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EVEN AND ODD INTEGRAL PARTS OF POWERS OF A REAL NUMBER

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Abstract. We define a subset $\mathcal{Z}$ of $(1, +\infty)$ with the property that for each $\alpha \in \mathcal{Z}$ there is a nonzero real number $\xi = \xi(\alpha)$ such that the integral parts $[\xi\alpha^n]$ are even for all $n \in \mathbb{N}$. A result of Tijdeman implies that each number greater than or equal to 3 belongs to $\mathcal{Z}$. However, Mahler's question on whether the number $3/2$ belongs to $\mathcal{Z}$ or not remains open. We prove that the set $\mathcal{S} := (1, +\infty) \setminus \mathcal{Z}$ is nonempty and find explicitly some numbers in $\mathcal{Z} \cap (5/4, 3)$ and in $\mathcal{S} \cap (1, 2)$.

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1. Introduction. Suppose that $\alpha > 1$ is a real number. Then *either* for any real number $\xi \neq 0$ the fractional parts $\{\xi\alpha^n\}$ are greater than or equal to $1/2$ for infinitely many $n \in \mathbb{N}$ *or* there exists a real number $\xi = \xi(\alpha) \neq 0$ such that $\{\xi\alpha^n\} < 1/2$ for every $n \in \mathbb{N}$. The set of $\alpha > 1$ for which the first possibility holds will be denoted by $\mathcal{S}$. Similarly, we will denote by $\mathcal{Z}$ the set of $\alpha > 1$ for which the second possibility holds, so $\mathcal{S} \cap \mathcal{Z} = \emptyset$ and $\mathcal{S} \cup \mathcal{Z} = (1, +\infty)$. Equivalently, $\alpha \in \mathcal{Z}$ if and only if there is a real number $\xi = \xi(\alpha) \neq 0$ such that the integral parts $[\xi\alpha^n]$, $n = 1, 2, \dots,$ are all even.

In 1968, Mahler [**16**] asked whether the number $3/2$ belongs to $\mathcal{S}$ or to $\mathcal{Z}$ (see also [**20**]). There are many reasons to believe that $3/2 \in \mathcal{S}$. Although for a 'random' pair ξ, α the fractional parts $\{\xi\alpha^n\}$, $n = 1, 2, \dots,$ are uniformly distributed in $[0, 1)$ (see [**13**], [**21**]), the behavior of the sequence $\{\xi\alpha^n\}$, $n = 1, 2, \dots,$ for almost any 'specific' pair ξ, α, where $\xi \neq 0$, $\alpha \notin \mathbb{N}$, is not known. See, however, [**3**], [**4**], [**6**]–[**12**], [**22**], [**23**] for some results in this direction. In this note, we shall collect several results about the sets $\mathcal{S}$ and $\mathcal{Z}$.

Trivially, $\{2, 3, 4, \dots\} \subset \mathcal{Z}$, because, for any integer $g \geq 2$, by taking $\xi = 2$ we see that the numbers $[2g^n] = 2g^n$, $n = 1, 2, \dots,$ are all even. In general, one may expect that 'large' α belong to the set $\mathcal{Z}$ (see, for instance, Tijdeman's result [**19**] stated in Theorem 1(*i*) below) whereas 'small' α lie in $\mathcal{S}$. In connection with this, one can ask whether there exist $\alpha > \alpha' > 1$ such that $\alpha \in \mathcal{S}$ and $\alpha' \in \mathcal{Z}$. If the answer to this question were negative then the sets $\mathcal{S}$ and $\mathcal{Z}$ would simply be two intervals. Unfortunately, the situation is not that simple, because such α and α' do exist. We will show, for instance, that the set $\mathcal{Z}$ contains a number smaller than 1.26 and that the set $\mathcal{S}$ contains the golden mean $(1 + \sqrt{5})/2 = 1.61803\dots$.

In order to state our theorems, we recall first that $\alpha > 1$ is called a *Pisot number* if it is an algebraic integer whose conjugates over $\mathbb{Q}$ different from α itself lie in the open unit disc. A Pisot number is called a *strong Pisot number* if it is not a rational integer and if its second largest (in modulus) conjugate is positive (see [**2**] and [**6**]). Also,

$\alpha > 1$ is called a *Salem number* if it is an algebraic integer whose conjugates over $\mathbb{Q}$ different from α itself lie in the closed unit disc $|z| \le 1$ with at least one conjugate lying on the unit circle $|z| = 1$. Finally, let $P_\alpha(x) \in \mathbb{Z}[x]$ denote the minimal polynomial of an algebraic number α. Note that $P_\alpha(1) \le -1$ for each α which is a Pisot number or a Salem number.

2. Results.

THEOREM 1. *We have*

(i) $[3, +\infty) \subset \mathcal{Z}$,

(ii) $3 - 2/q \in \mathcal{Z}$ *for any integer* $q \ge 2$,

(iii) $\alpha \in \mathcal{Z}$ *for any strong Pisot number* α,

(iv) $\alpha \in \mathcal{Z}$ *for any Pisot or Salem number* α *whose minimal polynomial satisfies* $P_\alpha(1) \le -3$.

By Theorem 1(iv), the thirteenth smallest known Salem number $\alpha = 1.2527759\ldots$ whose minimal polynomial is

$$P_\alpha(x) = x^{18} - x^{12} - x^{11} - x^{10} - x^9 - x^8 - x^7 - x^6 + 1$$

belongs to the set $\mathcal{Z}$, because $P_\alpha(1) = -5 < -3$. (See Mossinhoff's page on Lehmer's problem http://www.cecm.sfu.ca/~mjm/Lehmer/lists/SalemList.html for a list of small Salem numbers.) Most of the results stated in Theorem 1 have been published earlier or follow easily from **[19]**, **[10]**, **[11]**, **[22]**. Nevertheless, for the sake of completeness, we will give the proofs of (i)–(iii) below and derive (iv) from **[10]**, **[22]**.

The first example of a number $\alpha > 1$ lying in the set $\mathcal{S}$ was given recently by the author in **[8]**: for any $d \ge 4$ one can take $\alpha > 1$ that satisfies $\alpha^d - \alpha - 1 = 0$. So the set $\mathcal{S}$ is nonempty. Note that if $\alpha \in \mathcal{S}$ then $\alpha^{1/q} \in \mathcal{S}$ for each $q \in \mathbb{N}$, because the set of fractional parts $\{\xi\alpha^n\}$, where $n \in \mathbb{N}$, is a subset of the set $\{\xi\alpha^{n/q}\}$, $n \in \mathbb{N}$. The next theorem not only contains the example given above but also describes some new numbers in $\mathcal{S}$.

THEOREM 2. *We have*

(i) $2^{1/q} \in \mathcal{S}$ *for any integer* $q \ge 2$,

(ii) $\alpha \in \mathcal{S}$ *for any* $\alpha > 1$ *which is a root of an irreducible polynomial* $x^d - x^r - 1$, *where* $0 < r < d$,

(iii) $\alpha \in \mathcal{S}$ *for any* $\alpha > 1$ *which is a root of the polynomial* $x^d - x^m - x^r + 1$, *where* $0 < r \le m < d$, *but is not a Pisot number.*

Note that in case (iii) the polynomial $x^d - x^m - x^r + 1$ is reducible. Hence the degree of $\alpha > 1$ over $\mathbb{Q}$ is smaller than d. The requirement that α is not a Pisot number is necessary. If, for instance, $m = r = d - 1$ then

$$x^d - 2x^{d-1} + 1 = (x-1)(x^{d-1} - x^{d-2} - \cdots - x - 1).$$

The polynomial $P_\alpha(x) = x^{d-1} - x^{d-2} - \cdots - x - 1$ is irreducible and defines a Pisot number $\alpha > 1$ for each $d \ge 3$. Since $P_\alpha(1) \le -3$ for every $d \ge 5$, Theorem 1(iv) implies that $\alpha \in \mathcal{Z}$. However, for $d = 3$, $\alpha = (1+\sqrt{5})/2$ belongs to $\mathcal{S}$ by Theorem 2(ii). All irreducible polynomials of the form $x^d - x^r - 1$ have been described in **[15]**.

Since $\sqrt{2} \in \mathcal{S}$ and $\sqrt{m} \in \mathcal{Z}$ for each integer $m \ge 4$, it is natural to ask the following:

PROBLEM 3. *Determine whether* $\sqrt{3}$ *belongs to* $\mathcal{S}$ *or to* $\mathcal{Z}$.

We remark that if $\sqrt{3} \in \mathcal{S}$ then writing $\sqrt{3}$ in its base 3 expansion $\sqrt{3} = 1 + \sum_{j=1}^{\infty} b_j 3^{-j}$, where $b_1, b_2, \ldots \in \{0, 1, 2\}$, and taking $\xi = 1$ we would derive that

$$\{\sqrt{3} \cdot 3^j\} = b_{j+1}3^{-1} + b_{j+2}3^{-2} + \cdots \geq 1/2$$

for infinitely many $j \in \mathbb{N}$. Hence $b_m = 2$ for infinitely many $m \in \mathbb{N}$. Such results, however, are completely out of reach. See, for instance, [**1**] for a recent progress on the distribution of digits in the expansions of algebraic irrational numbers in base $b \geq 2$.

The next problem seems to be quite difficult too.

Problem 4. *Is it true that if $\alpha \in \mathcal{S}$ then for each nonzero real number ξ the sequence $[\xi\alpha^n]$, $n = 1, 2, \ldots$, contains infinitely many even numbers?*

By the definition of $\mathcal{S}$, the sequence $[\xi\alpha^n]$, $n = 1, 2, \ldots$, contains infinitely many odd numbers. It is easy to see that the answer to Problem 4 is affirmative precisely when there is a nonzero real number ξ such that $\{\xi\alpha^n\} \geq 1/2$ for each $n \in \mathbb{N}$, where $\{\xi\alpha^n\} > 1/2$ for infinitely many n and $\{\xi\alpha^n\} = 1/2$ for infinitely many $n \in \mathbb{N}$. Taking, for instance, $\xi = 1/2$ and $\alpha = \sqrt{3}$ we are back to a similar question about the distribution of digits in base 3 expansion again. This time, the number in question is $\sqrt{3}/2$.

Note that all numbers of $\mathcal{S}$ described in Theorem 2 are algebraic integers and lie in the interval $(1, 2)$. We thus conclude this section with the following problem.

Problem 5. *Is there an element of $\mathcal{S}$ greater than 2?*

3. Proof of Theorem 1. We shall prove (*i*) and (*ii*) using the method of nested intervals as in [**19**]. Suppose first that $\alpha > 3$. We claim that there is a $\xi > 0$ such that $\{\xi\alpha^n\} \leq \beta := 1/(\alpha - 1)$ for every $n \in \mathbb{N}$. Clearly, $\{\xi\alpha^n\} \leq \beta$ if and only if there is an integer k_n such that $k_n\alpha^{-n} \leq \xi \leq (k_n + \beta)\alpha^{-n}$. Let k_1 be an arbitrary integer greater than α. Set $I_1 = [k_1\alpha^{-1}, (k_1 + \beta)\alpha^{-1}]$. The sequence of intervals $I_j = [k_j\alpha^{-j}, (k_j + \beta)\alpha^{-j}]$, where $k_j \in \mathbb{N}$, $j = 1, 2, \ldots$, is nested if and only if $k_j\alpha^{-j} \leq k_{j+1}\alpha^{-j-1}$ and $(k_j + \beta)\alpha^{-j} \geq (k_{j+1} + \beta)\alpha^{-j-1}$ for each $j \in \mathbb{N}$. This happens precisely when for each $j \in \mathbb{N}$ the interval $[\alpha k_j, \alpha(k_j + \beta) - \beta]$ contains the integer k_{j+1}. Since the length of this interval is $\alpha\beta - \beta = 1$, such an integer k_{j+1} exists for every $j \in \mathbb{N}$. Hence, setting $\xi := \cap_{j=1}^{\infty} I_j$, we have that $\{\xi\alpha^n\} \leq 1/(\alpha - 1) < 1/2$ for every $n \in \mathbb{N}$. Therefore, each $\alpha > 3$ lies in $\mathcal{Z}$. Trivially, $2, 3 \in \mathcal{Z}$. This proves (*i*) and also (*ii*) for $\alpha = 2$, where $q = 2$.

Next, we will prove (*ii*) for $\alpha = (3q - 2)/q$, where $q \geq 3$ is an integer. As above, $\{\xi(3 - 2/q)^n\} \leq 1/2$ if and only if there is an integer k_n such that $k_n(3 - 2/q)^{-n} \leq \xi \leq (k_n + 1/2)(3 - 2/q)^{-n}$. Fix an integer k_1 greater than 3. Set $I_1 = [k_1(3 - 2/q)^{-1}, (k_1 + 1/2)(3 - 2/q)^{-1}]$. The sequence of intervals $I_j = [k_j(3 - 2/q)^{-j}, (k_j + 1/2)(3 - 2/q)^{-j}]$, where $k_j \in \mathbb{N}$, $j = 1, 2, \ldots$, is nested if and only if

$$(3 - 2/q)k_j \leq k_{j+1} \leq (3 - 2/q)(k_j + 1/2) - 1/2 = (3 - 2/q)k_j + 1 - 1/q.$$

It is easy to see that the interval $[(3q - 2)k_j, (3q - 2)k_j + q - 1]$ contains an integer divisible by q, say qu. So we can take $k_{j+1} := u$. Hence, setting $\xi := \cap_{j=1}^{\infty} I_j$, we derive that $\{\xi(3 - 2/q)^n\} \leq 1/2$ for each $n \in \mathbb{N}$. However, since $3 - 2/q$, $q \geq 3$, is not an integer, there are only finitely many $n \in \mathbb{N}$ (or no such n at all) for which $\{\xi(3 - 2/q)^n\} = 1/2$. (See, for instance, Lemma 4 in [**11**].) If n_0 is the largest among those n we can replace

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ξ by $\xi(3-2/q)^{n_0}$. With this new ξ, the inequality $\{\xi(3-2/q)^n\} < 1/2$ holds for every $n \in \mathbb{N}$. This completes the proof of (ii).

For a strong Pisot number α, we have $\{\alpha^n\} \to 1$ as $n \to \infty$ (see [**6**]). Indeed, since $S_n := \alpha^n + \alpha_2^n + \cdots + \alpha_d^n \in \mathbb{Z}$, where $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_d$ are the conjugates of α labelled so that $\alpha > 1 > \alpha_2 > |\alpha_j|$ for $j > 2$, we deduce that $S_n - \alpha^n$ is positive for each n sufficiently large. Clearly, $S_n - \alpha^n \to 0$ as $n \to \infty$. It follows that $\{\alpha^n\} = 1 - \alpha_2^n - \cdots - \alpha_d^n$ for each sufficiently large integer n. Hence $\{\alpha^n\} \to 1$ as $n \to \infty$. In particular, by taking $\xi = -\alpha^{n_0}$ with n_0 sufficiently large, we obtain that $\{-\alpha^{n_0}\alpha^n\} < 1/2$ for each $n \in \mathbb{N}$. This proves (iii).

The proof of (iv) for Pisot and Salem numbers follows [**10**] and [**22**], respectively. To be precise, it was shown in [**10**] that if the minimal polynomial of a Pisot number α satisfies $P_\alpha(1) \le -2$ then, setting $\xi = 1/(P_\alpha'(\alpha)(\alpha-1))$, we have $\lim_{n\to\infty}\{\xi\alpha^n\} = 1/|P_\alpha(1)|$. Similarly, Zaimi [**22**] showed that if the minimal polynomial of a Salem number α satisfies $P_\alpha(1) \le -2$ then, for any $\varepsilon > 0$, there is a nonzero $\xi = \xi(\alpha, \varepsilon) \in \mathbb{Q}(\alpha)$ such that $1/|P_\alpha(1)| - \varepsilon < \{\xi\alpha^n\} < 1/|P_\alpha(1)| + \varepsilon$ for each $n \in \mathbb{N}$ large enough. So in both (Pisot and Salem) cases one can find a positive integer n_0 such that, by taking $\xi\alpha^{n_0} \in \mathbb{Q}(\alpha)$ instead of ξ, we obtain that $1/|P_\alpha(1)| - \varepsilon < \{\xi\alpha^n\} < 1/|P_\alpha(1)| + \varepsilon$ for each $n \in \mathbb{N}$. Clearly, this implies the inequality $\{\xi\alpha^n\} < 1/2$ for each $n \in \mathbb{N}$ under the stronger condition $P_\alpha(1) \le -3$ if $\varepsilon < 1/6$. This proves (iv). The proof of Theorem 1 is completed.

Since in [**10**] and in [**22**] the statements concerning the fractional parts $\{\xi\alpha^n\}$ mentioned in the proof of (iv) are not given explicitly, let us summarize them here as follows.

THEOREM 6. *Suppose that α is a Pisot number or a Salem number with minimal polynomial $P_\alpha(x) \in \mathbb{Z}[x]$. If $P_\alpha(1) \le -2$ then for any $\varepsilon > 0$ there is a real number $\xi \in \mathbb{Q}(\alpha)$ (which depends on ε in the case α is a Salem number) such that*

$$1/|P_\alpha(1)| - \varepsilon < \{\xi\alpha^n\} < 1/|P_\alpha(1)| + \varepsilon$$

for any $n \in \mathbb{N}$.

We remark that the fractional parts $\{\xi\alpha^n\}$, $n \in \mathbb{N}$, can be quite small for some Salem numbers that are not too large. Take, for instance, the Salem number $\alpha = 1.6733248\ldots$ given in [**14**] whose minimal polynomial is

$$P_\alpha(x) = x^{14} - x^{12} - x^{11} - x^{10} - x^9 - 2x^8 - 3x^7 - 2x^6 - x^5 - x^4 - x^3 - x^2 + 1,$$

so that $P_\alpha(1) = -13$. Then, by Theorem 6, for any $\varepsilon > 0$, there exists a real number $\xi = \xi(\alpha, \varepsilon) \in \mathbb{Q}(\alpha)$ such that $1/13 - \varepsilon < \{\xi\alpha^n\} < 1/13 + \varepsilon$ for each $n \in \mathbb{N}$. This not only implies that $\alpha \in \mathcal{Z}$ but also that every integral part $[\zeta\alpha^n]$, where $n \in \mathbb{N}$ and $\zeta = 12\xi$, is divisible by 12.

4. Proof of Theorem 2.

In all three cases it suffices to show that, for any $\xi \neq 0$, the integral parts $x_n := [\xi\alpha^n]$, $n = 1, 2, \ldots$, cannot all be even. Suppose they are, i.e. $\alpha \in \mathcal{Z}$. Setting $y_n := \{\xi\alpha^n\}$ for $n \in \mathbb{N}$, we have $x_{n+q} - 2x_n = 2y_n - y_{n+q}$ (case (i)) or $x_{n+d} - x_{n+r} - x_n = y_n + y_{n+r} - y_{n+d}$ (case (ii)) or $x_{n+d} - x_{n+m} - x_{n+r} + x_n = -y_n + y_{n+r} + y_{n+m} - y_{n+d}$ (case (iii)). A fractional part is a non-negative number smaller than 1. So the right-hand sides of all three equalities belong to the interval $(-2, 2)$. But all left-hand sides are even integers. Hence, for every $n \in \mathbb{N}$, we have $x_{n+q} - 2x_n = 2y_n - y_{n+q} = 0$

(case (i)), $x_{n+d} - x_{n+r} - x_n = y_n + y_{n+r} - y_{n+d} = 0$ (case (ii)), $x_{n+d} - x_{n+m} - x_{n+r} + x_n = -y_n + y_{n+r} + y_{n+m} - y_{n+d}$ (case (iii)).

In case (i) we deduce that $y_{n+qm} = 2^m y_n$ for any $m \in \mathbb{N}$. Taking m arbitrarily large we obtain that $y_n = \{\xi 2^{n/q}\} = 0$ for every $n \in \mathbb{N}$. Next, by considering, firstly, the subsequence $n = qk$, $k = 1, 2, \dots$, and, secondly, the subsequence $n = qk + 1$, $k = 1, 2, \dots$, we derive that $\xi 2^{n/q}$ is an integer for every $n \in \mathbb{N}$ if and only if $\xi = 0$, a contradiction. Hence $2^{1/q} \in \mathcal{S}$ for each integer $q \geq 2$. This proves (i).

In case (iii) the sequence $s_n := -y_n + y_{n+r} + y_{n+m} - y_{n+d} = 0$ is periodic. So, by Lemma 3 of [**8**], $\alpha > 1$ must be a Pisot number or a Salem number. It cannot be a Pisot number by the condition of (iii). Hence α is a Salem number. But from $\alpha^d - \alpha^m - \alpha^r + 1 = 0$ on replacing $\alpha \to \alpha^{-1}$ (Salem numbers are reciprocal) we obtain that $\alpha^d - \alpha^{d-r} - \alpha^{d-m} + 1 = 0$. Observe that if $m + r = d$ then $\alpha^d - \alpha^m - \alpha^r + 1 = (\alpha^m - 1)(\alpha^r - 1) = 0$, a contradiction with $\alpha > 1$. If $m + r \neq d$ then

$$\begin{aligned}\alpha^d - \alpha^m - \alpha^r + 1 - (\alpha^d - \alpha^{d-r} - \alpha^{d-m} + 1) &= \alpha^{d-m} - \alpha^r + \alpha^{d-r} - \alpha^m \\ &= (\alpha^r + \alpha^m)(\alpha^{d-r-m} - 1) = 0,\end{aligned}$$

a contradiction again. This proves (iii). (Note that we proved the following statement: each irreducible reciprocal factor of $x^d - x^m - x^r + 1$ is cyclotomic. See [**15**] for more about irreducible factors of such quadrinomials.)

In case (ii) the sequence $s_n := -y_n + y_{n+r} - y_{n+d} = 0$ is periodic. As above, Lemma 3 of [**8**] implies that α is a Pisot number or a Salem number and $\xi \in \mathbb{Q}(\alpha)$. Since $\alpha > 1$ is a root of an irreducible nonreciprocal polynomial $x^d - x^r - 1$, it can only be a Pisot number. (Indeed it can: for instance, if $r = 1$ and $d = 2$ or $d = 3$.) Note that it is not a strong Pisot number, because the polynomial $x^d - x^r - 1$ has no roots in the interval $[0, 1]$. Suppose that the conjugates of $\alpha = \alpha_1 > 1$ over $\mathbb{Q}$ are $\alpha_2, \dots, \alpha_d$, where $|\alpha_1| > 1 > |\alpha_2| \geq |\alpha_3| \geq \dots \geq |\alpha_d|$. Since $x_{n+d} - x_{n+r} - x_n = 0$ for every $n \in \mathbb{N}$, we have that $x_n = \xi_1 \alpha_1^n + \dots + \xi_d \alpha_d^n$. Moreover (see [**5**] or the proof of Theorem 3 in [**8**]), $\xi_j \in \mathbb{Q}(\alpha_j)$, $j = 1, \dots, d$, and the numbers $\xi_1, \dots, \xi_d$ are conjugate over $\mathbb{Q}$. Similarly, from the linear recurrence $y_{n+d} - y_{n+r} - y_n = 0$, $n = 1, 2, \dots$, we obtain that there exist certain complex numbers $\eta_1, \dots, \eta_d$ such that $y_n = \eta_1 \alpha_1^n + \dots + \eta_d \alpha_d^n$ for each $n \in \mathbb{N}$. But $x_n + y_n = \xi \alpha_1^n$, so that $\eta_1 = \xi - \xi_1$, $\eta_2 = -\xi_2, \dots, \eta_d = -\xi_d$. If $\eta_1 \neq 0$ then $|y_n| \to \infty$ as $n \to \infty$, a contradiction. It follows that η_1 must be equal to zero, so $\xi_1 = \xi$. Summarizing, we have that $y_n = -\xi_2 \alpha_2^n - \xi_3 \alpha_3^n - \dots - \xi_d \alpha_d^n$, where $\xi_2 \in \mathbb{Q}(\alpha_2), \dots, \xi_d \in \mathbb{Q}(\alpha_d)$ are conjugate over $\mathbb{Q}$ and $\xi_2 \neq 0$.

In order to get a contradiction it suffices to show that the sums $y_n = -\xi_2 \alpha_2^n - \xi_3 \alpha_3^n - \dots - \xi_d \alpha_d^n$ are negative for infinitely many $n \in \mathbb{N}$. Indeed, since every Pisot number $\alpha = \alpha_1$ has at most two conjugates of largest modulus in the unit disc (see [**18**]) which is $|\alpha_2|$, but α is not a strong Pisot number, i.e. $\alpha_2 \notin (0, 1)$ there are only two possibilities. Either α_2 is a real negative number in $(-1, 0)$ and $|\alpha_2| > |\alpha_j|$ for $j > 2$ or α_2 and α_3 are complex conjugate numbers, i.e. $\alpha_3 = \overline{\alpha_2}$ and $|\alpha_2| > |\alpha_j|$ for $j > 3$. In both cases, since $-\xi_2 \alpha_2^n - \xi_3 \alpha_3^n = -2\Re(\xi_2 \alpha_2^n)$, the sign of y_n is the same as that of $-\Re(\xi_2 \alpha_2^n)$ for each n sufficiently large. Of course, if $\alpha_2 \in (-1, 0)$ then $-\xi_2 \alpha_2^n$ is positive for infinitely many $n \in \mathbb{N}$ and negative for infinitely many $n \in \mathbb{N}$. Assume that α_2 is complex. Let us write $\alpha_2 = \varrho e^{i\theta}$ and $\xi_2 = \varrho' e^{i\vartheta}$. Then $\Re(\xi_2 \alpha_2^n) = \varrho\varrho' \cos(n\theta + \vartheta)$. Since θ/π is irrational (see [**17**] or derive a contradiction from $\alpha_2^m = \alpha_3^m$, where $m \in \mathbb{N}$, by mapping α_2 to α), Kronecker's theorem [**5**] yields that the fractional parts $\{n\theta/2\pi + \vartheta/2\pi\}$, $n = 1, 2, \dots$, are dense in $[0, 1)$. It follows that $\cos(n\theta + \vartheta)$ is positive for infinitely many $n \in \mathbb{N}$ and negative for infinitely many $n \in \mathbb{N}$. This completes the proof of Theorem 2.

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