\ldots,x_{m-1}) \in P^m \mid x_i = c\} \text{ for all } c \in P \text{ and } i \in m, \\ P^m[i,j] &= \{(x_0,\ldots,x_{m-1}) \in P^m \mid x_i = x_j\} \text{ for all } i,j \in m \text{ with } i \neq j, \\ P^m[i,B] &= \{(x_0,\ldots,x_{m-1}) \in P^m \mid x_i \in B\} \text{ for all } i \in m. \end{aligned}$$

We call these sets and all their subsets *small.* Let us recall that p_σ is homeomorphism of $X^{\mathrm{ar}\,\sigma}$ onto $(B_\sigma, t \restriction B_\sigma)$ for every $\sigma \in \Sigma \setminus (\Sigma_0 \cup \Omega)$. If $\sigma \in \Omega$, then p_σ considered as a map of $X^{\mathrm{ar}\,\sigma}$ onto $(B_\sigma, t \restriction B_\sigma)$ is *not* continuous because it is the composite

$$(P^{\mathrm{ar}\,\sigma}, t^{\mathrm{ar}\,\sigma}) \xrightarrow{\ f\ } (P^{\mathrm{ar}\,\sigma}, z_\sigma) \xrightarrow{\ g\ } (B_\sigma, t \restriction B_\sigma)$$

where f is carried by the identity map and g is a homeomorphism satisfying $g(x) = p_\sigma(x)$ for all $x \in P^{\mathrm{ar}\,\sigma}$, and f is *not* continuous at the point $a^{(\mathrm{ar}\,\sigma)} = (a_0, \ldots, a_{\mathrm{ar}\,\sigma-1})$. On the other hand, the restriction of p_σ to any small set *is* continuous. In fact, the continuity of p_σ is violated *only* at $a^{(\mathrm{ar}\,\sigma)}$, and hence p_σ is continuous on the sets $P[i,B]$, $P[i,j]$, and $P[i,c]$ with $c \neq a_i$. The continuity of p_σ on all $P[i,a_i]$ with $i \in \mathrm{ar}\,\sigma$ follows from the construction of the neighborhood systems $\mathscr{M}^{(\mathrm{ar}\,\sigma)}$.

We show that every member of $L^{(m)} = \bigcup_{k=0}^{\infty} L_k^{(m)}$ belongs to $\mathscr{C}(X^m, X)$, by an induction on k. The claim is evident for each $f \in L_0^{(m)}$ because $L_0^{(m)}$ consists of projections and constants only. If $f \in L_1^{(m)}$, then $f = p_\sigma \circ (f_0 \dot{\times} \cdots \dot{\times} f_{\mathrm{ar}\,\sigma-1})$, where each f_i is either a projection or a constant, so that $g = f_0 \dot{\times} \cdots \dot{\times} f_{\mathrm{ar}\,\sigma-1}$ is continuous. If $\sigma \notin \Omega$, then $p_\sigma : X^{\mathrm{ar}\,\sigma} \to X$ is continuous, and hence $f = p_\sigma \circ g$ is also continuous. If $\sigma \in \Omega$, then either $f_i = f_j$ for some distinct $i, j \in \mathrm{ar}\,\sigma$, and hence the image $\operatorname{Im} g$ of g is contained in $P^{\mathrm{ar}\,\sigma}[i,j]$, or else f_i is a constant for some $i \in \mathrm{ar}\,\sigma$, and hence $\operatorname{Im} g$ is another small set. Since the restriction of p_σ to any small set is continuous, $f = p_\sigma \circ g$ must be continuous as well.

If $k \geq 1$ and $f \in L_{k+1}^{(m)} \setminus L_k^{(m)}$, then $f = p_\sigma \circ (f_0 \dot{\times} \cdots \dot{\times} f_{\mathrm{ar}\,\sigma-1})$ where all $f_i \in L_k^{(m)}$. Thus each f_i is continuous by the induction hypothesis and so is $g = f_0 \dot{\times} \cdots \dot{\times} f_{\mathrm{ar}\,\sigma-1}$ and, for some $j \in \mathrm{ar}\,\sigma$, the map f_j does *not* belong to $L_{k-1}^{(m)}$. Whence f_j has the form $f_j = p_{\sigma'} \circ (f_0' \dot{\times} \cdots \dot{\times} f'_{\mathrm{ar}\,\sigma'-1})$, so that $\operatorname{Im} g \subseteq P^{\mathrm{ar}\,\sigma}[j, B]$. But then $\operatorname{Im} g$ is a small set, and we conclude that $f = p_\sigma \circ g$ is continuous again. Therefore $L^{(m)} \subseteq \mathscr{C}(X^m, X)$.

5. The comprehensivity of topological modifications

In this section, we complete proofs of Theorems 3.3 and 3.8. In Section 4, we constructed a space $X = (P, t) = \langle (A, u), (\Sigma, \Omega) \rangle$ that determines a representation of the algebraic theory $\mathscr{T}(\Sigma, \Omega)$ in Top, and such that $(P, mt) = \langle (A, mu), (\Sigma, \Omega) \rangle$ for a lower modification m satisfying the requirements of Theorem 3.3, or for an upper

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modification m satisfying the requirements of Theorem 3.8 (and such that (A, mu) is metrizable). Furthermore, for any Ω' satisfying

$$\Omega \subseteq \Omega' \subseteq \Sigma \setminus (\Sigma_0 \cup \Sigma_1),$$

we shall also construct a topology τ on the set P such that

(α) (P, τ) determines a representation of $\mathscr{T}(\Sigma, \Omega')$ in Top, and

(β) τ is between t and mt.

The latter fact then implies that $m\tau = mt$, so that the space $(P, m\tau)$ then determines a representation of $\mathscr{T}(\Sigma, \Omega)$ (this is because (P, mt) determines it). This will complete all proofs. We employ the notation and the general method of Section 4. The initial space (A, u), however, is the space X_0 from Theorem 3.3(c) or 3.8(c), that is, one for which

$$u \neq mu \text{ and } (A, mu) \text{ is metrizable.}$$

A detailed proof is given for the case of a lower modification m, and outlined for an upper modification in the conclusion of the paper.

While constructing the space $X = (P, t) = \langle (A, u), (\Sigma, \Omega) \rangle$ in Section 4, we used the chains of topologies

$$\begin{array}{ll} u_0 \leq u_1 \leq u_2 \leq \cdots & \text{on } C = A \cup D \cup B \text{ and} \\ t_0 \leq t_1 \leq t_2 \leq \cdots & \text{on } P \end{array}$$

with $t_\alpha = \varrho * u_\alpha$ for every α.

In the construction of $\langle (A, mu), (\Sigma, \Omega) \rangle$, the corresponding chains of topologies are denoted

$$\begin{array}{ll} \tilde{u}_0 \leq \tilde{u}_1 \leq \tilde{u}_2 \leq \cdots & \text{on } C = A \cup D \cup B \text{ and} \\ \tilde{t}_0 \leq \tilde{t}_1 \leq \tilde{t}_2 \leq \cdots & \text{on } P. \end{array}$$

We have $\tilde{u}_\alpha = mu_\alpha$ and $\tilde{t}_\alpha = mt_\alpha$ for all α and $mX = (P, mt) = \langle (A, mu), (\Sigma, \Omega) \rangle$, by Proposition 4.7.

Depending on the given Ω' with $\Omega \subseteq \Omega' \subseteq \Sigma \setminus (\Sigma_0 \cup \Sigma_1)$, we now intend to define chains of topologies

$$\begin{array}{ll} w_0 \leq w_1 \leq w_2 \leq \cdots & \text{on } C = A \cup D \cup B \text{ and} \\ \tau_0 \leq \tau_1 \leq \tau_2 \leq \cdots & \text{on } P \end{array}$$

(in which $\tau_\alpha = \varrho * w_\alpha$ will hold again) such that $mu_\alpha \leq w_\alpha \leq u_\alpha$ and $mt_\alpha \leq \tau_\alpha \leq t_\alpha$ for every α. The resulting topology τ will be the τ_α whose index α is large enough for all chains to become stationary. As in Corollary 4.5, this will occur for $\alpha = \omega$.

The construction of w_α will be similar to that of u_α. We 'destroy' the continuity of the maps p_σ with $\sigma \in \Omega' \setminus \Omega$ at an additional point by means of the system $\mathcal{N}^{(\mathrm{ar}\,\sigma)}$ defined as follows.

Since $u \neq mu$, there exists a point $o \in A$ and its mu-neighborhood O which is not its u-neighborhood. For every integer $n \geq 1$, let $S^{(n)}$ denote the set of all n-tuples $s = (s_0, \dots, s_{n-1})$ of elements of $A \setminus O$ such that $s_i \neq s_j$ for any two distinct $i, j \in n$. Since o belongs to the u-closure of $A \setminus O$ and (A, u) is an infinite Hausdorff space, the set $S^{(n)}$ is infinite.

Let $\mathcal{U}$ denote the system of all u-open neighbourhoods of o in (A, u). For every $U \in \mathcal{U}$ and $\varepsilon > 0$ set

$$U_\varepsilon = \{z \in P \mid \varrho(z, U) < \varepsilon\}.$$

Then $\{U_\varepsilon \mid U \in \mathcal{U},\ \varepsilon > 0\}$ is a local open basis of o in $(P, t) = \langle (A, u), (\Sigma, \Omega)\rangle$, see Construction 4.2. Since $o \in A \subseteq G_0$, the n-tuple $o^{(n)} = (o, \dots, o)$ is in G_0^n and

$$\{U_\varepsilon^n \mid U \in \mathcal{U},\ \varepsilon > 0\}$$

is its local open basis in $(P, t)^n$. Define

$$\mathcal{N}^{(n)} = \{U_\varepsilon^n \setminus S^{(n)} \mid U \in \mathcal{U},\ \varepsilon > 0\}.$$

Observe that for small sets $P^n[i, j]$, $P^n[i, B]$, $P^n[i, o]$ (for the definition of small sets, see Section 4), we have

$$\begin{aligned} U_\varepsilon^n \cap P^n[i, j] &= (U_\varepsilon^n \setminus S^{(n)}) \cap P^n[i, j] \\ U_\varepsilon^n \cap P^n[i, o] &= (U_\varepsilon^n \setminus S^{(n)}) \cap P^n[i, o] \\ U_\varepsilon^n \cap P^n[i, B] &= \emptyset \qquad \text{for } \varepsilon < 1. \end{aligned}$$

Next we define the chains of topologies

$$\begin{aligned} &w_0 \leq w_1 \leq w_2 \leq \cdots \qquad \text{on } C = A \cup D \cup B \text{ and} \\ &\tau_0 \leq \tau_1 \leq \tau_2 \leq \cdots \qquad \text{on } P \end{aligned}$$

as follows. With the topologies u_α and t_α as in Construction 4.2, we set

$$w_0 = u_0 i \quad \text{and} \quad \tau_0 = \varrho * w_0 = t_0.$$

Suppose that w_γ and τ_γ have already been defined for all $\gamma \leq \alpha$.

For $\alpha = \beta + 1$, we let w_α be the topology on C which coincides with u on A, is discrete on D, and the subspaces $(A, w_\alpha \restriction A)$, $(D, w_\alpha \restriction D)$ and all spaces $(B_\sigma, w_\alpha \restriction B_\sigma)$ with $\sigma \in \Sigma \setminus \Sigma_0$ are clopen in (C, w_α), and $w_\alpha \restriction B_\sigma$ is defined so that

(A) p_σ is a homeomorphism of $(P^{\operatorname{ar}\sigma}, \tau_\beta^{\operatorname{ar}\sigma})$ onto $(B_\sigma, w_\alpha \restriction B_\sigma)$ for any $\sigma \in \Sigma \setminus (\Sigma_0 \cup \Omega')$,

(B) p_σ is a homeomorphism of $(P^{\operatorname{ar}\sigma}, y_{\beta,\sigma})$ onto $(B_\sigma, w_\alpha \restriction B_\sigma)$ for any $\sigma \in \Omega' \setminus \Omega$, and $y_{\beta,\sigma}$ is the topology in which $\mathscr{N}^{(\operatorname{ar}\sigma)}$ is a local open basis of the point $o^{(\operatorname{ar}\sigma)}$, while for all $x \in P^{\operatorname{ar}\sigma} \setminus \{o^{(\operatorname{ar}\sigma)}\}$ the $y_{\beta,\sigma}$-neighborhoods of x are precisely all its $\tau_\beta^{\operatorname{ar}\sigma}$-neighborhoods,

(C) p_σ is a homeomorphism of $(P^{\operatorname{ar}\sigma}, \bar{z}_{\beta,\sigma})$ onto $(B_\sigma, w_\alpha \restriction B_\sigma)$ for any $\sigma \in \Omega$, and $\bar{z}_{\beta,\sigma}$ is the topology in which $\mathscr{M}^{(\operatorname{ar}\sigma)}$ is a local open basis of the point $a^{(\operatorname{ar}\sigma)}$ just as in 4.2, while for all $x \in P^{\operatorname{ar}\sigma} \setminus \{a^{(\operatorname{ar}\sigma)}\}$ the $\bar{z}_{\beta,\sigma}$-neighborhoods of x are precisely all its $\tau_\beta^{\operatorname{ar}\sigma}$-neighborhoods.

These conditions define the topology w_α uniquely, and we set $\tau_\alpha = \varrho * w_\alpha$.

If α is a limit ordinal, we set $w_\alpha = \sup\{w_\beta \mid \beta < \alpha\}$ and $\tau_\alpha = \varrho * w_\alpha$.

Clearly, $\tilde{t}_\alpha \leq \tau_\alpha \leq t_\alpha$ for all α; since $\tilde{t}_\alpha = mt_\alpha$, we get $\tilde{t}_\alpha = m\tau_\alpha$, by Proposition 4.7.

Let $\tau = \tau_\alpha$ for α large enough to have $\tau_\alpha = \tau_{\alpha+1} = \cdots$, $t_\alpha = t_{\alpha+1} = \cdots$ and $\tilde{t}_\alpha = \tilde{t}_{\alpha+1} = \cdots$.

Denote $Y = (P, \tau)$. Then $mY = (P, m\tau) = (P, mt) = \langle (A, mu), (\Sigma, \Omega) \rangle$. Whence mY determines a representation of $\mathscr{T}(\Sigma, \Omega)$ in Top as in Section 4.

Now it remains to show that Y determines a representation of $\mathscr{T}(\Sigma, \Omega')$ in Top. In other words, it remains to show that $\mathscr{C}(Y^m, Y)$ is the set $(L^{(m)})'$ of maps $P^m \to P$ where $(L^{(m)})'$ is defined just as $L^{(m)}$ was in 4.9, except that Ω' now replaces Ω of 4.9. The proof that

$$\mathscr{C}(Y^m, Y) = (L^{(m)})'$$

is similar to the proof that $\mathscr{C}(X^m, X) = L^{(m)}$. For both spaces, the principal question is whether or not p_σ is continuous when regarded as a map from the $(\operatorname{ar}\sigma)$-th power of the space into it. For the space X, the map $p_\sigma : X^{\operatorname{ar}\sigma} \to X$ is continuous when $\sigma \in \Sigma \setminus (\Omega \cup \Sigma_0)$, and discontinuous at $a^{(\operatorname{ar}\sigma)}$ when $\sigma \in \Omega$. For the space Y, the map $p_\sigma : Y^{\operatorname{ar}\sigma} \to Y$ is continuous when $\sigma \in \Sigma \setminus (\Omega' \cup \Sigma_0)$, and discontinuous when $\sigma \in \Omega'$ because it is discontinuous at $o^{(\operatorname{ar}\sigma)}$ when $\sigma \in \Omega' \setminus \Omega$ and at $a^{(\operatorname{ar}\sigma)}$ when $\sigma \in \Omega$. In all cases, the inverse p_σ^{-1} maps B_σ continuously onto $P^{\operatorname{ar}\sigma}$ and the restriction of p_σ to small sets *is* continuous. Hence the proof that $\mathscr{C}(Y^m, Y) = (L^{(m)})'$ uses the same arguments as those presented in Section 4.

Now we turn to the case of upper modifications.

In a proof of Theorem 3.8, the roles of u and mu are interchanged. We define two chains of topologies

$$\begin{aligned} &w_0 \leq w_1 \leq w_2 \leq \cdots && \text{on } C = A \cup D \cup B \text{ and} \\ &\tau_0 \leq \tau_1 \leq \tau_2 \leq \cdots && \text{on } P \end{aligned}$$

again, but now $t_\alpha \leq \tau_\alpha \leq mt_\alpha$.

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If $X_0 = (A, u)$ is the space for which (A, mu) is metrizable and $u \neq mu$, we find a point $o \in A$ and its u-neighborhood O which is not its mu-neighborhood. Now, we let $\mathscr{U}$ be the system of all mu-open neighborhoods of o in (A, mu) and, for every $U \in \mathscr{U}$ and $\varepsilon > 0$, we define $U_\varepsilon = \{z \in P \mid \varrho(z, U) < \varepsilon\}$ again. Then $\{U_\varepsilon \mid U \in \mathscr{U}, \varepsilon > 0\}$ is a local open basis of o in $(P, mt) = \langle (A, mu), (\Sigma, \Omega) \rangle$. As in the case of lower modifications, we let $S^{(n)}$ be the set of all n-tuples $s = (s_0, \ldots, s_{n-1})$ of elements of $A \setminus O$ such that $s_i \neq s_j$ for some distinct $i, j \in n$, and set

$$\mathscr{N}^{(n)} = \{U_\varepsilon^n \setminus S^{(n)} \mid U \in \mathscr{U}, \varepsilon > 0\}.$$

This time, however, we subtract $S^{(n)}$ in chains $\tilde{u}_\alpha$ and $\tilde{t}_\alpha$, that is, we set

$$w_0 = \tilde{u}_0 \quad \text{and} \quad \tau_0 = \varrho * w_0 = \tilde{t}_0.$$

The remainder of the construction and the subsequent arguments are analogous to those used for lower modifications.

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