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STRUCTURE OF SOME NOETHERIAN INJECTIVE MODULES

B. SARATH

Introduction. The main object of this paper is to study when injective noetherian modules are artinian. This question was first raised by J. Fisher and an example of an injective noetherian module which is not artinian is given in [**9**]. However, it is shown in [**4**] that over commutative rings, and over hereditary noetherian P.I rings, injective noetherian does imply artinian. By combining results of [**6**] and [**4**] it can be shown that the above implication is true over any noetherian P.I ring. It is shown in this paper that injective noetherian modules are artinian over rings finitely generated as modules over their centers, and over semiprime rings of Krull dimension 1. It is also shown that every injective noetherian module over a P.I ring contains a simple submodule. Since any noetherian injective module is a finite direct sum of indecomposable injectives it suffices to study when such injectives are artinian. If Q is an injective indecomposable noetherian module, then Q contains a non-zero submodule Q_0 such that the endomorphism rings of Q_0 and all its submodules are skewfields. Over a commutative ring, such a Q_0 is simple. In the non-commutative case very little can be concluded, and many of the difficulties seem to arise here.

All rings in this paper have a unit. All properties are left properties unless otherwise specified, and homomorphisms are written opposite scalars. Finally, I would like to thank Dr. K. Varadarajan for the discussions I had with him, and the referee for his many helpful and instructive comments.

1. Noetherian injective modules.

We recall some definitions and results, and introduce some notation.

Definition 1.1. Let $0 \neq M$ be an R-module. Then M is called a *monoform* if for every $N \subseteq M$ and homomorphism $f : N \to M$, f is either zero or a monomorphism.

Definition 1.2. Let M be an R-module, S a subset of M;

(i) $E(M)$ denotes the injective hull of M.

(ii) $l(S) = \{a \in R \mid aS = 0\}$ is the (left) annihilator of S.

(iii) $S \subset M$ means S is contained in but not equal to M.

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Definition 1.3. Let M be an R-module. M is said to have *q.f.d.* (homomorphic images of finite (Goldie) dimension) if every independent family of non-zero submodules of any factor module of M is finite.

A more special notation which will be adhered to for the rest of this paper is the following:

Definition 1.4. Let Q be an injective indecomposable R-module. Then

$$Q_0 = \{q \in Q| \quad qs = 0 \text{ for every } s \in \operatorname{End}_R Q \text{ with } \ker s \neq 0\}.$$

The next proposition summarizes material contained in [**8**, 2.6] and [**5**, 2.6].

PROPOSITION 1.5. *Let Q be an injective indecomposable module $0 \neq C \subseteq Q$ a monoform submodule.*

(i) *Q is uniform;*

(ii) *$Q = E(C)$;*

(iii) *$S = \operatorname{End}_R Q$ is a local ring with maximal ideal $\{s \in S| \ker s \neq 0\}$;*

(iv) *Q_0 is the socle of Q when Q is considered as a right S-module;*

(v) *Q_0 is monoform with $C \subseteq Q_0$;*

(vi) *If $N \subseteq Q_0$ and $Ns \subseteq N$ for every $s \in S$ then $\operatorname{End}_R N$ is a skewfield. In particular $\operatorname{End}_R Q_0$ is a skewfield.*

Proof. (i), (ii) and (iii) are implied in [**8**, 2.6]. (iv) is simply the definition. (v) $C \subseteq Q_0$ follows from [**5**, 2.6]. If $f : N \rightarrow Q_0$ is given with $\ker f \neq 0$, we can extend f to a map $g : Q \rightarrow Q$ whose kernel is then automatically non-zero. Hence $Q_0 g = 0$ by definition and since $N \subseteq Q_0$, $f = 0$. So f is either zero or monic and Q_0 is a monoform. (vi) If $Ns \subseteq N$ for every $s \in S$, then the restriction map induces a ring homomorphism from $S = \operatorname{End}_R Q$ to $\operatorname{End}_R N$. This map is onto since Q is injective and its kernel is the maximal ideal of S hence $\operatorname{End}_R N$ is a skewfield. Since Q_0 is an S-submodule of Q from (iv), $Q_0 s \subseteq Q_0$ for every $s \in S$.

The above proposition yields a generalisation of [**8**, 2.5] which is perhaps of independent interest.

COROLLARY 1.6. *Let H be an injective module over R and suppose that the collection of annihilator ideals of subsets of H satisfies the ascending chain condition. Then $H = \bigoplus_{\alpha \in J} E(T_\alpha)$ where J is some arbitrary indexing set and each T_α is monoform.*

Proof. We follow [**8**, 2.5]. From [**11**], direct sums of independent families of injective submodules of H are injective. It therefore suffices to show that if $0 \neq T \subseteq H$ then $E(T)$ contains a monoform submodule. Let $x \in E(T)$ be such that $l(x)$ is maximal among annihilators of non-zero elements of $E(T)$. Let $N \subseteq Rx$, $0 \neq f : N \rightarrow Rx$. Suppose $\ker f \neq 0$. Let $a \notin l(x)$, $axf = 0$. Extend f to a map $g : E(T) \rightarrow E(T)$. Then $axf = 0$ implies $axg = 0$ and hence that $l(x) \subset l(xg)$, a contradiction.

For the rest of this paper Q denotes a Noetherian injective indecomposable module, Q_0 the corresponding submodule. Every non-zero noetherian module contains a non-zero monoform submodule, hence $Q_0 \neq 0$ [**5**].

The next theorem is useful in investigating noetherian injective modules.

THEOREM 1.7. *With Q, Q_0 as above,* $\mathrm{End}_R\, N$ *is a skewfield for every* $N \subseteq Q_0$.

Proof. Let $0 \neq f \in \mathrm{End}_R\, N$. From (1.4(v)) f is monic. Suppose $Nf \subset N$. Then there exists a map $g : Q \to Q$ with $g|N = f$. g is monic since Q is uniform, and g is an isomorphism since Q is injective and indecomposable. Let $h = g^{-1}$. Then

$$Nh^{n+1} \supset Nfh^{n+1} = Nh^n.$$

So we have $N \subset Nh \subset Nh^2 \ldots$ contradicting Q noetherian. Hence $Nf = N$ and f is a unit so $\mathrm{End}_R\, N$ is a skewfield.

It is known that $S = \mathrm{End}_R\, Q$ is a right and left perfect ring [**3**]. Hence Q has an ascending S-socle series $Q_0 \subset Q_1 \subset Q_2 \ldots \subset Q$; that is Q_{i+1}/Q_i is semisimple as right S-modules [**1**, Theorem P]. The submodules Q_i are invariant under right S-endomorphism of Q and in particular under multiplication by elements of R. Hence the terms in the socle series are $R - S$ bimodules. This yields the following.

PROPOSITION 1.8. *Let $Q_0 \subset Q_1 \subset \ldots$ be the S-socle series of Q, where $S = \mathrm{End}_R\, Q$ and Q is considered as a right S-module. Then* (i) [**3**, 2.5] *$Q = Q_n$ for some n;*
(ii) *If $P = l(Q_0)$, then $P^{n+1}Q = 0$ where n is such that $Q_n = Q$.*

Proof. (i) Since Q is R-noetherian, the sequence $Q_0 \subset Q_1 \subset \ldots Q$ is stationary past some n. From [**1**, Theorem P] this is possible only if $Q_n = Q$. (ii) We show that for $p \in P$,

$$pQ_i \subseteq Q_{i-1} \quad 1 \leqq i \leqq n.$$

Since $pQ_1 \cong Q_1/(Q_1 \cap \ker p)$ as right S-modules, and $pQ_0 = 0$, it follows that pQ_1 is semisimple as a right S-module; hence $pQ_1 \subseteq Q_0$. If $pQ_i \subseteq Q_{i-1}$ for $1 \leqq i \leqq r$, then p induces a map from Q_{r+1}/Q_r onto pQ_{r+1}/pQ_r, but Q_{r+1}/Q_r is semisimple and $pQ_r \subseteq Q_{r-1}$ hence pQ_{r+1}/Q_{r-1} is semisimple. So $pQ_{r+1} \subseteq Q_r$, and the proof is completed by induction and by noting that

$$P^{n+1}Q = P^{n+1}Q_n = PQ_0 = 0.$$

COROLLARY 1.9. *If $R/l(Q_0)$ is artinian then Q is artinian.*

Proof. Let $l(Q_0) = P$. Observe that in the exact sequence

$$0 \to P^iQ/P^{i+1}Q \to Q/P^{i+1}Q \to Q/P^iQ \to 0,$$

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$P^iQ/P^{i+1}Q$ is artinian from [**12**, 6]. A simple induction and the fact that $P^{i+1}Q = 0$ completes the proof.

The next theorem requires a modification of the main theorem in [**7**] which we state as a proposition.

PROPOSITION 1.10. [**7**]. *Let ${}_RM_S$ be an $R - S$ bimodule, with M noetherian both as an R-module and as an S-module. If M is S-artinian then M is R-artinian and $R/l(M)$ is a two sided artinian ring.*

Proof. It is easily verified from the finite generation of M as both R and S modules that $R/l(M)$, $S/\Omega(M)$ are noetherian where

$$\Omega(M) = \{s \in S| \quad Ms = 0\}.$$

Letting $\bar{R} = R/l(M)$, $\bar{S} = S/\Omega(M)$ and passing to the situation ${}_{\bar{R}}M_{\bar{S}}$, it is straightforward to modify the proof of [**7**] to complete the proof of this proposition.

THEOREM 1.11. *Let $M \leqq Q_0$, $D = \mathrm{End}_R\, Q_0$. Then MD finite dimensional as a D-vector space implies that M is R-artinian and $R/l(M)$ is artinian. In particular if Q_0 is finite dimensional over D then Q is artinian.*

Proof. Since (MD) is both noetherian and artinian as a D module and MD is noetherian as an R-module we have that MD is artinian as an R-module and $R/l(MD)$ is artinian from (1.10). Hence M and $R/l(M)$ are artinian. If Q_0 is finite dimensional over D, then $R/l(Q_0)$ is artinian and by (1.9) Q is artinian.

Remark 1.12. The converse to (1.11) does not hold; that is, Q artinian does not imply that Q_0 is finite dimensional over D. To see this let R be the differential polynomial ring studied in [**2**]. Let Q be its unique injective simple module. Then $Q = Q_0$ and an application of (4.1) shows that Q is infinite dimensional over $\mathrm{End}_R\, Q$. ([**2**] considers right modules, but is valid, with the obvious modifications, for left modules.)

2. Commutative rings. As mentioned in the introduction it has been shown in [**4**, 3.3] that over a commutative ring every injective noetherian module is artinian. This is deductible also from (1.6) and (1.9) and we give a brief proof.

PROPOSITION 2.1. [**4**, 3.3]. *Let R be commutative. Then Q_0 is simple and Q is artinian. Hence every injective noetherian R-module is artinian.*

Proof. Let $0 \neq x \in Q_0$. Since $\mathrm{End}_R\, Rx$ is a skewfield and R is commutative, Rx is simple. Hence Q_0 is semisimple and since Q is uniform, Q_0 is simple. The result follows from (1.9).

We take a closer look at the annihilator of Q_0.

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LEMMA 2.2. *Let R be commutative, $M = l(Q_0)$, $m \in M$. Then for some integer $n \geqq 1$, either $m^n = 0$ or $l(m^n) \nsubseteq M$. In either case, $m^nQ = 0$.*

Proof. Suppose that for some $m \in M$, $m^n \neq 0$ and $l(m^n) \subseteq M$ for every integer $n \geqq 1$. Pick an $x \neq 0$ in Q_0 and define $f_n : Rm^n \to Q$ for every integer $n \geqq 1$ by

$$am^nf_n = ax \quad \text{for } a \in R.$$

Since Q is injective, there exists $\{y_n\}_{n=1}^{\infty} \subseteq Q$ such that $m^ny_n = x$. Let A_n be the submodule generated by $y_1, \ldots, y_n$. Then

$$A_n \subset A_{n+1}$$

since

$$m^{n+1}A_n = 0 \quad \text{and} \quad 0 \neq x \in m^{n+1}A_{n+1},$$

but this contradicts Q noetherian. If $m^n = 0$ then $m^nQ = 0$. If $a \notin M$ then multiplication by a induces an isomorphism from Q to Q. Hence $l(m^n) \nsubseteq M$ implies there is an $a \notin M$ with $am^nQ = 0$ and hence $m^nQ = 0$.

THEOREM 2.3. *Let R be commutative, $M = l(Q_0)$;*

(i) *R semiprime implies Q is simple.*

(ii) *M finitely generated implies $R = T \oplus M^n$ for some integer n where T is local with maximal (nilpotent) ideal $T \cap M$.*

Proof. (i) R semiprime implies R has no nilpotent elements so by (2.2), $l(m^n) \nsubseteq M$ for some integer n for every $m \in M$. However R semiprime implies $l(m) = l(m^n)$ for every integer $n \geqq 1$. So $l(m) \nsubseteq M$ for every $m \in M$ and hence $MQ = 0$. Since R is commutative M is maximal and Q is semisimple but then Q is indecomposable and hence simple.

(ii) Let $m_1, \ldots, m_r$ generate M. Then by (2.2), for a suitably large n, M^n is generated by $y_1, \ldots, y_t$ with $l(y_i) \nsubseteq M$ $1 \leqq i \leqq t$. Let $v_iy_i = 0$, $v_i \notin M$. Then if $v = v_1 \ldots v_t$ we have (a) $v \notin M$ and (b) $vM^n = 0$. Together these imply that

$$R = Rv \oplus M^n.$$

Further if $a \notin M$, then

$$Ra + M^n = R.$$

So if $a \in Rv$, $a \notin Rv \cap M$ then a is a unit in Rv. Hence Rv is a local ring with maximal nilpotent ideal $Rv \cap M$.

Remark 2.4. Theorem (2.3) shows that over a commutative noetherian ring R, the only noetherian injective modules that exist come from artinian local direct summands.

LEMMA 2.5. *Let R be commutative, M noetherian over R, and $s \in R$ such that $sM = M$. Then for every $m \in M$, $Rsm = Rm$.*

Proof. Since $sM = M$ we can define $y_i \in M$ by

$$y_0 = m, \quad sy_i = y_{i-1} \quad \text{for } i \geqq 1.$$

Since M is noetherian

$$y_k = \sum_{j=0}^{k-1} r_j y_j \quad \text{for some } k, \text{ with } r_j \in R.$$

Hence

$$m = s^k y_k = \sum_{j=0}^{k-1} r_j s^{k-j} m \in Rsm.$$

So $Rsm = Rm$.

THEOREM 2.6. *Let T be a ring, R a subring of the center of T with T finitely generated as an R-module. Let Q be an injective indecomposable noetherian T-module. Then Q is artinian.*

Proof. Q is a noetherian R-module since T is finitely generated over R. Let $P_1 = l(Q_0)$ in T, and $P = P_1 \cap R$. If $a \in R$, $a \notin P$ then $aQ_0 = Q_0$ since multiplication by a induces a non-zero map from Q_0 to Q_0 and this has to be an automorphism. Let $\bar{R} = R/P$. Then Q_0 is a noetherian $\bar{R}$ module such that $aQ_0 = Q_0$ for every $a \in \bar{R}$. From 2.5, Q_0 is semi-simple artinian as an $\bar{R}$ and hence as an R-module. Since R is commutative it follows that R/P is artinian. Finally T/P_1 is finitely generated as an $\bar{R}$-module and hence is artinian. The theorem follows from (1.9).

3. Semiprime rings with Krull dimension. The advantage of working over semiprime rings is that if an ideal $I \subset R$ has endomorphism ring a skewfield, then I is simple. The proof is given below for completeness.

LEMMA 3.1. *Let R be a semiprime ring. Then* $\text{End}_R\, I$ *is a skewfield if and only if I is simple.*

Proof. If I is simple $\text{End}_R\, I$ is a skewfield. Conversely let $0 \neq i \in I$. Then $iri \neq 0$ for some $r \in R$. The map $f : I \to I$ induced by right multiplication by ri is not zero, hence $Iri = I$. So $Ri = I$, and this shows I is simple.

LEMMA 3.2. *Let R be a semiprime ring. If $l(x)$ is not essential for some $x \in Q_0$, then Q contains a simple module.*

Proof. Suppose $x \in Q_0$ and $I \cap l(x) = 0$ for some $0 \neq I \subset R$. Then $Ix \cong I$ and $\text{End}_R\, Ix$ is a skewfield by (1.6), and so I is simple by (3.1). Hence $Ix \subseteq Q$ is simple.

We state the next proposition without proof. All the material is contained in [**5**, Chapters 1 and 3].

PROPOSITION 3.3. *Let R be a semiprime ring with Krull dimension α, M a finitely generated R-module,*

(i) *If* $x \in M$ *and* $l(x)$ *is essential, then* $l(x)$ *contains a regular element and Krull dimension* $Rx < \alpha$.

(ii) *If Krull dimension* $Rx < \alpha$ *for every* $x \in M$ *then Krull dimension* $M < \alpha$.

THEOREM 3.4. *Let* R *be a semiprime ring of Krull dimension* $\alpha \geqq 1$. *Then Krull dimension* $Q < \alpha$. *In particular if* α *is* 1, Q *is artinian.*

Proof. Suppose Krull dimension $R = \alpha$ and that if $x \in Q_0$, then $l(x)$ is essential. Let $q \in Q$. Since $\{a \in R|\ aq \in Q_0\}$ is essential, there is a regular $d \in R$ with $dq \in Q_0$, (3.3(i)). Since $l(dq)$ is essential, there is a regular $c \in l(dq)$, and hence $cd \in l(q)$ is regular. Hence $l(q)$ is essential, and so Krull dimension $Rq < \alpha$ by (3.3(i)), and Krull dimension of $Q < \alpha$ by (3.3(ii)). If $l(x)$ is not essential for some $x \in Q_0$, then Q contains a simple module T. For any $q \in Q$,

$$I(q) = \{a \in R|\quad aq \in T\}$$

is essential. From the exact sequence

$$0 \rightarrow I(q)/l(q) \rightarrow R/l(q) \rightarrow R/I(q) \rightarrow 0,$$

and the fact that $I(q)/l(q)$ is simple, while Krull dimension $R/I(q) < \alpha$, it follows that Krull dimension $Rq < \alpha$. Again by (3.3(ii)), Krull dimension $Q < \alpha$.

4. *P.I.* **Rings.** A ring R is called a *P.I* ring if it satisfies a non trivial polynomial identity over its centre. In such a case, by using [**10**, II; 4.1] and the linearization process given in [**10**, I; 3.11], we can conclude that the polynomial identity is of type

$$\sum_{\sigma \in S} \pm x_{\sigma(1)} \dots x_{\sigma(n)}$$

where S is some subset of the group of permutations on n letters.

We present some technical lemmas, the first of which is [**8**, 2.8].

LEMMA 4.1. *Let* $D = \operatorname{End}_R Q_0, x_1, \dots, x_n, y \in Q_0$. *Then* $y \in \sum_{i=1}^n x_i D$ *if and only if*

$$l(y) \supseteq l(x_1) \cap \dots \cap l(x_n).$$

Proof. If $y \in \sum_{i=1}^n x_i D$ then clearly $l(x_1) \cap \dots \cap l(x_n) \subseteq l(y)$. Suppose

$$l(x_1) \cap \dots \cap l(x_n) \subseteq l(y).$$

We proceed by induction on n. If $n = 1$ define $\varphi : Rx_1 \rightarrow Ry$ by $ax_1\varphi = ay$ and extend to a $\psi : Q_0 \rightarrow Q_0$. Then $y = x_1\psi$. Now suppose the lemma is true for $r \leqq n - 1$. Then define

$$\varphi : [l(x_1) \cap \dots \cap l(x_{n-1})]x_n \rightarrow [l(x_1) \cap \dots \cap l(x_{n-1})]y$$

by

$$ax_n\varphi = ay.$$

Again extend φ to a $\psi : Q_0 \to Q_0$ and then

$$l(y - x_n\psi) \supseteq l(x_1) \cap \ldots \cap l(x_{n-1}).$$

Hence by induction

$$y = \sum_{i=1}^{n} x_i\varphi_i.$$

The following is [**12**, Lemma 6].

LEMMA 4.2. *Let M be a module over a ring R and suppose that for every $m \in M$, Rm is artinian. Then M has* q.f.d. *implies M is artinian.*

Proof. Suppose M is not artinian. Then M has a proper non-artinian submodule N, and there is an $x_1 \in M$ with $x_1 \notin N$. Suppose we have already constructed $x_1, \ldots, x_n \in M$, $N_1, \ldots, N_r$ submodules of M such that $x_i \notin N_i$, $x_i \in N_j$ for $j \neq i$, $1 \leqq i, j \leqq r$ and $\bigcap_{i=1}^r N_i$ not artinian. Let $\bar{M} = M/\sum Rx_i$ and $\eta : M \to \bar{M}$ the cannonical quotient map. Then $\eta(\bigcap_{i=1}^r N_i)$ is not artinian (since $\sum_{i=1}^n Rx_i$ is artinian) and hence, we can pick

$$\bar{N} \subset \eta\left(\bigcap_{i=1}^{r} N_i\right).$$

Let $N_{r+1} = \eta^{-1}(\bar{N})$ and pick $x_{r+1} \in \bigcap_{i=1}^r N_i$ with $\eta(x_{r+1}) \notin \bar{N}$. Hence we can construct a set $\{x_i\}_{i=1}^\infty \subseteq M$, $\{N_i\}_{i=1}^\infty$ submodules of M with (a) $x_i \notin N_i$, (b) $x_i \in N_j$, $j \neq i$. Let

$$\Psi : M \to \prod_{i=1}^{\infty} M/N_i$$

be the map induced by the canonical quotient maps from $M \to M/N_i$. It is easily verified that $\Psi(\sum_{i=1}^\infty Rx_i)$ does not have finite uniform dimension.

LEMMA 4.3. *Let M be a module over a* P.I *ring R, $N \leqq M$, $0 \neq x \in M - N$. Then there exist $t_1, \ldots, t_n \in R$, $J \leqq MJ \nsubseteq N$ with $[\bigcap_{i=1}^n l(t_i \ldots t_1)x]J \subseteq N$.*

Proof. Suppose the contrary. We demonstrate a contradiction of the fact that R satisfies a P.I by constructing a set $\{u_i\}_{i=0}^\infty$ as follows;

(i) Let $u_0 = 1$.

(ii) Since $l(x)Px \nsubseteq N$ by hypothesis, pick $u_1 \in R$ such that $l(x)u_1x \nsubseteq N$.

(iii) Suppose we have picked $u_1, \ldots, u_r$ such that

$$\left[\bigcap_{i=0}^{r-1} l(u_i \ldots u_0x)\right]u_r \ldots u_1x \nsubseteq N \quad \text{and}$$

$$u_r \in \left[\bigcap_{i=0}^{r-2} l(u_i \ldots u_0x)\right], r \geqq 2.$$

Then by assumption

$$\left[\bigcap_{i=0}^{r-1} l(u_i \ldots u_0x)\right]\left[\bigcap_{i=0}^{r-2} l(u_i \ldots u_0x)\right]u_r \ldots u_1x \not\subseteq N.$$

Hence we can pick

$$u_{r+1} \in \left[\bigcap_{i=0}^{r-2} l(u_i \ldots u_0x)\right]$$

with

$$\left[\bigcap_{i=0}^{r-1} l(u_i \ldots u_0x)\right]u_{r+1} \ldots u_1x \not\subseteq N.$$

Therefore, by induction, we have a set $\{u_i\}_{i=0}^{\infty}$ satisfying

(a) $u_r \ldots u_1x \neq 0$,
(b) $u_ju_i \ldots u_1x = 0 \quad \text{if } j > i + 1$,
(c) $u_ix = 0 \quad \text{if } i \geqq 2$.

Let R satisfy the polynomial identity

$$f(x_1, \ldots, x_n) = x_n \ldots x_1 + \sum_{\sigma \in S} \pm x_{\sigma(n)} \ldots x_{\sigma(1)}$$

where $S \subseteq S_n$ does not contain the identity permutation. Then

$$f(u_1, \ldots, u_n)x = u_n \ldots u_1x$$

by (b), and

$$f(u_1, \ldots, u_n)x = u_n \ldots u_1x \neq 0$$

by (a), and this contradicts the fact that R satisfies f.

THEOREM 4.4. *Let R be a* P.I *ring satisfying a non-trivial identity. Then*
(i) *Q contains a simple module.*
(ii) *If I/I^2 has* q.f.d. *for every ideal $I \leqq R$, then Q is artinian.*

Proof. (i) Setting $N = 0$, $M = Q_0$ in (4.3) we get a $J \neq 0, x_1, \ldots, x_n \in Q_0$ with

$$\left[\bigcap_{i=1}^{n} l(x_i)\right]J = 0.$$

Hence by (4.1) JD is finite dimensional over $D = \operatorname{End}_R Q_0$. So by (1.11), J is artinian and hence J contains a simple submodule and so does Q.

(ii) Let $A = \sum\{A_i \subseteq Q_0 | A_i \text{ is artinian}\}$. Since homomorphic images of artinian submodules are artinian, A is a D-module. Since Q_0 is noetherian, A is artinian and since R satisfies a polynomial identity, $\bar{R} = R/l(A)$ is artinian.

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If $Q_0 = A$ we are done by (1.9). If $Q_0 \neq A$, then by (4.3) there exist $x_1, \ldots, x_n \in Q_0$, $J \nsubseteq A$ such that

$$[l(x_1) \cap \ldots \cap l(x_n)]J \subseteq A.$$

Let

$$l(A) = l(a_1) \cap \ldots \cap l(a_m),$$
$$I = [l(x_1) \cap \ldots \cap l(x_n) \cap l(A)] + l(J).$$

Then $IJ \subseteq A$ and hence $(l(A)I)J = 0$ and so $I^2J = 0$. Hence

$$I^2 \subseteq l(J) \quad \text{and} \quad l(A)I \subseteq l(J).$$

From the assumption that I/I^2 has q.f.d. and the fact that $\bar{R}$ is artinian it follows that $I/l(J)$ is artinian. If $I \neq l(J)$ there is a $\gamma_1 \in J$ with $I\gamma_1 \neq 0$. Hence

$$I \supsetneqq I \cap l(\gamma_1) \supseteq l(J).$$

Repeating the above argument and using the fact that $I/l(J)$ is artinian we can find $\gamma_1, \ldots, \gamma_P \in J$ with $I \cap l(\gamma_1) \cap \ldots \cap l(\gamma_P) \subseteq l(J)$. It follows that

$$[l(x_1) \cap \ldots \cap l(x_n)] \cap [l(a_1) \cap \ldots \cap l(a_n)] \cap [l(\gamma_1) \cap \ldots \cap l(\gamma_P)] \subseteq l(J).$$

This implies (1.11) that $J \subseteq A$, a contradiction. Hence $A = Q_0$ and so Q is artinian.

Remark 4.5. The condition about I/I^2 having h.f.d. in Theorem (4.4) appears somewhat artificial. However, (in the notation of the theorem), the passage from $(l(x_1) \cap \ldots \cap l(x_n))J \subseteq A$ to the fact that JD is finite dimensional over D appears difficult without some hypothesis on R. It is also perhaps worth pointing out that (4.3) includes, as special cases, the known results that over noetherian P.I rings and over V-rings satisfying a P.I, injective and noetherian implies artinian.

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Shiraz University,
Shiraz, Iran