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COMPACT COMPOSITION OPERATORS ON BLOCH SPACES

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The present paper proposes new criteria for compactness of a composition operator $C_\phi f = f \circ \phi$ on the Bloch space and the little Bloch space. For the case when ϕ is univalent, a criterion given by K. Madigan and A. Matheson is generalised.

1. Introduction

Let D denote the unit disk in the complex plane. A function f holomorphic on D is said to belong to the Bloch space $\mathcal{B}$ if

$$\sup_{z\in D}\left(1-|z|^2\right)\left|f'(z)\right| < \infty$$

and to the little Bloch space $\mathcal{B}_0$ if

$$\lim_{|z|\to 1^-}\left(1-|z|^2\right)\left|f'(z)\right| = 0.$$

It is well known that $\mathcal{B}$ is a Banach space under the norm

$$\|f\|_{\mathcal{B}} = \left|f(0)\right| + \sup_{z\in D}\left(1-|z|^2\right)\left|f'(z)\right|$$

and that $\mathcal{B}_0$ is a closed subspace of $\mathcal{B}$. A good source for results and references concerning Bloch functions is the paper of Anderson, Clunie and Pommerenke ([**1**]).

It is known that a holomorphic mapping ϕ of D into itself induces a bounded composition operator $C_\phi f = f \circ \phi$ on $\mathcal{B}$, and C_ϕ is a composition operator on $\mathcal{B}_0$ if and only if $\phi \in \mathcal{B}_0$ ([**2**]). Madigan and Matheson ([**2**]) studied the compactness of composition operators and obtained the following results.

Theorem A. *If ϕ is a holomorphic mapping of D into itself, then C_ϕ is a compact operator on $\mathcal{B}_0$ if and only if*

$$\lim_{z\to\partial D}\frac{1-|z|^2}{1-|\phi(z)|^2}\left|\phi'(z)\right| = 0.$$

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Theorem B. *If ϕ is a holomorphic mapping of D into itself, then C_ϕ is a compact operator on $\mathcal{B}$ if and only if for any $\varepsilon > 0$ there exists r, $0 < r < 1$, such that*

$$\sup_{|\phi(z)|>r} \frac{1-|z|^2}{1-|\phi(z)|^2}\left|\phi'(z)\right| < \varepsilon.$$

Theorem C. *If ϕ is univalent and if $G = \phi(D)$ has a nontangential cusp at 1 and touches ∂D at no other point, then C_ϕ is a compact operator on $\mathcal{B}_0$.*

In the same paper, Madigan and Matheson proposed two questions:

1. If $\phi \in \mathcal{B}_0$ and C_ϕ is compact, then $\log\big(1/1 - \overline{w}\phi(z)\big) \in \mathcal{B}_0$ for all $w \in \partial D$. Is the converse of this true?
2. Is there a $\phi \in \mathcal{B}_0$ such that $\overline{\phi(D)} \cap \partial D$ is infinite and C_ϕ is compact on $\mathcal{B}_0$?

In Section 2 of this paper, we discuss Question 1 and give an example which yields a negative anwser. At the same time, we show that C_ϕ is compact on $\mathcal{B}_0$ if and only if $\log\big(1/1 - \overline{w}\phi(z)\big) \in \mathcal{B}_0$ uniformly for all $w \in \partial D$. Section 3 is devoted to the situation when ϕ is a univalent holomorphic self-mapping of D. Theorem C is generalised to the case that $\overline{\phi(D)} \cap \partial D$ contains countably many points, and an example which satisfies the conditions in Question 2 is constructed.

2. Question one

We shall construct a function ϕ which furnishes a negative answer for Question 1. Let

$$G = \left\{re^{i\theta} : 0 < \theta < \frac{\pi}{16},\ 1 - 3\theta^2 < r < 1 - \theta^2\right\}.$$

By the Riemann mapping theorem, there exists a conformal mapping ϕ of D onto G. It is known that ϕ belongs to $\mathcal{B}_0$ ([4]) and C_ϕ is a bounded composition operator on $\mathcal{B}_0$.

First, we show that C_ϕ is not compact on $\mathcal{B}_0$. For $0 < \theta < \pi/16$, let $z(\theta) \in D$ be such that $\phi\big(z(\theta)\big) = w(\theta) = (1 - 2\theta^2)e^{i\theta} \in G$. Then

$$\begin{aligned}\operatorname{dist}\big(w(\theta), \partial G\big) &= \inf_{0<\alpha<\pi/16}\left\{\left|(1-2\theta^2)e^{i\theta} - (1-\alpha^2)e^{i\alpha}\right|, \left|(1-2\theta^2)e^{i\theta} - (1-3\alpha^2)e^{i\alpha}\right|\right\}\\ &\geqslant \theta^2.\end{aligned}$$

Using the Koebe distortion theorem [3] for ϕ, we have

$$\left(1 - \left|z(\theta)\right|^2\right)\left|\phi'\big(z(\theta)\big)\right| \geqslant \operatorname{dist}\Big(\phi\big(z(\theta)\big), \partial G\Big) \geqslant \theta^2.$$

So,

$$\frac{1-|z(\theta)|^2}{1-|w(\theta)|^2}\left|\phi'\big(z(\theta)\big)\right| \geqslant \frac{\theta^2}{1-(1-2\theta^2)^2} \geqslant \frac{1}{4} \quad \text{for } 0 < \theta < \frac{\pi}{16}.$$

Note that $\left|z(\theta)\right| \to 1$ as $\theta \to 0$. By theorem A, C_ϕ is not compact.

Now, let $w \in \partial D$ be fixed and $f(z) = \log\bigl(1/1 - \overline{w}\phi(z)\bigr)$. Then

$$(1 - |z|^2)\bigl|f'(z)\bigr| = \frac{1 - |z|^2}{\bigl|1 - \overline{w}\phi(z)\bigr|}\bigl|\phi'(z)\bigr|.$$

If $w \neq 1$, there exists a positive number δ such that $\bigl|1 - \overline{w}\phi(z)\bigr| > \delta > 0$ for $z \in D$. Thus $f \in \mathcal{B}_0$ since $\phi \in \mathcal{B}_0$. If $w = 1$, assume for the sake of contradiction that $f(z) \notin \mathcal{B}_0$. By Theorem A, there exists a sequence $\{z_n\}$ and a positive number ε, such that $|z_n| \to 1$ and

$$(1 - |z_n|^2)\bigl|f'(z_n)\bigr| = \frac{1 - |z_n|^2}{\bigl|1 - \phi(z_n)\bigr|}\bigl|\phi'(z_n)\bigr| > \varepsilon.$$

Without loss of generality, we may assume $w_n = \phi(z_n) \to w_0$. If $w_0 \neq 1$, the above inequality contradicts the fact $\phi(z) \in \mathcal{B}_0$. If $w_0 = 1$, using the Schwarz-Pick lemma, we have

$$\begin{aligned}(1 - |z_n|^2)\bigl|f'(z_n)\bigr| &= \frac{1 - |z_n|^2}{1 - |\phi(z_n)|^2}\bigl|\phi'(z_n)\bigr| \cdot \frac{1 - |\phi(z_n)|^2}{\bigl|1 - \phi(z_n)\bigr|} \\ &\leqslant \bigl(1 + |w_n|\bigr)\frac{1 - |w_n|}{|1 - w_n|} \to 0,\end{aligned}$$

which leads to a contradiction. This shows that $\log\bigl(1/1 - \overline{w}\phi(z)\bigr) \in \mathcal{B}_0$ for all $w \in \partial D$.

The above example shows that the condition $f(z, w) = \log\bigl(1/1 - \overline{w}\phi(z)\bigr) \in \mathcal{B}_0$ for all $w \in \partial D$ does not guarantee the compactness of C_ϕ. However, if we assume that $f(z, w)$ belongs to $\mathcal{B}_0$ uniformly for $w \in \partial D$, that is, for any $\varepsilon > 0$, there exists r, $0 < r < 1$, such that $\bigl(1 - |z|^2\bigr)\bigl|f'_z(z, w)\bigr| < \varepsilon$ for $r < |z| < 1$ and $w \in \partial D$, then C_ϕ will be compact.

Theorem 1. *If $\phi \in \mathcal{B}_0$, then the composition operator C_ϕ on $\mathcal{B}_0$ is compact if and only if $f(z, w) \in \mathcal{B}_0$ uniformly for $w \in \partial D$.*

Proof: First, assume that $\phi \in \mathcal{B}_0$ and C_ϕ is compact on $\mathcal{B}_0$. A direct computation gives

$$\begin{aligned}\bigl(1 - |z|^2\bigr)\bigl|f'_z(z, w)\bigr| &= \bigl(1 - |z|^2\bigr)\left|\frac{\phi'(z)}{1 - \overline{w}\phi(z)}\right| \\ &= \frac{(1 - |z|^2)|\phi'(z)|}{1 - |\phi(z)|^2} \cdot \frac{1 - |\phi(z)|^2}{|1 - \overline{w}\phi(z)|} \\ &\leqslant \frac{2(1 - |z|^2)|\phi'(z)|}{1 - |\phi(z)|^2}.\end{aligned}$$

Since C_ϕ is compact on $\mathcal{B}_0$, using theorem A, we have

$$\lim_{z \to \partial D} \frac{1 - |z|^2}{1 - |\phi(z)|^2}\bigl|\phi'(z)\bigr| = 0.$$

Thus, $f(z, w)$ belongs to $\mathcal{B}_0$ uniformly for $w \in \partial D$. The necessity is proved.

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Now, let $f(z,w)$ belong to $\mathcal{B}_0$ uniformly for $w \in \partial D$. Then, for every $\varepsilon > 0$, there exists r, $0 < r < 1$, such that

$$\begin{aligned}(1-|z|^2)\left|f_z'(z,w)\right| &= \frac{(1-|z|^2)|\phi'(z)|}{1-|\phi(z)|^2}\cdot\frac{1-|\phi(z)|^2}{|1-\overline{w}\phi(z)|}\\ &< \varepsilon \quad \text{for } |z|>r,\ w\in\partial D.\end{aligned}$$

For $|z| > r$, letting $w = \phi(z)/\left|\phi(z)\right|$ in the above inequality, we have

$$\frac{(1-|z|^2)|\phi'(z)|}{1-|\phi(z)|^2} < \varepsilon.$$

By Theorem A, C_ϕ is compact on $\mathcal{B}_0$. The theorem is proved. □

In the same way as above, we can prove the following theorem for composition operators on $\mathcal{B}$.

Theorem 2. *For any holomorphic mapping ϕ of D into itself, the composition operator C_ϕ is compact on $\mathcal{B}$ if and only if for any $\varepsilon > 0$ there exists r, $0 < r < 1$, such that*

$$\sup_{|\phi(z)|>r}\left(1-|z|^2\right)\left|f_z'(z,w)\right| < \varepsilon \quad \text{for } r<|z|<1,\ w\in\partial D.$$

3. Univalent case

In this section, we consider only the case that $\phi(z)$ is a univalent mapping of D into itself. It is well known that such a ϕ belongs to the little Bloch space $\mathcal{B}_0$ ([**4**]), and C_ϕ is a bounded composition operator on $\mathcal{B}_0$. Denote $G = \phi(D) \subset D$. For $w_0 \in \overline{G}\cap\partial D$, we say G has a cusp at w_0 if there exists a neighbourhood U of w_0 such that $U\cap\overline{G}\cap\partial D = \{w_0\}$, and

$$\operatorname{dist}(w,\partial G) = o\left(|w_0-w|\right)$$

as $w \to w_0$ in G. The cusp w_0 is said to be nontangential if $U \cap G$ lies inside a Stolz angle near w_0, that is, there exist $r, M > 0$ such that

$$|w_0-w| \leqslant M\left(1-|w|^2\right) \quad \text{for } |w_0-w|<r,\ w\in G\cap U.$$

In Theorem C, it is assumed that $\overline{G}$ intersects ∂D at one point 1. It is obvious that Theorem C is valid when G touches the unit circle at only a finite number of points all of which are nontangential cusps. Now, we generalise this to the case that $\overline{G}\cap\partial D$ contains infinitely many points.

Theorem 3. *Let ϕ be a univalent holomorphic mapping of D into itself and $G = \phi(D)$. If $\operatorname{dist}(w,\partial G) = o\left(1-|w|\right)$ as $w \to \partial D$, then C_ϕ is compact on $\mathcal{B}_0$.*

PROOF: By the hypothesis of the theorem, for $\varepsilon > 0$, there exists r, $0 < r < 1$, such that

$$\operatorname{dist}(w, \partial G) < \varepsilon\bigl(1 - |w|\bigr) \quad \text{for } \ |w| > r.$$

Since $\phi \in \mathcal{B}_0$, there exists r', $0 < r' < 1$, such that

$$\bigl(1 - |z|^2\bigr)\bigl|\phi'(z)\bigr| < (1 - r^2)\varepsilon \quad \text{for } \ |z| > r'.$$

Let z be a point such that $|z| > r'$ and $w = \phi(z)$. If $|w| > r$, let ψ be a holomorphic mapping of D onto itself such that $\psi(0) = z$. Then, by Koebe's one-quarter theorem,

$$\bigl(1 - |z|^2\bigr)\bigl|\phi'(z)\bigr| = (\phi \circ \psi)'(0) \leqslant 4 \operatorname{dist}(w, \partial G).$$

Thus,

$$\frac{(1 - |z|^2)|\phi'(z)|}{1 - |\phi(z)|^2} < \frac{4\varepsilon}{1 + |\phi(z)|} < 4\varepsilon.$$

If $\bigl|\phi(z)\bigr| \leqslant r$, we have

$$\frac{(1 - |z|^2)|\phi'(z)|}{1 - |\phi(z)|^2} < \frac{(1 - r^2)\varepsilon}{1 - |\phi(z)|^2} \leqslant \varepsilon.$$

By Theorem A, C_ϕ is compact. The theorem is proved. ∎

COROLLARY 1. *Let ϕ be a univalent holomorphic mapping of D into itself and $G = \phi(D)$. For $0 < r < 1$, denote the maximum length of all arcs contained in $G \cap \bigl\{z : |z| = r\bigr\}$ by $l(r)$. If $l(r)/(1 - r) \to 0$ as $r \to 1$, then C_ϕ is compact on $\mathcal{B}_0$.*

We now present an example of a ϕ for which $G = \phi(D)$ satisfies the condition in the above corollary and $\overline{G} \cap \partial D$ contains infinitely many points. This gives a positive answer to Question 2. Let

$$D_0 = \Bigl\{z : |z| < \frac{1}{2}\Bigr\},$$
$$A(\delta, \theta_0) = \Bigl\{re^{i\theta} : \theta_0 - \delta < \theta < \theta_0 + \delta,\ \frac{1}{2} \leqslant r < 1 - \frac{1}{2\sqrt{\delta}}\sqrt{|\theta - \theta_0|}\Bigr\}.$$

Denote

$$G = D_0 \cup A\Bigl(\frac{\pi}{8}, \frac{\pi}{8}\Bigr) \cup A\left(\frac{\pi}{16}, \frac{5\pi}{16}\right) \cup A\left(\frac{\pi}{32}, \frac{13\pi}{32}\right) \cup \cdots$$
$$= D_0 \bigcup_{i=1}^{\infty} A\Bigl(\frac{1}{2^{i+2}}\pi, \frac{2^{i+1} - 3}{2^{i+2}}\pi\Bigr).$$

It is easy to verify that $G \subset D$ is a simply connected domain and satisfies the condition in Corollary 1. Let ϕ be a conformal mapping of D onto G. Then, by Corollary 1, C_ϕ is compact on $\mathcal{B}_0$.

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