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A HILBERT LATTICE WITH A SMALL AUTOMORPHISM GROUP

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ABSTRACT. We construct an orthomodular inner product space to answer the questions posed by R. P. Morash in his paper "Angle bisection and orthoautomorphisms in Hilbert lattices" [6]. For example we show that every automorphism of the Hilbert lattice belonging to our inner product space has the property, that no atom is orthogonal to its image.

0. **Introduction and terminology**. A Hilbert lattice is a complete, orthocomplemented, atomic, irreducible, orthomodular, infinite-dimensional lattice with the covering property (for the definition of these properties see e.g. [5]). Hilbert lattices are closely related to orthomodular spaces, as is outlined below.

Let E be a K-vector space, equipped with a sesquilinear form $\langle , \rangle$ which is hermitian with respect to an antiautomorphism $*: K \to K$. A subspace $U \subset E$ is called orthoclosed iff $U = U^{\perp\perp} := (U^{\perp})^{\perp}$. The set of all orthoclosed subspaces of E is denoted by $L_{\perp\perp}(E)$. If $U \oplus U^{\perp} = E$ for all orthoclosed subspaces of E then E is said to be an orthomodular space. For an infinite-dimensional orthomodular space E $L_{\perp\perp}(E)$, ordered by inclusion and equipped with the natural orthocomplementation, becomes a Hilbert lattice. Conversely every Hilbert lattice is isomorphic (as orthocomplemented lattice) to $L_{\perp\perp}(E)$ for some infinite-dimensional orthogonal space E. Furthermore every isomorphism $L_{\perp\perp}(E_1) \to L_{\perp\perp}(E_2)$ is induced by a similarity $\Psi: E_1 \to E_2$ (Ψ is called a similarity iff it is a bijection and there are $\mu \in K\setminus\{0\}$ and a division ring automorphism $\alpha: K \to K$ such that the following properties are satisfied:

$$\forall x, y \in E \forall \lambda \in K: \Psi(x + \lambda y) = \Psi(x) + \alpha(\lambda)\Psi(y) \tag{1}$$

$$\forall x, y \in E: \langle \Psi x, \Psi y \rangle = \alpha\langle x, y \rangle \tag{2}$$

All the facts cited above concerning the relationship between Hilbert lattices and orthomodular spaces follow from the fundamental theorems of projective geometry, see e.g. [10] or [7].

By a theorem of Amemiya-Araki-Piron [1] an inner product space E with basefield $K = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ is orthomodular iff E is a Hilbert space. In this case $L := L_{\perp\perp}(E)$ is easily seen by geometric considerations to have the following properties:

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(o) L is isomorphic to the interval $[0, a]$ (equipped with the relative orthocomplementation $u \mapsto u^{\perp} \cap a$) iff $\dim(L) = \dim(a)$.

(i) $\forall x, y \in E \backslash \{0\}\ \exists \lambda \in K: \langle x, x\rangle = \langle \lambda y, \lambda y\rangle$ (a purely lattice-theoretic formulation of this geometric statement is the angle bisection property of Morash [6].)

(ii) For all $a, b \in L$ with $\dim(a) = \dim(b)$ and $a \perp b$ there is an automorphism of L mapping a to b.

(iii) For all $a, b \in L$ with $a \cap b = a^{\perp} \cap b^{\perp} = \{0\}$ there is an automorphism of L mapping a to $b^{\perp}$.

In [6] the question is discussed whether (i), (ii) and (iii) are valid for all Hilbert lattices. We shall construct an example in the next section to show that this is not the case.

1. **The basefield**. Consider

$$G_i := \left\{\frac{p}{q} \in \mathbb{Q} \,\middle|\, p, q \in \mathbb{N} \ \&\ q \text{ is odd } \&\ q \text{ is not divisible by the } (i+1)\text{th power of any prime}\right\} \tag{3}$$

$$\Gamma := \{(g_i)_{i \in \mathbb{N}} \in \mathbb{O}^{\mathbb{N}} | g_i \in G_i\}. \tag{4}$$

Γ is an ordered abelian group with respect to componentwise addition and anti-lexicographic order. The elements $d_i := (\delta_{ij})_{j \in \mathbb{N}}$ (Kronecker-δ) can be internally characterized: d_i is the smallest element greater than 0 and divisible by i-th powers of all odd primes. Γ is generated by elements of the form gd_i with $g \in G_i$, so that Γ admits no non-trivial order preserving automorphism.

Now let K be a complete and henselian valued field (with valuation $\varphi: K \rightarrow \Gamma \cup \{\infty\}$) satisfying $\varphi(2) = 0$ and the residue class field $\bar{K}$ be quadratically closed. (Such fields exist: if $\bar{K}$ is any given field and Γ any abelian ordered group, then there is a maximal complete valued field K with value group Γ and residue class field $\bar{K}$. This field is of course complete and henselian [8].) The valuation ring of K can be described in a purely algebraic way by

$$\varphi(x) = 0 \leftrightarrow \forall \cap \exists y: x = y^{2^n}, \text{ and} \tag{5}$$

$$\varphi(x) \geqslant 0 \leftrightarrow \varphi(x) = 0 \vee \varphi(1 + x) = 0. \tag{6}$$

(6) holds trivially for all valuations. The implication from left to right in (5) is a consequence of the henselian property of K and the fact that $\bar{K}$ is a quadratic element of Γ other than 0 is divisible by all powers of 2. Now using (5) and (6), if $\alpha: K \rightarrow K$ is a field automorphism, then $\varphi(x) \geqslant 0 \rightarrow \varphi\alpha(x) \geqslant 0$. Hence $\varphi \circ \alpha$ is a valuation equivalent to φ and so, by general valuation theory, there is an order preserving group isomorphism $\Psi: \Gamma \rightarrow \Gamma$ such that $\varphi \circ \alpha = \Psi \circ \varphi$. But it was just shown that such a Ψ must be the identity. Hence all field automorphisms of K preserve the valuation.

2. **The space**. For each $n \in \mathbb{N}$ choose an $a_n \in K$ satisfying $(a_n) = d_n$. Then put

$$E := \{(\lambda_i)_{i \in \mathbb{N}} \in K^{\mathbb{N}} | \lim_{i \rightarrow \infty} \lambda_i^2 a_i = 0\}. \tag{7}$$

E is a K-vectorspace with respect to componentwise addition and scalar multiplication. Defining

(8) $$\langle (\lambda_i)_{i\in\mathbb{N}}, (\eta_i)_{i\in\mathbb{N}}\rangle := \sum_{i\in\mathbb{N}} \lambda_i a_i \eta_i$$

makes E into symmetric orthomodular space (cf. [3], Remark 29). E has the following property ([3], Lemma 25).

(9) If $(e_i)_{i\in\mathbb{N}}$ is a maximal orthogonal family in E, then there is a bijection $T:I\to\mathbb{N}$ such that for all $i\in I$ $\varphi\langle e_i, e_i\rangle \equiv d_{T(i)} \pmod{2\Gamma}$.

An immediate consequence of (9) is

(10) $$\forall x,y\in E: \langle x,y\rangle = 0 \to \varphi\langle x,x\rangle \not\equiv \varphi\langle y,y\rangle \pmod{2\Gamma}.$$

It follows that if x and y are orthogonal nonzero elements of E, then $\langle x,x\rangle \not\equiv \langle y,y\rangle$, and so (i) fails for $L_{\perp\!\perp}(E)$.

Now let $\Psi:E\to U\subset E$ be a similarity with respect to a field automorphism $\alpha:K\to K$ and the constant μ. $\varphi(\mu)$ has the form $(m_0,\dots,m_r,0,0,\dots)$. Next choose $x\in E$ such that $\varphi\langle x,x\rangle = d_n$ for some $n>r$. Then $\varphi\langle\Psi x,\Psi x\rangle = \varphi\alpha\langle x,x\rangle + \varphi(\mu) = d_n + \varphi(\mu)$. Hence it follows by (9) that $\varphi(\mu)\in 2\Gamma$, so μ is a square and therefore $\mu = 1$ can be assumed without loss of generality, implying

(11) $$\forall y\in E: \varphi\langle\Psi y,\Psi y\rangle = \varphi\langle y,y\rangle.$$

Applying (10), one sees that y is orthogonal to Ψy only if $y=0$, thus (ii) does not hold for $L_{\perp\!\perp}(E)$.

In order to treat (iii) consider

$$A := \{(\lambda_i)_{i\in\mathbb{N}}\in E \mid \lambda_{2i} = \lambda_{2i+1}\}, \text{ and}$$

$$B := \{(\lambda_i)_{i\in\mathbb{N}}\in E \mid \lambda_{2i+1} = 0\}.$$

The orthogonals to these spaces are

$$A^\perp = \{(\eta_i)_{i\in\mathbb{N}}\in E \mid \eta_{2i}a_{2i} + \eta_{2i+1}a_{2i+1} = 0\}, \text{ and}$$

$$B^\perp = \{(\eta_i)_{i\in\mathbb{N}}\in E \mid \eta_{2i} = 0\}.$$

It is easily seen that $A,B\in L_{\perp\!\perp}(E)$ and $A\cap B = \{0\} = A^\perp\cap B^\perp$. For any $(\eta_i)_{i\in\mathbb{N}}\in B^\perp\backslash\{0\}$ there is an $n\in\mathbb{N}$ with

$$\varphi\langle(\eta_i)_{i\in\mathbb{N}},(\eta_i)_{i\in\mathbb{N}}\rangle = \varphi\Big(\sum_{i\in\mathbb{N}} \eta_{2i+1}^2 a_{2i+1}\Big) = \operatorname*{Min}_{i\in\mathbb{N}} \varphi(\eta_{2i+1}^2 a_{2i+1}) \equiv d_{2n+1} \pmod{2\Gamma},$$

and for any $(\lambda_i)_{i\in\mathbb{N}}\in A\backslash\{0\}$ there is an $m\in N$ with

$$\varphi\langle(\lambda_i)_{i\in\mathbb{N}},(\lambda_i)_{i\in\mathbb{N}}\rangle = \varphi\Big(\sum_{i\in 2\mathbb{N}} \lambda_i^2(a_i + a_{i+1})\Big) = \operatorname*{Min}_{i\in 2\mathbb{N}} \varphi(\lambda_i^2 a_i) \equiv d_{2m} \pmod{2\Gamma}.$$

(11) now shows that (iii) fails for $L_{\perp\!\perp}(E)$.

Finally consider (o): Recall that Ψ is a similarity between E and some subspace U of E. Then U is also an orthomodular space. As in [1] it can be shown that U is a closed subset of E with respect to the vector space topology having $(\{x \in E \mid \varphi\langle x, x\rangle \geqslant E\})_{E \in \Gamma}$ as basis for the neighborhoods of 0. By Theorem 28 of [3], $U \in L_{\perp\perp}(E)$. Now let $(e_i)_{i \in \mathbb{N}}$ be a maximal orthogonal family in E. By (9) and (11), $(\Psi e_i)_{i \in \mathbb{N}}$ is also a maximal orthogonal family in E. But $\Psi e_i \in U = U^{\perp\perp}$, so $U = E$.

3. **Remarks**. The space constructed in section 2 can be modified in several ways (see e.g. [2], [4]). The interest in (iii) comes from the fact that it implies the o-symmetry. But even though (iii) fails in our example, $L_{\perp\perp}(E)$ is o-symmetric. This follows from §XII of [3] and from Remark 3 of [9]. It is still an open problem whether there are Hilbert lattices which are not o-symmetric.

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