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LAURICELLA HYPERGEOMETRIC FUNCTIONS, UNIPOTENT FUNDAMENTAL GROUPS OF THE PUNCTURED RIEMANN SPHERE, AND THEIR MOTIVIC COACTIONS

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Abstract. The goal of this paper is to raise the possibility that there exists a meaningful theory of '*motives*' associated with certain hypergeometric integrals, viewed as functions of their parameters. It goes beyond the classical theory of motives, but should be compatible with it. Such a theory would explain a recent and surprising conjecture arising in the context of scattering amplitudes for a motivic Galois group action on Gauss's ${}_2F_1$ hypergeometric function, which we prove in this paper by direct means. More generally, we consider Lauricella hypergeometric functions and show, on the one hand, how the coefficients in their Taylor expansions can be promoted, via the theory of motivic fundamental groups, to motivic multiple polylogarithms. The latter are periods of ordinary motives and admit an action of the usual motivic Galois group, which we call the *local* action. On the other hand, we define lifts of the full Lauricella functions as matrix coefficients in a Tannakian category of twisted cohomology, which inherit an action of the corresponding Tannaka group. We call this the *global* action. We prove that these two actions, local and global, are compatible with each other, even though they are defined in completely different ways. The main technical tool is to prove that metabelian quotients of generalised Drinfeld associators on the punctured Riemann sphere are hypergeometric functions. We also study single-valued versions of these hypergeometric functions, which may be of independent interest.

Contents

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§1. Introduction

Let $\Sigma = \{\sigma_0, \sigma_1, \ldots, \sigma_n\}$ be distinct points in $\mathbb{C}$, where $\sigma_0 = 0$. In this paper, we study the Lauricella hypergeometric functions with singularities in Σ which are defined by

$$(L_\Sigma)_{ij} \;=\; -s_j \int_0^{\sigma_i} x^{s_0} \prod_{k=1}^{n} (1 - x\sigma_k^{-1})^{s_k} \frac{dx}{x - \sigma_j}\,, \qquad \text{for } 1 \leq i, j \leq n. \tag{1}$$

The most familiar examples are Euler's beta function $\beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ (see §1.2) and Gauss's hypergeometric function $F = {}_2F_1$ (see §9), which satisfies the integral formula

$$\beta(b, c-b)\, F(a,b,c;y) = \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-yx)^{-a} dx \tag{2}$$

whenever it converges. It can be expressed in terms of (1) for $\Sigma = \{0, 1, y^{-1}\}$.

There are two possible ways in which one might try to define a '*motivic*' Galois group acting on these functions using Tannakian theory, by viewing them in one of the following ways.

(G) Globally, for generic values of the exponents $s_k \in \mathbb{C}$, where genericity means that the $s_k \notin \mathbb{Z}$ for each k and $s_0 + \cdots + s_n \notin \mathbb{Z}$. For such generic s_k, it is known [A1], [DM] how to interpret (1) as '*periods*'[1] of the cohomology of $X_\Sigma = \mathbb{A}^1 \backslash \Sigma$ with coefficients in a rank-one algebraic vector bundle with integrable connection (or local system). One can interpret these twisted cohomology groups as realizations of objects in a suitable Tannakian category, and hence interpret them as representations of a *global* Tannaka group.

(L) Locally, as formal power series in the s_k around the non-generic point $s_0 = \cdots = s_n = 0$. Even though the integral in (1) is divergent at that point, the prefactor s_j compensates the pole and (1) has a Taylor expansion in the s_k at the origin. Its coefficients are generalized polylogarithms, which can be lifted to periods of mixed Tate motives. They admit an action of the usual motivic Galois group, which acts term by term on coefficients in the series expansion.

As is customary in the Tannakian formalism, it is easier to compute the *motivic coaction* of the Hopf algebra of functions on the motivic Galois group, which is dual to the group action. In this article, we compute the global (G) and local (L) motivic coactions and prove

[1] These numbers are not periods in the sense of Kontsevich and Zagier [KZ2] unless the exponents s_i are rational.

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that they are formally identical. This strongly suggests that there exists an underlying '*Lauricella motive*', *viewed as a function of the* s_k, of which (G) and (L) are two different incarnations, or realizations. The realization (G) is obtained by specializing s_k to generic complex numbers; the realization (L) is obtained by expanding locally in the s_k. Such a theory has a consequence for ordinary motives: it implies that the ordinary motivic Galois group acts on the coefficients of the power series expansion (L) in a uniform way. This is surprising since in general these coefficients are periods of unrelated motives. For example, in the case of the Euler beta function, it implies that the motivic Galois group acts uniformly on all (motivic) zeta values of all weights (see §1.2). For the general Lauricella function (1), the main geometric object in the local framework (L) is the mixed Tate motivic fundamental groupoid of the punctured Riemann sphere X_Σ with respect to suitable tangential basepoints.

The impetus for this work came from a remarkable conjecture [ABD+3], [ABD+4] arising in the study of dimensionally regularized one-loop and two-loop Feynman amplitudes in $4-2\varepsilon$ space-time dimensions, which can be expressed in terms of hypergeometric functions [ABD+1], [ABD+2]. It was observed experimentally that the motivic coaction (L), computed order by order in an ε-expansion of $F(n_1+a_1\varepsilon, n_2+a_2\varepsilon, n_3+a_3\varepsilon; y)$, where $n_1, n_2, n_3, a_1, a_2, a_3$ are integers, could, at least to low orders in ε, be succinctly packaged into a coaction formula on the hypergeometric function itself with only two terms. We give a rigorous sense to these statements and prove a complete (global and local) coaction formula for the hypergeometric function.

A large part of this paper is also devoted to studying the single-valued versions of the integrals $(L_\Sigma)_{ij}$, defined by the following complex integrals:

$$(L^{\mathbf{s}}_\Sigma)_{ij} = \frac{s_j}{2\pi i}\iint_{\mathbb{C}} |z|^{2s_0}\prod_{k=1}^{n}|1-z\sigma_k^{-1}|^{2s_k}\left(\frac{d\overline{z}}{\overline{z}-\overline{\sigma_i}}-\frac{d\overline{z}}{\overline{z}}\right)\wedge\frac{dz}{z-\sigma_j}\,, \qquad \text{for } 1\le i,j\le n. \quad (3)$$

The prototype for such functions is the single-valued (or '*complex*') beta function

$$\beta^{\mathbf{s}}(a,b) = \frac{1}{2\pi i}\iint_{\mathbb{C}} |z|^{2a}|1-z|^{2b}\frac{d\overline{z}}{\overline{z}(1-\overline{z})}\wedge\frac{dz}{z(1-z)} = \frac{\Gamma(a)\,\Gamma(b)\,\Gamma(1-a-b)}{\Gamma(a+b)\,\Gamma(1-a)\,\Gamma(1-b)}\,.$$

They are closely related to constructions in conformal field theory (see [BPZ, (E)], [KZ1, §4], [DF]) and have been studied from the global point of view (G) in [A2], [Mi], [MY]. We recast them and their double copy formulae in the general framework of single-valued integration developed in [BD1], [BD2]. In particular, we prove that they can also be interpreted as single-valued versions of Lauricella hypergeometric functions in the local sense (L), that is, by applying the single-valued period homomorphism term by term to the coefficients in the series expansion. As a special case, we define and study two single-valued versions of the hypergeometric function (2), one of which may be new.

In summary, we prove that both the Galois coaction and single-valued period map coincide for (G) and (L); in other words, they '*commute*' with Taylor expanding at $s_0 = \cdots = s_n = 0$.

1.1 Contents

This paper is in two parts, corresponding to the two points of view (G) and (L). In the first part (G), we interpret the Lauricella function as a matrix coefficient in a Tannakian

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category of Betti and de Rham realizations of cohomology with coefficients. To make this a little more precise, consider the trivial algebraic vector bundle of rank one on X_Σ with the integrable connection

$$\nabla_{\underline{s}} = d + \sum_{k=0}^{n} s_k \frac{dx}{x-\sigma_k} \cdot$$

Let $\mathcal{L}_{\underline{s}}$ be the rank one local system generated by $x^{-s_0}\prod_{k=1}^{n}(1-x\sigma_k^{-1})^{-s_k}$, which is a flat section of $\nabla_{\underline{s}}$. For generic $s_0,\dots,s_n$, integration defines a canonical pairing between algebraic de Rham cohomology and locally finite homology groups

$$H^1_{\mathrm{dR}}(X_\Sigma, \nabla_{\underline{s}}) \qquad \text{and} \qquad H_1^{\mathrm{lf}}(X_\Sigma(\mathbb{C}), \mathcal{L}^\vee_{\underline{s}})$$

which are both of rank n. The period matrix, with respect to suitable bases, is exactly the $(n\times n)$ matrix (1). Its entries can be promoted to equivalence classes

$$(L^{\mathfrak{m}}_\Sigma)_{ij} = \Big[M_\Sigma\ ,\ \delta_i \otimes x^{s_0}\prod_{k=1}^{n}(1-x\sigma_k^{-1})^{s_k}\ ,\ -s_j\, d\log(x-\sigma_j)\Big]^{\mathfrak{m}}$$

of Betti–de Rham matrix coefficients (a.k.a. motivic periods), where δ_i is a path from 0 to σ_i, and M_Σ is an object of a Tannakian category encoding the data of the Betti and de Rham cohomology together with the integration pairing. They map to (1) under the period homomorphism:

$$\mathrm{per}(L^{\mathfrak{m}}_\Sigma) = L_\Sigma.$$

We also define de Rham versions of $(L^{\mathfrak{m}}_\Sigma)_{ij}$ as equivalence classes of matrix coefficients as follows. Consider the (classes of the) logarithmic 1-forms:

$$\nu_i = \frac{dx}{x-\sigma_i} - \frac{dx}{x} \qquad \text{for } 1\le i\le n,$$

with residues at 0 and σ_i only (see Remark 3.2 for an interpretation of these forms as de Rham '*versions*' of the paths δ_i). They define de Rham cohomology classes in the space $H^1_{\mathrm{dR}}(X_\Sigma, \nabla_{-\underline{s}})$, which is isomorphic, via the de Rham intersection pairing, to the dual $H^1_{\mathrm{dR}}(X_\Sigma, \nabla_{\underline{s}})^\vee$. Consider the de Rham–de Rham matrix coefficients (a.k.a. de Rham periods)

$$\big(L^{\mathfrak{dr}}_\Sigma\big)_{ij} = [M_\Sigma\ ,\ \nu_i,\ -s_j\, d\log(x-\sigma_j)]^{\mathfrak{dr}}.$$

Comultiplication of matrix coefficients immediately implies a global coaction formula which takes the very simple matrix form:

$$\Delta L^{\mathfrak{m}}_\Sigma = L^{\mathfrak{m}}_\Sigma \otimes L^{\mathfrak{dr}}_\Sigma. \tag{4}$$

The period map cannot be applied to the de Rham Lauricella functions, but one can replace them with variants $\widetilde{L}^{\mathfrak{dr}}_\Sigma$ by passing to a slightly different Tannakian category (where a coaction formula similar to (4) holds) which takes into account the real Frobenius, that

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is, complex conjugation. This added feature yields a *single-valued period* homomorphism $\mathbf{s}$ which can be applied to de Rham periods, whenever all the s_i are real.[2] We recover in this way the single-valued Lauricella hypergeometric functions (3):

$$\mathbf{s}(\widetilde{L}^{\mathfrak{dr}}_{\Sigma}) = L^{\mathbf{s}}_{\Sigma}.$$

This implies, by the definition of the single-valued period homomorphism, the identity

$$L^{\mathbf{s}}_{\Sigma}(\underline{s}) = L_{\overline{\Sigma}}(-\underline{s})^{-1} L_{\Sigma}(\underline{s}), \tag{5}$$

where $\overline{\Sigma} = \{\overline{\sigma_0}, \overline{\sigma_1}, \dots, \overline{\sigma_n}\}$, and $L_{\Sigma}(\underline{s})$ denotes the matrix (1) with dependence on $\underline{s} = (s_0, \dots, s_n)$ made explicit. After applying twisted period relations [CM], (5) can be rewritten as a quadratic formula for single-valued Lauricella functions, which we call a *double copy formula*:

$$L^{\mathbf{s}}_{\Sigma}(\underline{s}) = \frac{1}{2\pi i}\, {}^{t}L_{\overline{\Sigma}}(\underline{s})\, I^{\mathrm{B}}_{\overline{\Sigma}}(-\underline{s})\, L_{\Sigma}(\underline{s}), \tag{6}$$

where $I^{\mathrm{B}}_{\overline{\Sigma}}(-\underline{s})$ is a *Betti intersection matrix* computed in terms of intersection numbers on $X_{\Sigma}(\mathbb{C})$.

In §4, we make the transition from the global to the local picture and explain how to renormalise the integrals (1), following a similar procedure to [BD2], to expose their poles in the $s_0, \dots, s_n$ in a neighborhood of the origin. These poles are compensated by the prefactors s_j in the definition (1), yielding Taylor expansions for both the functions $(L_{\Sigma})_{ij}$ (Proposition 4.4) and their single-valued versions $(L^{\mathbf{s}}_{\Sigma})_{ij}$ (Proposition 4.8).

In the remaining, *local*, part of the paper, we compute the periods, single-valued periods, and motivic coaction order by order in the Taylor expansion with respect to the s_i. For this, we assume that the σ_i lie in a number field $k \subset \mathbb{C}$ and work in the Tannakian category $\mathcal{MT}(k)$ of mixed Tate motives over k [DG]. It has a canonical fiber functor ϖ. Alternatively, one can avoid fixing the σ_i by working in the category of mixed Tate motives over the moduli space of configurations Σ, which leads to identical formulae. The motivic torsor of paths $\pi_1^{\mathrm{mot}}(X_{\Sigma}, t_0, -t_i)$, where t_0 is the tangent vector 1 at 0, and t_i is the tangent vector σ_i at σ_i, is dual to an ind-object of $\mathcal{MT}(k)$ whose periods are regularized iterated integrals of logarithmic 1-forms. Since it is a torsor over the motivic fundamental group based at t_0, one can define its metabelian (or double-commutator) quotient. It turns out that the periods of the latter are very closely related to generalized beta integrals of the form (1). To make this more precise, for any formal power series $F \in R\langle\langle e_0, \dots, e_n\rangle\rangle$ in non-commuting variables $e_0, \dots, e_n$ with coefficients in a commutative ring R, consider its *abelianisation* and *jth beta quotient*

$$\overline{F} \quad \text{and} \quad \overline{F_j} \quad \in \quad R[[s_0, \dots, s_n]],$$

where $\overline{S}$ denotes the image of a formal power series S under the abelianization map $e_k \mapsto s_k$, where the s_k are commuting variables, and the F_j are the unique power series such that

$$F = F_{\varnothing} + F_0 e_0 + \cdots + F_n e_n,$$

[2] This is an unnatural technical assumption that is forced upon us by the fact that the parameters s_i are not treated as formal variables (see §3.5), but formulae such as (5) and (6) remain valid for complex s_i.

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where $F_\varnothing \in R$ is simply the constant term of F. The motivic torsor of paths from t_0 to $-t_i$ defines formal power series $\mathcal{Z}^{\mathfrak{m},i}$, $\mathcal{Z}^{\varpi,i}$ in the $e_0,\dots,e_n$ whose coefficients are motivic periods and (canonical) de Rham periods of $\mathcal{MT}(k)$, respectively,[3] and whose abelianizations and beta quotients are of interest. We assemble them into a matrix of formal power series in $s_0,\dots,s_n$:

$$(FL_\Sigma^{\mathfrak{m}})_{ij} = \mathbf{1}_{i=j}\overline{\mathcal{Z}^{\mathfrak{m},i}} - s_j\overline{\mathcal{Z}_j^{\mathfrak{m},i}},$$

for $1 \leq i,j \leq n$ (resp. with $\mathfrak{m}$ replaced with ϖ). The following result (Theorems 6.18 and 7.11) states that these are indeed *local motivic lifts* of the expanded Lauricella functions (viewed as power series in the s_k). This justifies the notation FL_Σ where the letter F stands for *formal*.

THEOREM 1.1. *We have the equalities of formal power series in $s_0,\dots,s_n$:*

$$(i) \quad \mathrm{per}\,(FL_\Sigma^{\mathfrak{m}}) = L_\Sigma,$$
$$(ii) \quad \mathbf{s}\,(FL_\Sigma^{\varpi}) = L_\Sigma^{\mathbf{s}}\ ,$$

where per *and* $\mathbf{s}$ *are the period map and single-valued period map of $\mathcal{MT}(k)$ applied order by order to the coefficients in the expansion in the variables s_k.*

In the special case $\Sigma = \{0,1\}$, part (i) of the theorem reduces to Drinfeld's well-known computation of the metabelian quotient of the Drinfeld associator in terms of Euler's beta function.

Having therefore established that $FL_\Sigma^{\mathfrak{m}}$ is a local motivic lift of the Lauricella functions viewed as power series, we then prove that its local motivic coaction is given by the same formula as the global motivic coaction (4) of the global motivic lift $L_\Sigma^{\mathfrak{m}}$.

THEOREM 1.2. *The (local) motivic coaction on the formal power series $FL_\Sigma^{\mathfrak{m}}$ reads:*

$$\Delta FL_\Sigma^{\mathfrak{m}}(s_0,\dots,s_n) = FL_\Sigma^{\mathfrak{m}}(s_0,\dots,s_n) \otimes FL_\Sigma^{\varpi}(s_0(\mathbb{L}^{\varpi})^{-1},\dots,s_n(\mathbb{L}^{\varpi})^{-1}), \tag{7}$$

where $\mathbb{L}^{\varpi}$ is the (canonical) de Rham Lefschetz motivic period (motivic version of $2\pi i$). In this formula, the variables s_k should be viewed as having weight -2 and as such admit a nontrivial motivic coaction: $\Delta(s_k) = s_k\,(1 \otimes (\mathbb{L}^{\varpi})^{-1})$.

In the last two sections, we apply our results to Gauss's hypergeometric function $F = {}_2F_1$. Via the integral formula (2) we define global and local lifts of the function $F(a,b,c;y)$. In the global setting, a,b,c are generic complex numbers, that is, $a,b,c,c-a,c-b \notin \mathbb{Z}$, whereas in the local setting, they are formal variables around $a=b=c=0$. The global motivic coaction formula reads:

$$\Delta F^{\mathfrak{m}}(a,b,c;y) = F^{\mathfrak{m}}(a,b,c;y) \otimes F^{\mathfrak{dr}}(a,b,c;y) - \frac{y}{1+c}\,F^{\mathfrak{m}}(a+1,b+1,c+2;y) \otimes G^{\mathfrak{dr}}(a,b,c;y),$$

and the local formula (Theorem 10.9) is formally similar, with $\mathfrak{dr}$ replaced with ϖ and extra $(\mathbb{L}^{\varpi})^{-1}$ factors inserted as in Theorem 1.2. The term $G^{\mathfrak{dr}}$ appearing in the formula is a de

[3] The reader should be warned that the superscripts $\mathfrak{m}$, ϖ, $\mathfrak{dr}$, corresponding to different types of matrix coefficients in Tannakian categories, have two different interpretations in this paper: one in the *global* setting, and one is the *local* setting. This should not create any ambiguity.

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Rham version of the function

$$G(a,b,c;y)=\frac{\sin(\pi a)\sin(\pi(c-a))}{\pi\sin(\pi c)}\,\beta(b,c-b)^{-1}\int_{\infty}^{y^{-1}}x^{b-1}(1-x)^{c-b-1}(1-yx)^{-a}\,dx,$$

which equals $y^{1-c}F(1+b-c,1+a-c,2-c;y)$ up to a prefactor. We also study single-valued versions of the functions F and G, both in the global and local settings, and prove double copy formulae relating them to F and G (see Proposition 9.12).

1.2 Example

Let $n=1$, $\Sigma=\{0,1\}$, $k=\mathbb{Q}$, and $X_\Sigma=\mathbb{P}^1\backslash\{0,1,\infty\}$. The canonical fiber functor ϖ on $\mathcal{MT}(\mathbb{Q})$ is simply the de Rham fiber functor.

1.2.1. Cohomology with coefficients

Let $s_0,s_1\in\mathbb{C}$ be generic, that is, $\{s_0,s_1,s_0+s_1\}\cap\mathbb{Z}=\varnothing$. The algebraic de Rham cohomology $H^1_{\mathrm{dR}}(X,\nabla_{\underline{s}})$ has rank one over $\mathbb{Q}(s_0,s_1)$ and is spanned by the class of $s_1\frac{dx}{1-x}$. The locally finite homology $H_1^{\mathrm{lf}}(X(\mathbb{C}),\mathcal{L}^\vee_{\underline{s}})$ also has rank one over $\mathbb{Q}(e^{2\pi i s_0},e^{2\pi i s_1})$, and is spanned by the class of $(0,1)\otimes x^{s_0}(1-x)^{s_1}$. The corresponding period matrix is the (1×1) matrix

$$L_{\{0,1\}}=\left(s_1\int_0^1 x^{s_0}(1-x)^{s_1}\frac{dx}{1-x}\right)=\left(\frac{s_0s_1}{s_0+s_1}\beta(s_0,s_1)\right).$$

Note that $L_{\{0,1\}}$ is a priori only defined for generic s_0,s_1. It turns out a posteriori that it admits a Taylor expansion at the point $(s_0,s_1)=(0,0)$, which is not generic. The lifted (global) period matrix $L^{\mathfrak{m}}_{\{0,1\}}$ also has a single entry and satisfies the (global) coaction formula

$$\Delta L^{\mathfrak{m}}_{\{0,1\}}=L^{\mathfrak{m}}_{\{0,1\}}\otimes L^{\mathfrak{dr}}_{\{0,1\}}. \tag{8}$$

This is immediate from the fact that the matrix has rank one.

1.2.2. Formal series expansion

Consider the Drinfeld associator

$$\mathcal{Z}=\sum_{w\in\{e_0,e_1\}^\times}\zeta(w)\,w=1+\zeta(2)(e_0e_1-e_1e_0)+\cdots,$$

where $\zeta(w)$ are shuffle regularized multiple zeta values. Its abelianization satisfies $\overline{\mathcal{Z}}=1$. The (1×1) matrix of formal expansions of Lauricella functions $FL_{\{0,1\}}$ is therefore

$$FL_{\{0,1\}}=\left(1-s_1\overline{\mathcal{Z}_1}\right).$$

Its entry is the formal power series

$$1-s_1\int_{\mathrm{dch}}x^{s_0}(1-x)^{s_1}\frac{dx}{x-1}=\frac{s_0s_1}{s_0+s_1}\beta(s_0,s_1),$$

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where dch is the straight line path between tangential basepoints at 0 and 1, and the second equality follows from Proposition 6.16. It is well known that

$$\frac{s_0 s_1}{s_0+s_1}\beta(s_0,s_1) = \exp\left(\sum_{n\geq 2}\frac{(-1)^{n-1}\zeta(n)}{n}\left((s_0+s_1)^n - s_0^n - s_1^n\right)\right). \tag{9}$$

The above objects have motivic and de Rham versions $\mathcal{Z}^{\mathfrak{m}}, FL^{\mathfrak{m}}_{\{0,1\}}$ (resp. $\mathcal{Z}^{\mathfrak{dr}}$, $FL^{\mathfrak{dr}}_{\{0,1\}}$), formally denoted by adding superscripts in the appropriate places, which are formal power series in s_0, s_1 whose coefficients are motivic (resp. de Rham) periods of $\mathcal{MT}(\mathbb{Q})$.[4] For example, the entry of the matrix $FL^{\mathfrak{m}}_{\{0,1\}}$ is exactly the right-hand side of (9), in which $\zeta(n)$ is replaced by $\zeta^{\mathfrak{m}}(n)$. The (local) motivic coaction satisfies[5]

$$\Delta\, FL^{\mathfrak{m}}_{\{0,1\}}(s_0,s_1) = FL^{\mathfrak{m}}_{\{0,1\}}(s_0,s_1)\otimes FL^{\mathfrak{dr}}_{\{0,1\}}((\mathbb{L}^{\mathfrak{dr}})^{-1}s_0, (\mathbb{L}^{\mathfrak{dr}})^{-1}s_1), \tag{10}$$

where Δ acts term by term and on the formal variables s_k via $\Delta(s_k) = s_k(1\otimes(\mathbb{L}^{\mathfrak{dr}})^{-1})$. The formula (10) is equivalent to the equation:

$$\Delta\,\zeta^{\mathfrak{m}}(n) = \zeta^{\mathfrak{m}}(n)\otimes(\mathbb{L}^{\mathfrak{dr}})^n + 1\otimes\zeta^{\mathfrak{dr}}(n), \tag{11}$$

for all $n\geq 2$, using a variant of the well-known fact that in a complete Hopf algebra, an element is group-like if and only if it is the exponential of a primitive element. Since $\zeta^{\mathfrak{dr}}(2n) = 0$, for $n\geq 1$, we retrieve the known coaction formulae on motivic zeta values [B1].

Remark 1.3. Equation (11) is equivalent to the fact that zeta values are periods of simple extensions in the category $\mathcal{MT}(\mathbb{Q})$. It is very interesting that this nontrivial statement shows up as the apparently simpler fact (8) that the cohomology group underlying $L_{\{0,1\}}$ has rank one. It also explains why the formula is uniform for all values of n.

1.2.3. Single-valued versions

The single-valued beta integral is

$$-\frac{s_1}{2\pi i}\iint_{\mathbb{C}}|z|^{2s_0}|1-z|^{2s_1}\left(\frac{d\overline{z}}{\overline{z}-1}-\frac{d\overline{z}}{\overline{z}}\right)\wedge\frac{dz}{1-z} \;=\; \frac{s_0 s_1}{s_0+s_1}\beta^{\mathbf{s}}(s_0,s_1), \tag{12}$$

where the single-valued (or '*complex*') beta function $\beta^{\mathbf{s}}(s_0,s_1)$ satisfies the formula

$$\frac{s_0 s_1}{s_0+s_1}\beta^{\mathbf{s}}(s_0,s_1) = -\frac{\beta(s_0,s_1)}{\beta(-s_0,-s_1)}.$$

The expansion of (12) can be expressed in the form

$$1 - s_1\,\mathbf{s}(\overline{\mathcal{Z}^{\mathfrak{dr}}_1}) = \exp\left(\sum_{n\geq 2}\frac{(-1)^{n-1}\zeta^{\mathbf{sv}}(n)}{n}\left((s_0+s_1)^n - s_0^n - s_1^n\right)\right),$$

where the single-valued zeta $\zeta^{\mathbf{sv}}(n)$ equals $2\zeta(n)$ for n odd ≥ 3 and vanishes for even n.

[4] They are actually motivic and de Rham periods of the Tannakian subcategory $\mathcal{MT}(\mathbb{Z})$.

[5] If one writes this in terms of the motivic beta function $\beta^{\mathfrak{m}}(s_0,s_1)$ defined by $(s_0^{-1}+s_1^{-1})$ times the entry of $FL^{\mathfrak{m}}_{\{0,1\}}$, then it takes the form

$$\Delta\beta^{\mathfrak{m}}(s_0,s_1) = \frac{s_0 s_1}{s_0+s_1}\beta^{\mathfrak{m}}(s_0,s_1)\otimes\beta^{\mathfrak{dr}}((\mathbb{L}^{\mathfrak{dr}})^{-1}s_0,(\mathbb{L}^{\mathfrak{dr}})^{-1}s_1).$$

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This discussion of the beta function generalizes to the case of the moduli spaces of curves of genus zero with marked points. It has been studied in [BD2], [SS1], [VZ].

1.3 Comments

1.3.1. Comparing local expansions

It is natural to ask what can be said about Taylor expansions of Lauricella functions around different (rational) values of the parameters s_k. Clearly, the expansions around different values of s_k are independent from each other: trying to compare them quickly leads to identities involving infinite sums of the kind

$$\sum_{n=2}^{\infty}(\zeta(n)-1)=1,$$

for which there is no motivic interpretation. In fact, the coefficients of expansions at different rational points are very different from the motivic point of view. For example, the value of $\beta(s_0,s_1)$ at non-integer rational values of s_0,s_1 is not a mixed Tate period in general. However, we believe that there should be a general Tannakian framework which, when correctly interpreted, controls the motivic Galois theory of all the expansions of Lauricella functions (and related hypergeometric-type integrals) around rational values of the parameters. This is beyond the scope of this article.

1.3.2. Divergences

The main technical point in this paper is, as usual, dealing with divergences. For cohomology with coefficients, this appears as non-genericity of the parameters s_k. For motivic fundamental groups, it takes the form of tangential basepoints. The following key example illustrates the point.

EXAMPLE 1.4. Suppose that $\mathrm{Re}(s)>0$. Then, viewed as a function of s,

$$I(s)=\int_0^1 x^s\,\frac{dx}{x}=\frac{1}{s}\ .$$

The renormalized version $I^{\mathrm{ren}}(s)$ of this integral (defined in §4) removes the pole in s, and hence $I^{\mathrm{ren}}(s)=0$. Now, consider the integral as a formal power series in s. We perform a Taylor expansion of the integrand and integrate term by term. Since the integrals diverge, they are regularized with respect to a tangent vector of length 1 at the origin, which is equivalent to integrating along the straight line path dch (for *droit chemin*):

$$I^{\mathrm{local}}(s)=\sum_{n\geq 0}\frac{s^n}{n!}\int_{\mathrm{dch}}\log^n(x)\,\frac{dx}{x}=0.$$

Thus, $I^{\mathrm{local}}(s)$ is indeed the Taylor expansion of $I^{\mathrm{ren}}(s)$, which would not be true if one were to regularize with respect to a different tangential basepoint. In general, our tangential basepoints are chosen to be consistent with the renormalization of divergent integrals.

1.3.3. Higher-dimensional generalizations

There are precursors in the physics literature to coaction formulae on generating series of motivic periods. Indeed, in [SS2], open string amplitudes in genus 0 (which can be computed

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in terms of associators [BSS+]) were recast in terms of series of motivic multiple zeta values, and some conjectures were formulated about their f-alphabet decomposition to all orders. For four particles, this is equivalent to Example 1.2.

We can make these conjectures precise as a simple application of the framework described here. Let $\mathcal{M}_{0,S}$ denote the moduli space of curves with points marked by a set S with $n+3$ elements, and let $\nabla_{\underline{s}}$, $\mathcal{L}_{\underline{s}}$ be the Koba–Nielsen connection and local system considered in [BD2, §6]. The periods and single-valued periods studied in that paper can be easily formalized using the Tannakian categories defined in this paper: in brief, the triple

$$M^S \;=\; \big(H^n_{\mathrm{B}}(\mathcal{M}_{0,S},\mathcal{L}_{\underline{s}})\;,\; H^n_{\mathrm{dR}}(\mathcal{M}_{0,S},\nabla_{\underline{s}})\;,\; \mathrm{comp}_{\mathrm{B,dR}}\big)$$

defines an object of any of the *global* Tannakian categories considered in §3. From this, one can define *global* Tannakian lifts of the closed and open superstring amplitudes in genus 0. *Local* lifts (after expanding in the variables s_{ij}) were worked out in [BD2]. We can define global matrix coefficients

$$I^{\mathfrak{m}}(\omega) = [M^S,\gamma,\omega]^{\mathfrak{m}} \qquad \text{and} \qquad I^{\mathfrak{d}\mathfrak{r}}(\nu,\omega) = [M^S,\nu,\omega]^{\mathfrak{d}\mathfrak{r}}$$

for suitable Betti homology classes γ and de Rham (resp. dual de Rham) classes ω (resp. ν). The period of $I^{\mathfrak{m}}(\omega)$ is an open string amplitude, and the single-valued period of (a slight variant of) $I^{\mathfrak{d}\mathfrak{r}}(\nu,\omega)$ is a closed string amplitude. The general coaction formalism (33) or [B3] yields

$$\Delta I^{\mathfrak{m}}(\omega) = \sum_{\eta} I^{\mathfrak{m}}(\eta)\otimes I^{\mathfrak{d}\mathfrak{r}}(\eta^{\vee},\omega),$$

where η ranges over a basis of $H^n_{\mathrm{dR}}(\mathcal{M}_{0,S},\nabla_{\underline{s}})$ and $\eta^{\vee}$ is the dual basis. The objects $I^{\mathfrak{m}}(\omega)$ can be viewed as versions of open string amplitudes, and the objects $I^{\mathfrak{d}\mathfrak{r}}(\eta^{\vee},\omega)$ can be viewed as versions of closed string amplitudes. In terms of the de Rham intersection pairing, this can equivalently be written

$$\Delta I^{\mathfrak{m}}(\omega) = \sum_{\eta,\eta'} \langle \eta,\eta'\rangle^{\mathrm{dR}}\, I^{\mathfrak{m}}(\eta)\otimes I^{\mathfrak{d}\mathfrak{r}}(\eta',\omega),$$

where η,η' range over bases of $H^n_{\mathrm{dR}}(\mathcal{M}_{0,S},\nabla_{\underline{s}})$ and $H^n_{\mathrm{dR}}(\mathcal{M}_{0,S},\nabla_{-\underline{s}})$, respectively. We proved in [BD2] that the Laurent expansions of open and closed string amplitudes admit (noncanonical) motivic lifts. In the light of the present paper, it is natural to expect that their coactions are compatible with the global formula written above. It would be interesting to see if this is equivalent to the conjectures of Stieberger and Schlotterer mentioned above.

1.3.4. Generalizations to other settings

There should be interesting possible generalizations of our results to the elliptic [Ma2] and ℓ-adic [IKY], [N] settings.

Convention

Throughout this paper, we use the following convention: we denote the coordinate on the affine line $\mathbb{C}$ by x when dealing with line integrals, and by z when dealing with double (single-valued) integrals.

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§2. Cohomology with coefficients of a punctured Riemann sphere

We first recall the interpretation of the integrals (1) as periods of the cohomology of the punctured Riemann sphere with coefficients in a rank-one algebraic vector bundle or local system. Most of the results of this section can be found in the classical literature [A1], [CM], [HK], [KN], [KY1], [KY2]. However, we provide an original treatment of the single-valued period homomorphism for cohomology with coefficients. It is based on the Frobenius at infinity, as in the general setting of [BD1], [BD2] and not on the complex conjugation of coefficients, as in [HY], [MY].

2.1 Periods of cohomology with coefficients

Let $k \subset \mathbb{C}$ be a field, and let $\Sigma = \{\sigma_0, \dots, \sigma_n\}$ be distinct points in k with $\sigma_0 = 0$. Write

$$X_\Sigma = \mathbb{A}^1_k \backslash \Sigma.$$

We consider a tuple $\underline{s} = (s_0, \dots, s_n)$ of complex numbers that we shall often assume to be *generic*, meaning that we have

$$\{s_0, s_1, \dots, s_n, s_0 + s_1 + \cdots + s_n\} \cap \mathbb{Z} = \varnothing. \tag{13}$$

An alternative point of view, that we do not develop here, would be to treat the s_i as formal variables (see §3.5).

2.1.1. Algebraic de Rham cohomology

For any subfield $k \subset \mathbb{C}$, denote by

$$k^{\mathrm{dR}}_{\underline{s}} = k(s_0, \dots, s_n).$$

The algebraic de Rham cohomology groups that we will consider are $k^{\mathrm{dR}}_{\underline{s}}$-vector spaces with a natural $\mathbb{Q}^{\mathrm{dR}}_{\underline{s}}$-structure. Define the following logarithmic 1-forms on $\mathbb{P}^1_k$:

$$\omega_i = \frac{dx}{x - \sigma_i} \qquad \text{for} \quad i = 0, \dots, n, \tag{14}$$

which have residue 0 or 1 at points of Σ, and -1 at ∞. They form a basis of the space of global logarithmic forms $\Gamma(\mathbb{P}^1_k, \Omega^1_{\mathbb{P}^1_k}(\log \Sigma \cup \{\infty\}))$, which maps isomorphically to $H^1_{\mathrm{dR}}(X_\Sigma / k)$.

Definition 2.1. Let $\mathcal{O}_{X_\Sigma}$ denote the trivial rank-one bundle on $X_\Sigma \times_k k^{\mathrm{dR}}_{\underline{s}}$, and consider the following logarithmic connection upon it:

$$\nabla_{\underline{s}} : \mathcal{O}_{X_\Sigma} \longrightarrow \Omega^1_{X_\Sigma} \qquad \text{given by} \qquad \nabla_{\underline{s}} = d + \sum_{i=0}^{n} s_i \omega_i.$$

It is automatically integrable since X_Σ has dimension one, and is in fact closely related to the abelianization of the canonical connection on the de Rham unipotent fundamental group of X_Σ. Consider the algebraic de Rham cohomology groups

$$H^r_{\mathrm{dR}}(X_\Sigma, \nabla_{\underline{s}}) = H^r_{\mathrm{dR}}(X_\Sigma, (\mathcal{O}_{X_\Sigma}, \nabla_{\underline{s}})),$$

which are finite-dimensional $k^{\mathrm{dR}}_{\underline{s}}$-vector spaces. The fact that the s_i are generic implies, by [D1, Prop. II.3.13], that one has a logarithmic comparison theorem for $(\mathcal{O}_{X_\Sigma}, \nabla_{\underline{s}})$. Since the cohomology of X_Σ is spanned by global logarithmic forms, this implies (see [ESV]) that

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the cohomology groups $H^r_{\mathrm{dR}}(X_\Sigma, \nabla_{\underline{s}})$ are computed by the complex of global logarithmic forms

$$0 \longrightarrow k_{\underline{s}}^{\mathrm{dR}} \xrightarrow{\nabla_{\underline{s}}} k_{\underline{s}}^{\mathrm{dR}}\omega_0 \oplus \cdots \oplus k_{\underline{s}}^{\mathrm{dR}}\omega_n \longrightarrow 0,$$

where $\nabla_{\underline{s}}(1) = \sum_{i=0}^n s_i\omega_i$. Again, by genericity of the s_i, $H^r_{\mathrm{dR}}(X_\Sigma, \nabla_{\underline{s}})$ vanishes for $r \neq 1$, and $H^1_{\mathrm{dR}}(X_\Sigma, \nabla_{\underline{s}})$ has dimension n. It is generated by the ω_i subject to the single relation

$$\sum_{i=0}^n s_i\,\omega_i = 0. \tag{15}$$

Since the forms ω_i have rational residues, they in fact define a natural $\mathbb{Q}_{\underline{s}}^{\mathrm{dR}}$-structure on $H^1_{\mathrm{dR}}(X_\Sigma, \nabla_{\underline{s}})$ which we shall denote by $H^1_{\varpi}(X_\Sigma, \nabla_{\underline{s}})$. We therefore have

$$H^1_{\varpi}(X_\Sigma, \nabla_{\underline{s}}) \cong \left(\bigoplus_{i=0}^n \mathbb{Q}_{\underline{s}}^{\mathrm{dR}}\omega_i\right) \Big/\ \mathbb{Q}_{\underline{s}}^{\mathrm{dR}} \sum_{i=0}^n s_i\omega_i. \tag{16}$$

We shall use the following basis for (16):

$$\{-s_i\omega_i, \text{ for } i = 1, \ldots, n\}. \tag{17}$$

2.1.2. Betti (co)homology

We introduce the subfield of $\mathbb{C}$ defined by

$$\mathbb{Q}_{\underline{s}}^{\mathrm{B}} = \mathbb{Q}(e^{2\pi i s_0}, \ldots, e^{2\pi i s_n}).$$

Definition 2.2. Let $\mathcal{L}_{\underline{s}}$ denote the rank-one local system of $\mathbb{Q}_{\underline{s}}^{\mathrm{B}}$-vector spaces on the complex points $X_\Sigma(\mathbb{C}) = \mathbb{C}\backslash\Sigma$ defined as

$$\mathcal{L}_{\underline{s}} = \mathbb{Q}_{\underline{s}}^{\mathrm{B}}\, x^{-s_0} \prod_{k=1}^n (1 - x\sigma_k^{-1})^{-s_k}.$$

The local system $\mathcal{L}_{\underline{s}}$ has monodromy $e^{-2\pi i s_k}$ around the point σ_k. After extending scalars to $\mathbb{C}$, it is identified with the horizontal sections of the (analytified) connexion $\nabla_{\underline{s}}$ on the trivial vector bundle of rank one on $X_\Sigma(\mathbb{C})$:

$$\mathcal{L}_{\underline{s}} \otimes_{\mathbb{Q}_{\underline{s}}^{\mathrm{B}}} \mathbb{C} \cong \left(\mathcal{O}^{\mathrm{an}}_{X_\Sigma}\right)^{\nabla_{\underline{s}}}.$$

We will be interested in its cohomology

$$H^r_{\mathrm{B}}(X_\Sigma, \mathcal{L}_{\underline{s}}) = H^r(\mathbb{C}\backslash\Sigma, \mathcal{L}_{\underline{s}}) \ \cong\ H_r(\mathbb{C}\backslash\Sigma, \mathcal{L}_{\underline{s}}^\vee)^\vee,$$

where $\mathcal{L}_{\underline{s}}^\vee$ is the dual local system

$$\mathcal{L}_{\underline{s}}^\vee = \mathbb{Q}_{\underline{s}}^{\mathrm{B}}\, x^{s_0} \prod_{k=1}^n (1 - x\sigma_k^{-1})^{s_k}.$$

The genericity assumption (13) implies that $\mathcal{L}_{\underline{s}}$ and $\mathcal{L}_{\underline{s}}^\vee$ have nontrivial monodromy around every point of Σ and around ∞, which implies that the natural map

$$H_r(\mathbb{C}\backslash\Sigma, \mathcal{L}_{\underline{s}}^\vee) \longrightarrow H_r^{\mathrm{lf}}(\mathbb{C}\backslash\Sigma, \mathcal{L}_{\underline{s}}^\vee)$$

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from ordinary homology to locally finite homology is an isomorphism. Its inverse is sometimes called regularization. An easy computation shows that all homology is concentrated in degree one and $H_1^{\mathrm{lf}}(\mathbb{C}\backslash\Sigma, \mathcal{L}_{\underline{s}}^{\vee})$ has rank n. It has a basis consisting of the (classes of the) locally finite chains

$$\delta_i \otimes x^{s_0} \prod_{k=1}^{n} (1 - x\sigma_k^{-1})^{s_k}, \tag{18}$$

for $i = 1, \ldots, n$, where $\delta_i : (0,1) \to \mathbb{C}\backslash\Sigma$ is a smooth path from 0 to σ_i that can be extended to a smooth path $\overline{\delta_i} : [0,1] \to \mathbb{C}$, and $x^{s_0} \prod_{k=1}^{n} (1 - x\sigma_k^{-1})^{s_k}$ denotes some choice of section of $\mathcal{L}_{\underline{s}}^{\vee}$ on δ_i.

REMARK 2.3. In all that follows, we assume that the choices of representatives are such that δ_i does not wind infinitely around 0 and σ_i, that is, that the argument of $\delta_i(t)$ is bounded as t approaches 0 and that the argument of $\delta_i(t) - \sigma_i$ is bounded as t approaches 1. This assumption will ensure the convergence of the integrals considered below.

2.1.3. Comparison isomorphism.

There is a canonical isomorphism [D1, §6]

$$\mathrm{comp}_{\mathrm{B,dR}}(\underline{s}) : H^1_{\mathrm{dR}}(X_\Sigma, \nabla_{\underline{s}}) \otimes_{k_{\underline{s}}^{\mathrm{dR}}} \mathbb{C} \xrightarrow{\sim} H^1_{\mathrm{B}}(X_\Sigma, \mathcal{L}_{\underline{s}}) \otimes_{\mathbb{Q}_{\underline{s}}^{\mathrm{B}}} \mathbb{C}, \tag{19}$$

whose restriction to the $\mathbb{Q}_{\underline{s}}^{\mathrm{dR}}$-structure we shall denote by

$$\mathrm{comp}_{\mathrm{B},\varpi}(\underline{s}) : H^1_{\varpi}(X_\Sigma, \nabla_{\underline{s}}) \otimes_{\mathbb{Q}_{\underline{s}}^{\mathrm{dR}}} \mathbb{C} \xrightarrow{\sim} H^1_{\mathrm{B}}(X_\Sigma, \mathcal{L}_{\underline{s}}) \otimes_{\mathbb{Q}_{\underline{s}}^{\mathrm{B}}} \mathbb{C}. \tag{20}$$

Assuming (13), we can identify Betti cohomology $H^1_{\mathrm{B}}(X_\Sigma, \mathcal{L}_{\underline{s}})$ with the dual of locally finite homology, which leads to a bilinear pairing

$$H_1^{\mathrm{lf}}(\mathbb{C}\backslash\Sigma, \mathcal{L}_{\underline{s}}^{\vee}) \times H^1_{\varpi}(X_\Sigma, \nabla_{\underline{s}}) \longrightarrow \mathbb{C}.$$

It is well-known that the comparison isomorphism is computed by integration when it makes sense.

LEMMA 2.4. *Assuming (13), a matrix representative for the comparison isomorphism in the bases (17) and (18) is the $(n \times n)$ period matrix L_Σ with entries*

$$(L_\Sigma)_{ij} = -s_j \int_{\delta_i} x^{s_0} \prod_{k=1}^{n} (1 - x\sigma_k^{-1})^{s_k} \frac{dx}{x - \sigma_j},$$

provided that $\mathrm{Re}\, s_0 > -1$ *and* $\mathrm{Re}\, s_i > \begin{cases} -1, & \text{if } i \neq j, \\ 0, & \text{if } i = j. \end{cases}$

Proof. By definition, the pairing that we wish to compute is

$$\langle \delta_i \otimes x^{s_0} \prod_{k=1}^{n} (1 - x\sigma_k^{-1})^{s_k}, \mathrm{comp}_{\mathrm{B,dR}}(\underline{s})(-s_j\omega_j) \rangle = -s_j \int_{\delta_i} x^{s_0} \prod_{k=1}^{n} (1 - x\sigma_k^{-1})^{s_k} \widetilde{\omega}_j,$$

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where $\widetilde{\omega}_j$ is a smooth form on $\mathbb{C}\setminus\Sigma$ with compact support, representing the cohomology class of $\omega_j = dz/(z-\sigma_j)$. In other words, we have

$$\widetilde{\omega}_j - \omega_j = \nabla_{\underline{s}}\phi = d\phi + \sum_{k=0}^{n} s_k\,\phi\, d\log(z-\sigma_k),$$

where ϕ is a smooth function on $\mathbb{P}^1(\mathbb{C})$. Since $\widetilde{\omega}_j$ vanishes in the neighborhood of $\Sigma\cup\infty$, taking residues in the previous equation along points of $\Sigma\cup\infty$ implies that ϕ vanishes at every σ_k, $k\neq j$, including at $\sigma_0=0$. Now, we only need to prove that the integral

$$\int_{\delta_i} x^{s_0}\prod_{k=1}^{n}(1-x\sigma_k^{-1})^{s_k}\,\nabla_{\underline{s}}\phi$$

vanishes. Since $x^{s_0}\prod_{k=1}^{n}(1-x\sigma_k^{-1})^{s_k}\,\nabla_{\underline{s}}\phi = d(x^{s_0}\prod_{k=1}^{n}(1-x\sigma_k^{-1})^{s_k}\,\phi)$, this integral is computed by Stokes' theorem and we need to prove that $x^{s_0}\prod_{k=1}^{n}(1-x\sigma_k^{-1})^{s_k}\phi$ vanishes at 0 and at σ_i.

– At 0, this amounts to proving that $\delta(t)^{s_0}\phi(\delta(t))$ goes to 0 when $t\to 0$. Since ϕ is smooth and vanishes at 0, we have $|\phi(\delta(t))| < C|\delta(t)|$ for some constant C, and so

$$|\delta(t)^{s_0}\phi(\delta(t))| < C|\delta(t)^{s_0+1}| = C|\delta(t)|^{\mathrm{Re}(s_0)+1}\exp(-\mathrm{Im}(s_0)\arg(\delta(t))).$$

  By assumption, $\mathrm{Re}(s_0)+1>0$ and $\arg(\delta(t))$ is bounded as t approaches zero (see Remark 2.3), which implies that the limit of $\delta(t)^{s_0+1}$, and also $\delta(t)^{s_0}\phi(\delta(t))$, is zero when $t\to 0$.
– At σ_i, the same argument gives the desired vanishing, by using the fact that ϕ vanishes at σ_i if $i\neq j$.

The result follows. □

2.2 Intersection pairings

For generic s_i (13), the natural map $H_1(\mathbb{C}\setminus\Sigma,\mathcal{L}_{\underline{s}}^{\vee})\to H_1^{\mathrm{lf}}(\mathbb{C}\setminus\Sigma,\mathcal{L}_{\underline{s}}^{\vee})$ is an isomorphism. Poincaré duality gives an isomorphism

$$H_1^{\mathrm{lf}}(\mathbb{C}\setminus\Sigma,\mathcal{L}_{\underline{s}}^{\vee}) \simeq H_1(\mathbb{C}\setminus\Sigma,\mathcal{L}_{\underline{s}})^{\vee} \simeq H_1(\mathbb{C}\setminus\Sigma,\mathcal{L}_{-\underline{s}}^{\vee})^{\vee},$$

where we set $-\underline{s}=(-s_0,\ldots,-s_n)$. By combining these two isomorphisms, we get a perfect pairing, called the *Betti intersection pairing* [KY1, §2], [CM], [MY, §2]:

$$\langle\ ,\ \rangle_{\mathrm{B}} : H_1(\mathbb{C}\setminus\Sigma,\mathcal{L}_{-\underline{s}}^{\vee})\otimes_{\mathbb{Q}_{\underline{s}}^{\mathrm{B}}} H_1(\mathbb{C}\setminus\Sigma,\mathcal{L}_{\underline{s}}^{\vee}) \longrightarrow \mathbb{Q}_{\underline{s}}^{\mathrm{B}},$$

or dually in cohomology:

$$\langle\ ,\ \rangle^{\mathrm{B}} : H_{\mathrm{B}}^1(X_\Sigma,\mathcal{L}_{-\underline{s}})\otimes_{\mathbb{Q}_{\underline{s}}^{\mathrm{B}}} H_{\mathrm{B}}^1(X_\Sigma,\mathcal{L}_{\underline{s}}) \longrightarrow \mathbb{Q}_{\underline{s}}^{\mathrm{B}}. \tag{21}$$

The de Rham counterpart is the *de Rham intersection pairing* [CM], [Ma1]:

$$\langle\ ,\ \rangle^{\mathrm{dR}} : H_{\mathrm{dR}}^1(X_\Sigma,\nabla_{-\underline{s}})\otimes_{k_{\underline{s}}^{\mathrm{dR}}} H_{\mathrm{dR}}^1(X_\Sigma,\nabla_{\underline{s}}) \longrightarrow k_{\underline{s}}^{\mathrm{dR}}, \tag{22}$$

which comes from Poincaré duality and the fact that the natural map

$$H_{\mathrm{dR,c}}^1(X_\Sigma,\nabla_{\underline{s}}) \longrightarrow H_{\mathrm{dR}}^1(X_\Sigma,\nabla_{\underline{s}}) \tag{23}$$

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is an isomorphism if the s_i are generic, where the subscript c denotes compactly supported cohomology. The map (22) respects the natural $\mathbb{Q}_{\underline{s}}^{\mathrm{dR}}$-structures and induces

$$\langle\ ,\rangle^{\varpi} : H^1_{\varpi}(X_\Sigma, \nabla_{-\underline{s}}) \otimes_{\mathbb{Q}_{\underline{s}}^{\mathrm{dR}}} H^1_{\varpi}(X_\Sigma, \nabla_{\underline{s}}) \longrightarrow \mathbb{Q}_{\underline{s}}^{\mathrm{dR}}.$$

We set

$$\nu_i = \frac{dx}{x-\sigma_i} - \frac{dx}{x} \qquad \text{for} \quad 1 \leq i \leq n \tag{24}$$

and use the same notation for the corresponding class in $H^1_{\varpi}(X_\Sigma, \nabla_{-\underline{s}})$.

LEMMA 2.5. *For all $1 \leq i, j \leq n$, and $s_0, \ldots, s_n$ satisfying (13),*

$$\langle \nu_i, \omega_j \rangle^{\mathrm{dR}} = -\frac{1}{s_i} \mathbf{1}_{i=j}. \tag{25}$$

Proof. By definition, the de Rham intersection pairing that we wish to compute is

$$\langle \nu_i, \omega_j \rangle^{\mathrm{dR}} = \frac{1}{2\pi i} \iint_{\mathbb{P}^1(\mathbb{C})} \widetilde{\nu}_i \wedge \omega_j,$$

where $\widetilde{\nu}_i$ is a smooth form on $\mathbb{C} \setminus \Sigma$ with compact support, representing the cohomology class of ν_i. In other words, we have

$$\widetilde{\nu}_i - \nu_i = \nabla_{-\underline{s}} \phi = d\phi - \sum_{k=0}^{n} s_k \, \phi \, d\log(x - \sigma_k),$$

where ϕ is a smooth function on $\mathbb{P}^1(\mathbb{C})$. Since $\widetilde{\nu}_i$ vanishes in the neighborhood of $\Sigma \cup \infty$, taking residues along points of $\Sigma \cup \infty$ implies that ϕ vanishes at every σ_k, $k \notin \{0, i\}$, and at ∞, and

$$\phi(0) = -\frac{1}{s_0}\ ,\ \phi(\sigma_i) = \frac{1}{s_i}.$$

By noticing that $\nu_i \wedge \omega_j = 0$ and $d\log(x - \sigma_k) \wedge \omega_j = 0$, we thus get

$$\langle \nu_i, \omega_j \rangle^{\mathrm{dR}} = \frac{1}{2\pi i} \iint_{\mathbb{P}^1(\mathbb{C})} d\phi \wedge \omega_j = \frac{1}{2\pi i} \iint_{\mathbb{P}^1(\mathbb{C})} d(\phi \omega_j).$$

By Stokes, this last integral can be computed as the limit when ε goes to zero of

$$-\frac{1}{2\pi i} \int_{\partial P_\varepsilon} \phi \omega_j,$$

where P_ε is the complement in $\mathbb{P}^1(\mathbb{C})$ of ε-disks around the points of $\Sigma \cup \infty$, and the sign comes from the orientation of ∂P_ε. By using the fact that $\phi(\infty) = 0$ and ω_j is regular at every σ_k, $k \neq j$, a local computation (variant of Cauchy's formula) thus gives

$$\langle \nu_i, \omega_j \rangle^{\mathrm{dR}} = -\mathrm{Res}_{\sigma_j}(\phi \omega_j) = -\frac{1}{s_i} \mathbf{1}_{i=j}.$$

□

This lemma implies that the dual basis to (17) is given by the classes

$$\nu_i \ \in \ H^1_{\varpi}(X_\Sigma, \nabla_{-\underline{s}}) \qquad \text{for } i = 1, \ldots, n.$$

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REMARK 2.6. The Betti and de Rham intersection pairings are compatible with the comparison isomorphism, which leads to quadratic relations among periods of cohomology with coefficients, known in the literature as *twisted period relations* [CM], [G]. In matrix form, they read:

$$ {}^t L_\Sigma(-\underline{s})\, I^{\mathrm{B}}_\Sigma(\underline{s})\, L_\Sigma(\underline{s}) = 2\pi i\, I^{\mathrm{dR}}_\Sigma(\underline{s}), \tag{26} $$

where $I^{\mathrm{B}}_\Sigma(\underline{s})$ and $I^{\mathrm{dR}}_\Sigma(\underline{s})$ are the matrices of the Betti and de Rham intersection pairings (21) and (22), respectively.

2.3 Single-valued periods of cohomology with coefficients

We can define and compute a period pairing on de Rham cohomology classes by transporting complex conjugation.

2.3.1. Definition of the single-valued period map

Let $\overline{\Sigma} = \{\overline{\sigma_0}, \overline{\sigma_1}, \ldots, \overline{\sigma_n}\}$ denote the complex conjugates of the points in Σ. We have an anti-holomorphic diffeomorphism

$$ \mathrm{conj} : \mathbb{C}\backslash\overline{\Sigma} \longrightarrow \mathbb{C}\backslash\Sigma $$

given by complex conjugation. We note that the induced map $H_1(\mathbb{C}\backslash\overline{\Sigma}) \to H_1(\mathbb{C}\backslash\Sigma)$ sends the class of a positively oriented loop around $\overline{\sigma_j}$ to the class of a negatively oriented loop around σ_j. Since a rank one local system on $\mathbb{C}\backslash\Sigma$ (resp. $\mathbb{C}\backslash\overline{\Sigma}$) is equivalent to a representation of the abelian group $H_1(\mathbb{C}\backslash\Sigma)$ (resp. $H_1(\mathbb{C}\backslash\overline{\Sigma})$), we see that we have an isomorphism of local systems:

$$ \mathrm{conj}^*\mathcal{L}_{\underline{s}} \simeq \mathcal{L}_{-\underline{s}}. \tag{27} $$

We thus get a morphism of local systems on $\mathbb{C}\backslash\Sigma$:

$$ \mathcal{L}_{\underline{s}} \longrightarrow \mathrm{conj}_*\mathrm{conj}^*\mathcal{L}_{\underline{s}} \simeq \mathrm{conj}_*\mathcal{L}_{-\underline{s}}, $$

which at the level of cohomology induces a morphism of $\mathbb{Q}^{\mathrm{B}}_{\underline{s}}$-vector spaces

$$ F_\infty : H^1_{\mathrm{B}}(\mathbb{C}\backslash\Sigma, \mathcal{L}_{\underline{s}}) \longrightarrow H^1_{\mathrm{B}}(\mathbb{C}\backslash\overline{\Sigma}, \mathcal{L}_{-\underline{s}}). $$

We call F_∞ the *real Frobenius* or *Frobenius at the infinite prime*. We will use the notation $F_\infty(\underline{s})$ when we want to make the dependence on $\underline{s}$ explicit. One checks that the Frobenius is involutive: $F_\infty(-\underline{s})F_\infty(\underline{s}) = \mathrm{id}$.

REMARK 2.7. The isomorphism (27) is induced by the trivialisation of the tensor product $\mathrm{conj}^*\mathcal{L}_{\underline{s}} \otimes \mathcal{L}_{\underline{s}}$ given by the section

$$ g_{\underline{s}} = |z|^{-2s_0} \prod_{k=1}^{n} |1 - z\sigma_k^{-1}|^{-2s_k}. $$

Thus, the homological real Frobenius

$$ F_\infty : H^{\mathrm{B}}_1(\mathbb{C}\setminus\overline{\Sigma}, \mathcal{L}_{\underline{s}}) \longrightarrow H^{\mathrm{B}}_1(\mathbb{C}\setminus\Sigma, \mathcal{L}_{-\underline{s}}) $$

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is computed at the level of representatives by the formula

$$\delta\otimes z^{-s_0}\prod_{k=1}^{n}(1-z\overline{\sigma_k}^{-1})^{-s_k}\mapsto\overline{\delta}\otimes\overline{z}^{-s_0}\prod_{k=1}^{n}(1-\overline{z}\,\overline{\sigma_k}^{-1})^{-s_k}\,g_{\underline{s}}^{-1}=\overline{\delta}\otimes z^{s_0}\prod_{k=1}^{n}(1-z\sigma_k^{-1})^{s_k}.$$

REMARK 2.8. A morphism similar to F_∞ was considered in [HY] and leads to similar formulae but has a different definition and a different interpretation. Our definition only uses the action of complex conjugation on the complex points of the variety X_Σ relative to two complex conjugate embeddings of k in $\mathbb{C}$, whereas the definition in [HY] conjugates the field of coefficients of the local systems, which requires the s_i to be real. Note that our definition does not require the assumption that the $s_i\in\mathbb{R}$.

In the rest of this article, however, we often assume that the s_i are real. (This is an unnatural assumption and would not be necessary if the s_i were treated as formal variables; see §3.5.) In this way, the complex conjugate of the field $k_{\underline{s}}^{\mathrm{dR}}$ inside $\mathbb{C}$ is the field $\overline{k}(s_1,\dots,s_n)$. We use the notation $(-)\otimes_{k_{\underline{s}}^{\mathrm{dR}}}\overline{\mathbb{C}}$ for the tensor product with $\mathbb{C}$, viewed as a $k_{\underline{s}}^{\mathrm{dR}}$-vector space via the complex conjugate embedding. We thus have an additional $\mathbb{C}$-linear comparison isomorphism:

$$\mathrm{comp}_{\overline{\mathrm{B}},\mathrm{dR}}(\underline{s}):H^1_{\mathrm{dR}}(X_\Sigma,\nabla_{\underline{s}})\otimes_{k_{\underline{s}}^{\mathrm{dR}}}\overline{\mathbb{C}}\longrightarrow H^1_{\mathrm{B}}(\mathbb{C}\setminus\overline{\Sigma},\mathcal{L}_{\underline{s}})\otimes_{\mathbb{Q}_{\underline{s}}^{\mathrm{B}}}\mathbb{C}.$$

DEFINITION 2.9. Assume that the s_i are real. The *single-valued period map* is the $\mathbb{C}$-linear isomorphism

$$\mathbf{s}:H^1_{\mathrm{dR}}(X_\Sigma,\nabla_{\underline{s}})\otimes_{k_{\underline{s}}^{\mathrm{dR}}}\mathbb{C}\longrightarrow H^1_{\mathrm{dR}}(X_\Sigma,\nabla_{-\underline{s}})\otimes_{k_{\underline{s}}^{\mathrm{dR}}}\overline{\mathbb{C}}$$

defined as the composite

$$\mathbf{s}=\mathrm{comp}^{-1}_{\overline{\mathrm{B}},\mathrm{dR}}(-\underline{s})\circ(F_\infty\otimes\mathrm{id})\circ\mathrm{comp}_{\mathrm{B},\mathrm{dR}}(\underline{s}).$$

In other words, it is defined by the commutative diagram

$$\begin{array}{ccc}H^1_{\mathrm{dR}}(X_\Sigma,\nabla_{\underline{s}})\otimes_{k_{\underline{s}}^{\mathrm{dR}}}\mathbb{C} & \xrightarrow{\mathrm{comp}_{\mathrm{B},\mathrm{dR}}(\underline{s})} & H^1_{\mathrm{B}}(\mathbb{C}\setminus\Sigma,\mathcal{L}_{\underline{s}})\otimes_{\mathbb{Q}_{\underline{s}}^{\mathrm{B}}}\mathbb{C}\\ \downarrow{\scriptstyle\mathbf{s}} & & \downarrow{\scriptstyle F_\infty\otimes\mathrm{id}}\\ H^1_{\mathrm{dR}}(X_\Sigma,\nabla_{-\underline{s}})\otimes_{k_{\underline{s}}^{\mathrm{dR}}}\overline{\mathbb{C}} & \xrightarrow[\mathrm{comp}_{\overline{\mathrm{B}},\mathrm{dR}}(-\underline{s})]{} & H^1_{\mathrm{B}}(\mathbb{C}\setminus\overline{\Sigma},\mathcal{L}_{-\underline{s}})\otimes_{\mathbb{Q}_{\underline{s}}^{\mathrm{B}}}\mathbb{C}.\end{array}$$

We will use the notation $\mathbf{s}(\underline{s})$ when we want to make the dependence on $\underline{s}$ explicit.

The single-valued period map is a transcendental comparison isomorphism that is naturally interpreted at the level of analytic de Rham cohomology via the isomorphisms

$$H^1_{\mathrm{dR}}(X_\Sigma,\nabla_{\underline{s}})\otimes_{k_{\underline{s}}^{\mathrm{dR}}}\mathbb{C}\simeq H^1_{\mathrm{dR,an}}(\mathbb{C}\setminus\Sigma,(\mathcal{O}_{\mathbb{C}\setminus\Sigma},\nabla_{\underline{s}}))$$

and

$$H^1_{\mathrm{dR}}(X_\Sigma,\nabla_{-\underline{s}})\otimes_{k_{\underline{s}}^{\mathrm{dR}}}\overline{\mathbb{C}}\simeq H^1_{\mathrm{dR,an}}(\mathbb{C}\setminus\overline{\Sigma},(\mathcal{O}_{\mathbb{C}\setminus\overline{\Sigma}},\nabla_{-\underline{s}})).$$

To avoid any confusion, we use the coordinate $w=\overline{z}$ on $\mathbb{C}\setminus\overline{\Sigma}$.

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LEMMA 2.10. *Assume that the s_i are real. In analytic de Rham cohomology, the single-valued period map is induced by the morphism of smooth de Rham complexes*

$$\mathbf{s}_{\mathrm{an}} : (\mathcal{A}^{\bullet}_{\mathbb{C}\setminus\Sigma}, \nabla_{\underline{s}}) \longrightarrow \qquad\qquad \mathrm{conj}_*(\mathcal{A}^{\bullet}_{\mathbb{C}\setminus\overline{\Sigma}}, \nabla_{-\underline{s}})$$

given on the level of sections by

$$\mathcal{A}^{\bullet}_{\mathbb{C}\setminus\Sigma}(U) \ni \omega \quad \mapsto \quad |w|^{2s_0}\prod_{k=1}^{n}|1-w\overline{\sigma}_k^{-1}|^{2s_k}\,\mathrm{conj}^*(\omega) \in \mathcal{A}^{\bullet}_{\mathbb{C}\setminus\overline{\Sigma}}(\overline{U}).$$

Proof. Let $P = |w|^{2s_0}\prod_{k=1}^{n}|1-w\overline{\sigma}_k^{-1}|^{2s_k}$. We first check that $\mathbf{s}_{\mathrm{an}}$ is a morphism of complexes:

$$\begin{aligned}
\nabla_{-\underline{s}}(\mathbf{s}_{\mathrm{an}}(\omega)) &= \nabla_{-\underline{s}}(P\,\mathrm{conj}^*(\omega))\\
&= P\left(\left(\sum_{k=0}^{n} s_k\, d\log(w-\overline{\sigma_k}) + \sum_{k=0}^{n} s_k\, d\log(\overline{w}-\sigma_k)\right)\wedge \mathrm{conj}^*(\omega) + d(\mathrm{conj}^*(\omega))\right)\\
&\quad - \sum_{k=0}^{n} s_k\, d\log(w-\overline{\sigma_k})\wedge(P\,\mathrm{conj}^*(\omega))\\
&= P\left(\sum_{k=0}^{n} s_k\, d\log(\overline{w}-\sigma_k)\wedge \mathrm{conj}^*(\omega) + d(\mathrm{conj}^*(\omega))\right)\\
&= P\,\mathrm{conj}^*\left(\sum_{k=0}^{n} s_k\, d\log(z-\sigma_k)\wedge\omega + d\omega\right)\\
&= \mathbf{s}_{\mathrm{an}}(\nabla_{\underline{s}}(\omega)).
\end{aligned}$$

On the level of horizontal sections, we compute

$$\begin{aligned}
\mathbf{s}_{\mathrm{an}}\left(z^{-s_0}\prod_{k=1}^{n}(1-z\sigma_k^{-1})^{-s_k}\right) &= |w|^{2s_0}\prod_{k=1}^{n}|1-w\overline{\sigma}_k^{-1}|^{2s_k}\,\overline{w}^{-s_0}\prod_{k=1}^{n}(1-\overline{w}\sigma_k^{-1})^{-s_k}\\
&= w^{s_0}\prod_{k=1}^{n}(1-w\overline{\sigma_k}^{-1})^{s_k}.
\end{aligned}$$

Thus, $\mathbf{s}_{\mathrm{an}}$ induces the morphism $\mathcal{L}_{\underline{s}} \to \mathrm{conj}_*\mathcal{L}_{-\underline{s}}$ and the result follows. ∎

2.3.2. Integral formula for single-valued periods

We derive a formula for the single-valued period map $\mathbf{s}$ using the de Rham intersection pairing (22), that is, for the *single-valued period pairing*,

$$H^1_{\mathrm{dR}}(X_\Sigma, \nabla_{\underline{s}}) \times H^1_{\mathrm{dR}}(X_\Sigma, \nabla_{\underline{s}}) \longrightarrow \mathbb{C} \quad , \quad (\nu,\omega) \mapsto \langle \nu, \mathbf{s}\omega\rangle^{\mathrm{dR}}. \tag{28}$$

Note that this pairing is $k_{\underline{s}}^{\mathrm{dR}}$-linear in each argument, where in the first slot the complex conjugate embedding of $k_{\underline{s}}^{\mathrm{dR}}$ inside $\mathbb{C}$ is understood: this means that we have, for $a, b \in k_{\underline{s}}^{\mathrm{dR}}$,

$$\langle a\nu, \mathbf{s}\, b\omega\rangle^{\mathrm{dR}} = \overline{a}b\,\langle \nu, \mathbf{s}\omega\rangle^{\mathrm{dR}}.$$

PROPOSITION 2.11. *Assume that $s_0, \ldots, s_n$ are real and generic (13). Let $\omega, \nu \in \Gamma(\mathbb{P}^1_k, \Omega^1_{\mathbb{P}^1_k}(\log\Sigma\cup\infty))$, and write ω and ν for their classes in $H^1_{\mathrm{dR}}(X_\Sigma, \nabla_{\underline{s}})$. Assume that ν*

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has no pole at ∞. Then the single-valued pairing is

$$\langle \nu, \mathbf{s}\omega\rangle^{\mathrm{dR}} = -\frac{1}{2\pi i}\iint_{\mathbb{C}} |z|^{2s_0}\prod_{k=1}^{n}|1-z\sigma_k^{-1}|^{2s_k}\,\overline{\nu}\wedge\omega \tag{29}$$

whenever $s_i > 0$ for all $0 \leq i \leq n$, and $s_0 + \cdots + s_n < 1/2$.

Proof. Let us first note that the integral in (29) converges under the assumptions on ω, ν, and the s_i. To check this, pass to local polar coordinates $z = \rho e^{i\theta}$ in the neighbourhood of every point in Σ, and verify that $|z|^{2s}\frac{dz\,d\overline{z}}{z\overline{z}}$ is proportional to $\rho^{2s-1}d\rho\,d\theta$ and is integrable when $\mathrm{Re}(s) > 0$. At ∞, use the fact that ν has no pole to obtain a local estimate of the form $\rho^{-2s_0-\cdots-2s_n}d\rho\,d\theta$, which is integrable for $s_0 + \cdots + s_n < 1/2$.

We use the notation $\omega = \omega_\sigma$ for the smooth form on $\mathbb{C}\setminus\Sigma$ induced by ω, and $\omega_{\overline{\sigma}}$ for the smooth form on $\mathbb{C}\setminus\overline{\Sigma}$ induced by ω. (It is obtained by replacing each occurrence of σ_j in ω by a $\overline{\sigma_j}$.) Then, by definition, we have $\langle \nu, \mathbf{s}\omega\rangle = \langle \nu_{\overline{\sigma}}, \mathbf{s}_{\mathrm{an}}\omega_\sigma\rangle$. By Lemma 2.10 and the definition of the de Rham intersection pairing, this equals the integral

$$\frac{1}{2\pi i}\iint_{\mathbb{P}^1(\mathbb{C})} |w|^{2s_0}\prod_{k=1}^{n}|1-w\overline{\sigma_k}^{-1}|^{2s_k}\,\widetilde{\nu_{\overline{\sigma}}}\wedge\mathrm{conj}^*(\omega_\sigma), \tag{30}$$

where $\widetilde{\nu_{\overline{\sigma}}}$ is a smooth form on $\mathbb{C}\setminus\overline{\Sigma}$ with compact support, representing the cohomology class of $\nu_{\overline{\sigma}}$. In other words, we have

$$\widetilde{\nu_{\overline{\sigma}}} - \nu_{\overline{\sigma}} = \nabla_{\underline{s}}\phi = d\phi + \sum_{k=0}^{n} s_k\,\phi\,d\log(w-\overline{\sigma_k}),$$

where ϕ is a smooth function on $\mathbb{P}^1(\mathbb{C})$. The assumption that ν has no pole at ∞ and the fact that $s_0 + \cdots + s_n \neq 0$ imply, by taking residues, that $\phi(\infty) = 0$. We first prove that we may remove the tilde in (30), that is, that the integral

$$\iint_{\mathbb{P}^1(\mathbb{C})} |w|^{2s_0}\prod_{k=1}^{n}|1-w\overline{\sigma_k}^{-1}|^{2s_k}\,\nabla_{\underline{s}}\phi\wedge\mathrm{conj}^*(\omega_\sigma) \tag{31}$$

vanishes. Its integrand equals

$$d\left(|w|^{2s_0}\prod_{k=1}^{n}|1-w\overline{\sigma_k}^{-1}|^{2s_k}\,\phi\,\mathrm{conj}^*(\omega)\right)$$

because $d\overline{w}\wedge\mathrm{conj}^*(\omega_\sigma) = \mathrm{conj}^*(dz\wedge\omega_\sigma) = 0$. By Stokes, the integral (31) can be computed as the limit when ε goes to zero of

$$-\int_{\partial P_\varepsilon} |w|^{2s_0}\prod_{k=1}^{n}|1-w\overline{\sigma_k}^{-1}|^{2s_k}\,\phi\,\mathrm{conj}^*(\omega),$$

where P_ε is the complement in $\mathbb{P}^1(\mathbb{C})$ of ε-disks around the points of $\overline{\Sigma}\cup\infty$, and the sign comes from the orientation of ∂P_ε. The contribution of each point of $\overline{\Sigma}$ vanishes, as can be seen from a computation in a local coordinate, because of the assumption that $s_i > 0$ for all i. The contribution of the point ∞ also vanishes because of the fact that $\phi(\infty) = 0$ and

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the assumption that $s_0+\cdots+s_n<1/2$. Thus, we have

$$\begin{aligned}\langle \nu, \mathbf{s}\omega\rangle^{\mathrm{dR}} &= \frac{1}{2\pi i}\iint_{\mathbb{P}^1(\mathbb{C})} |w|^{2s_0}\prod_{k=1}^{n}|1-w\overline{\sigma_k}^{-1}|^{2s_k}\,\nu_{\overline{\sigma}}\wedge \mathrm{conj}^*(\omega_\sigma)\\ &= -\frac{1}{2\pi i}\iint_{\mathbb{P}^1(\mathbb{C})} |\overline{z}|^{2s_0}\prod_{k=1}^{n}|1-\overline{z\sigma_k}^{-1}|^{2s_k}\,\mathrm{conj}^*(\nu_{\overline{\sigma}})\wedge\omega_\sigma\\ &= -\frac{1}{2\pi i}\iint_{\mathbb{P}^1(\mathbb{C})} |z|^{2s_0}\prod_{k=1}^{n}|1-z\sigma_k^{-1}|^{2s_k}\,\overline{\nu_\sigma}\wedge\omega_\sigma.\end{aligned}$$

The second equality follows from performing a change of variables via $\mathrm{conj}:\mathbb{C}\backslash\Sigma\to\mathbb{C}\backslash\overline{\Sigma}$, which reverses the orientation of $\mathbb{C}$, hence the minus sign. The third equality relies on $\mathrm{conj}^*(\nu_{\overline{\sigma}})=\overline{\nu_\sigma}$, which is obvious. The result follows. □

COROLLARY 2.12. *Assume that $s_0,\dots,s_n$ are real and generic (13). Then, for all $1\le i,j\le n$, the single-valued Lauricella function $(L^{\mathbf{s}}_{\Sigma})_{ij}$ (3) is a single-valued period of cohomology with coefficients:*

$$(L^{\mathbf{s}}_{\Sigma})_{ij} = \langle \nu_i, \mathbf{s}(-s_j\omega_j)\rangle^{\mathrm{dR}}$$

for $s_k>0$ for all $0\le k\le n$ and $s_0+\cdots+s_n<1/2$.

Proof. This follows from Proposition 2.11. □

COROLLARY 2.13. *Under the assumptions of the previous corollary, we have the matrix equality*

$$L^{\mathbf{s}}_{\Sigma}(s_0,\dots,s_n) = L_{\overline{\Sigma}}(-s_0,\dots,-s_n)^{-1}L_{\Sigma}(s_0,\dots,s_n)\ , \tag{32}$$

where it is understood that the Lauricella functions $(L_\Sigma)_{ij}$ and $(L_{\overline{\Sigma}})_{ij}$ are computed via choices of paths which are complex conjugate to each other.

Proof. This follows from the definition of the single-valued period homomorphism and Corollary 2.12 since the classes ν_i are the dual basis, with respect to the de Rham pairing, of the basis $(-s_i\omega_i)$ by Lemma 2.5. □

REMARK 2.14. The inverse of the matrix $L_{\overline{\Sigma}}(-\underline{s})$ appearing in (32) can be computed in terms of $L_{\overline{\Sigma}}(\underline{s})$ from the twisted period relation (26). This leads to quadratic expressions of the single-valued Lauricella function in terms of ordinary Lauricella functions, which we call *double copy formulae*. They read, in matrix form,

$$L^{\mathbf{s}}_{\Sigma}(\underline{s}) = \frac{1}{2\pi i}\,{}^tL_{\overline{\Sigma}}(\underline{s})\,I^{\mathrm{B}}_{\overline{\Sigma}}(-\underline{s})\,L_{\Sigma}(\underline{s}).$$

This is because the matrix $I^{\mathrm{dR}}_{\overline{\Sigma}}(-\underline{s})$ is the identity matrix in the bases $(+s_i\omega_i)$ and (ν_i) by Lemma 2.5. They are related to double copy formulae in the physics literature such as the Kawai–Lewellen–Tye formula [KLT], which was interpreted in the framework of cohomology with coefficients in [Mi2].

§3. Tannakian interpretations and *global* coaction

Using some simple Tannakian formalism, we compute a global coaction formula on Tannakian lifts of Lauricella functions. We also consider a more refined Tannakian formalism

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which takes into account the Frobenius at infinity and interpret single-valued Lauricella functions in this more refined framework.

3.1 Minimalist version

Let $k_{\mathrm{dR}} \subset \mathbb{C}$ and $\mathbb{Q}_{\mathrm{B}} \subset \mathbb{C}$ be two subfields of $\mathbb{C}$. Consider the $\mathbb{Q}$-linear abelian category $\mathcal{T}$ whose objects consist of triples $V = (V_{\mathrm{B}}, V_{\mathrm{dR}}, c)$ where $V_{\mathrm{B}}, V_{\mathrm{dR}}$ are finite-dimensional vector spaces over $\mathbb{Q}_{\mathrm{B}}$ and k_{dR}, respectively, and $c : V_{\mathrm{dR}} \otimes_{k_{\mathrm{dR}}} \mathbb{C} \xrightarrow{\sim} V_{\mathrm{B}} \otimes_{\mathbb{Q}_{\mathrm{B}}} \mathbb{C}$ is a $\mathbb{C}$-linear isomorphism. The morphisms ϕ in $\mathcal{T}$ are pairs of linear maps ϕ_{B}, ϕ_{dR} compatible with the isomorphisms c. The category $\mathcal{T}$ is Tannakian with two fiber functors

$$\omega_{\mathrm{dR}} : \mathcal{T} \longrightarrow \mathrm{Vec}_{k_{\mathrm{dR}}} \qquad \text{and} \qquad \omega_{\mathrm{B}} : \mathcal{T} \longrightarrow \mathrm{Vec}_{\mathbb{Q}_{\mathrm{B}}}.$$

The ring $\mathcal{P}^{\mathfrak{m}}_{\mathcal{T}} = \mathcal{O}(\mathrm{Isom}^{\otimes}_{\mathcal{T}}(\omega_{\mathrm{dR}}, \omega_{\mathrm{B}}))$ is the $(\mathbb{Q}_{\mathrm{B}}, k_{\mathrm{dR}})$-bimodule spanned by equivalence classes of matrix coefficients $[V, \sigma, \omega]^{\mathfrak{m}}$ where $\sigma \in V_{\mathrm{B}}^{\vee}$ and $\omega \in V_{\mathrm{dR}}$. The k_{dR}-algebra $\mathcal{P}^{\mathfrak{dr}}_{\mathcal{T}} = \mathcal{O}(\mathrm{Aut}^{\otimes}_{\mathcal{T}}(\omega_{\mathrm{dR}}))$ is spanned by equivalence classes of matrix coefficients $[V, f, \omega]^{\mathfrak{dr}}$ where $f \in V_{\mathrm{dR}}^{\vee}$ and $\omega \in V_{\mathrm{dR}}$. The multiplicative structure is given by tensor products. There is a natural coaction

$$\Delta : \mathcal{P}^{\mathfrak{m}}_{\mathcal{T}} \longrightarrow \mathcal{P}^{\mathfrak{m}}_{\mathcal{T}} \otimes_{k_{\mathrm{dR}}} \mathcal{P}^{\mathfrak{dr}}_{\mathcal{T}},$$

which expresses $\mathcal{P}^{\mathfrak{m}}_{\mathcal{T}}$ as an algebra comodule over the Hopf algebra $\mathcal{P}^{\mathfrak{dr}}_{\mathcal{T}}$. It is given by the formula

$$\Delta[V, \sigma, \omega]^{\mathfrak{m}} = \sum_i [V, \sigma, e_i]^{\mathfrak{m}} \otimes [V, e_i^{\vee}, \omega]^{\mathfrak{dr}}, \tag{33}$$

where the sum ranges over a k_{dR}-basis $\{e_i\}$ of V_{dR}, and $e_i^{\vee}$ denotes the dual basis of $V_{\mathrm{dR}}^{\vee}$.

Definition 3.1. For generic complex numbers s_i (13), let $k_{\mathrm{dR}} = \mathbb{Q}^{\mathrm{dR}}_{\underline{s}}$ and $\mathbb{Q}_{\mathrm{B}} = \mathbb{Q}^{\mathrm{B}}_{\underline{s}}$, and define

$$M_\Sigma = \big(\ H^1_{\mathrm{B}}(X_\Sigma, \mathcal{L}_{\underline{s}}) \ , \ H^1_{\varpi}(X_\Sigma, \nabla_{\underline{s}}) \ , \ \mathrm{comp}_{\mathrm{B},\varpi}(\underline{s}) \ \big) \quad \in \quad \mathrm{Ob}(\mathcal{T}).$$

Define a matrix $L^{\mathfrak{m}}_\Sigma \in M_{n\times n}(\mathcal{P}^{\mathfrak{m}}_{\mathcal{T}})$ by

$$(L^{\mathfrak{m}}_\Sigma)_{ij} = \Big[M_\Sigma \ , \ \delta_i \otimes x^{s_0} \prod_{k=1}^{n} (1 - x\sigma_k^{-1})^{s_k} \ , \ -s_j \omega_j \Big]^{\mathfrak{m}},$$

where the basis elements are given by (17) and (18). We will use the notation $M_\Sigma(\underline{s})$ and $L^{\mathfrak{m}}_\Sigma(\underline{s})$ when we want to make the dependence on $\underline{s}$ explicit.

The image of $L^{\mathfrak{m}}_\Sigma$ under the period homomorphism

$$\begin{aligned} \mathrm{per} : \mathcal{P}^{\mathfrak{m}}_{\mathcal{T}} &\longrightarrow \mathbb{C} \\ [(V_{\mathrm{B}}, V_{\mathrm{dR}}, c), \sigma, \omega]^{\mathfrak{m}} &\mapsto \langle \sigma \, , \, c(\omega) \rangle \end{aligned}$$

is precisely the matrix of Lauricella functions (when the integral converges; see Lemma 2.4):

$$\mathrm{per}\,(L^{\mathfrak{m}}_\Sigma)_{ij} = -s_j \int_{\delta_i} x^{s_0} \prod_{k=1}^{n} (1 - x\sigma_k^{-1})^{s_k} \frac{dx}{x - \sigma_j} \ .$$

For this reason, we think of $L^{\mathfrak{m}}_\Sigma$ as a (global) motivic lift of the matrix of Lauricella functions (1). Define a (global) de Rham version

$$(L^{\mathfrak{dr}}_\Sigma)_{ij} = [M_\Sigma \ , \ \nu_i \ , \ -s_j \omega_j]^{\mathfrak{dr}},$$

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where the forms ν_i, defining classes in $H^1_{\varpi}(X_\Sigma, \nabla_{-\underline{s}})$, were defined in (24). Recall that the de Rham pairing (23) induces an isomorphism $H^1_{\varpi}(X_\Sigma, \nabla_{-\underline{s}}) \simeq H^1_{\varpi}(X_\Sigma, \nabla_{\underline{s}})^\vee$ and that Lemma 2.5 implies that the basis $\{\nu_i\}$ is dual to the basis $\{-s_i\omega_i\}$.

REMARK 3.2. The class of ν_i is the image of the relative homology class (viewed in homology with trivial coefficients) of the path δ_i under the map $c_0^\vee$ studied in [BD1, §4.5.1]. It would be interesting to know whether this map can be naturally defined at the level of cohomology with coefficients by relying on Hodge theoretic arguments as in [BD1].

EXAMPLE 3.3. For all $n \in \mathbb{Z}$, one has '*Tate*' objects

$$\mathbb{Q}_{\underline{s}}(-n) = (\mathbb{Q}^{\mathrm{B}}_{\underline{s}}, \mathbb{Q}^{\mathrm{dR}}_{\underline{s}}, 1 \mapsto (2\pi i)^n).$$

The Betti and de Rham pairings (21) and (22), along with their compatibility with the comparison map (19) (a.k.a. the *twisted period relations*), can be succinctly encoded as a perfect pairing

$$M_\Sigma(-\underline{s}) \otimes M_\Sigma(\underline{s}) \longrightarrow \mathbb{Q}_{\underline{s}}(-1)$$

in the category $\mathcal{T}$. Equivalently, $M_\Sigma(\underline{s})^\vee \simeq M_\Sigma(-\underline{s})(1)$, where (n) is the standard notation for Tate twist, that is, tensor product with $\mathbb{Q}_{\underline{s}}(n)$.

PROPOSITION 3.4. *The (global) coaction satisfies*

$$\Delta L^{\mathfrak{m}}_\Sigma = L^{\mathfrak{m}}_\Sigma \otimes L^{\mathfrak{dr}}_\Sigma.$$

Proof. This is an immediate consequence of the formula (33). □

REMARK 3.5. The coproduct Δ in the Hopf algebra $\mathcal{P}^{\mathfrak{dr}}_{\mathcal{T}}$ is given by a formula similar to (33) and satisfies $\Delta L^{\mathfrak{dr}}_\Sigma = L^{\mathfrak{dr}}_\Sigma \otimes L^{\mathfrak{dr}}_\Sigma$.

3.2 Version with real Frobenius involutions

In order to incorporate the real Frobenius isomorphism, and hence single-valued periods, into a Tannakian framework, we are obliged to consider a more complicated version of the previous categorical construction. This is somewhat artificial, but is forced upon us by the fact that complex conjugation does not define an involution on $H^1_{\mathrm{B}}(X, \mathcal{L}_{\underline{s}})$, but rather relates the cohomology of $\mathcal{L}_{\underline{s}}$ with that of $\mathcal{L}_{-\underline{s}}$. For this reason, we must consider two de Rham and two Betti realizations corresponding to coefficients $\nabla_{\underline{s}}$, $\nabla_{-\underline{s}}$ and $\mathcal{L}_{\underline{s}}$, $\mathcal{L}_{-\underline{s}}$, which we denote by superscripts $+/-$.

Let $s_i \in \mathbb{R}$ be real numbers satisfying the genericity conditions (13). Let $k_{\mathrm{dR}} \subset \mathbb{C}$ and $\mathbb{Q}_{\mathrm{B}} \subset \mathbb{C}$ be two subfields of $\mathbb{C}$. We let $(-) \otimes_{k_{\mathrm{dR}}} \overline{\mathbb{C}}$ denote the tensor product taken with respect to the complex conjugate embedding. One solution is to consider a category $\mathcal{T}_\infty$ defined in a similar manner as $\mathcal{T}$, except that the objects are given by:

1. a pair of finite-dimensional k_{dR}-vector spaces V^+_{dR}, V^-_{dR};
2. two pairs of finite-dimensional $\mathbb{Q}_{\mathrm{B}}$-vector spaces V^+_{B}, V^-_{B} and $V^+_{\overline{\mathrm{B}}}$, $V^-_{\overline{\mathrm{B}}}$;
3. two $\mathbb{C}$-linear comparison isomorphisms

$$c^+_{\mathrm{B,dR}} : V^+_{\mathrm{dR}} \otimes_{k_{\mathrm{dR}}} \mathbb{C} \xrightarrow{\sim} V^+_{\mathrm{B}} \otimes_{\mathbb{Q}_{\mathrm{B}}} \mathbb{C}\ ,$$
$$c^+_{\overline{\mathrm{B}},\mathrm{dR}} : V^+_{\mathrm{dR}} \otimes_{k_{\mathrm{dR}}} \overline{\mathbb{C}} \xrightarrow{\sim} V^+_{\overline{\mathrm{B}}} \otimes_{\mathbb{Q}_{\mathrm{B}}} \mathbb{C}\ ,$$

and another two defined in the same way with all $+$'s replaced by $-$,

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4. two $\mathbb{Q}_B$-linear real Frobenius maps

$$V_B^+ \xrightarrow{\sim} V_{\overline{B}}^- \qquad \text{and} \qquad V_{\overline{B}}^+ \xrightarrow{\sim} V_B^-,$$

and another two with the $+$'s and $-$'s interchanged. These maps will simply be denoted by F_∞, since the source and, hence, the target will be clear from context. The composition of any two such maps, when defined, is the identity: $F_\infty F_\infty = \mathrm{id}$.

The morphisms ϕ between objects are given by the data of two k_{dR}-linear maps $\phi_{\mathrm{dR}}^+, \phi_{\mathrm{dR}}^-$, and four $\mathbb{Q}_{\mathrm{B}}$-linear maps ϕ_{B}^+, ϕ_{B}^-, $\phi_{\overline{\mathrm{B}}}^+$, $\phi_{\overline{\mathrm{B}}}^-$ which are compatible with (3) and (4). One checks that this category is Tannakian, equipped with six fiber functors $\omega_{\mathrm{dR}}^{\pm}, \omega_{\mathrm{B}}^{\pm}, \omega_{\overline{\mathrm{B}}}^{\pm}$ (over $k_{\mathrm{dR}}, \mathbb{Q}_{\mathrm{B}}$, respectively) which are obtained by forgetting all data except one of the vector spaces in (1) or (2).

REMARK 3.6. For objects of $\mathcal{T}_\infty$ coming from geometry such as the ones that we will shortly define, there are compatibilities between the real Frobenius isomorphisms F_∞ and the comparison isomorphisms. We do not include such compatibilities in our definition because they will be irrelevant to the computations that we will be performing.

The category $\mathcal{T}_\infty$ admits various rings of periods defined in a similar manner as before. Consider the eight rings of periods:

$$\mathcal{P}_{\mathcal{T}_\infty}^{\mathrm{B}^\pm,\mathrm{dR}^\pm} = \mathcal{O}(\mathrm{Isom}_{\mathcal{T}_\infty}^{\otimes}(\omega_{\mathrm{dR}}^{\pm}, \omega_{\mathrm{B}}^{\pm})) \qquad \text{and} \qquad \mathcal{P}_{\mathcal{T}_\infty}^{\overline{\mathrm{B}}^\pm,\mathrm{dR}^\pm} = \mathcal{O}(\mathrm{Isom}_{\mathcal{T}_\infty}^{\otimes}(\omega_{\mathrm{dR}}^{\pm}, \omega_{\overline{\mathrm{B}}}^{\pm})),$$

corresponding to all possible choices of signs. The four comparison isomorphisms define four period homomorphisms:

$$\mathrm{per} : \mathcal{P}_{\mathcal{T}_\infty}^{\mathrm{B}^+,\mathrm{dR}^+} \longrightarrow \mathbb{C} \qquad \text{and} \qquad \mathrm{per} : \mathcal{P}_{\mathcal{T}_\infty}^{\overline{\mathrm{B}}^+,\mathrm{dR}^+} \longrightarrow \mathbb{C}$$

and similarly with $+$ replaced with $-$. The four Frobenius maps define isomorphisms:

$$F_\infty \ : \ \mathcal{P}_{\mathcal{T}_\infty}^{\mathrm{B}^+,\bullet} \ \cong \ \mathcal{P}_{\mathcal{T}_\infty}^{\overline{\mathrm{B}}^-,\bullet} \quad \text{and} \quad F_\infty \ : \ \mathcal{P}_{\mathcal{T}_\infty}^{\mathrm{B}^-,\bullet} \ \cong \ \mathcal{P}_{\mathcal{T}_\infty}^{\overline{\mathrm{B}}^+,\bullet},$$

where $\bullet \in \{\mathrm{dR}^+, \mathrm{dR}^-\}$. By composing with the period homomorphism per, one obtains a full set of eight period homomorphisms for each of our period rings, for example, $\mathrm{per}\, F_\infty : \mathcal{P}_{\mathcal{T}_\infty}^{\mathrm{B}^+,\mathrm{dR}^-} \to \mathbb{C}$.

There are also four possible rings of de Rham periods. We shall mainly consider two of them:

$$\mathcal{P}_{\mathcal{T}_\infty}^{\mathrm{dR}^-,\mathrm{dR}^+} = \mathcal{O}(\mathrm{Isom}_{\mathcal{T}_\infty}^{\otimes}(\omega_{\mathrm{dR}}^{+}, \omega_{\mathrm{dR}}^{-})) \qquad \text{and} \qquad \mathcal{P}_{\mathcal{T}_\infty}^{\mathrm{dR}^+,\mathrm{dR}^-} = \mathcal{O}(\mathrm{Isom}_{\mathcal{T}_\infty}^{\otimes}(\omega_{\mathrm{dR}}^{-}, \omega_{\mathrm{dR}}^{+})).$$

Duality induces a canonical isomorphism $\mathcal{P}_{\mathcal{T}_\infty}^{\mathrm{dR}^-,\mathrm{dR}^+} \cong \mathcal{P}_{\mathcal{T}_\infty}^{\mathrm{dR}^+,\mathrm{dR}^-}$ which we shall not make use of here. Both of these rings admit a single-valued period map, which can be used to detect the nonvanishing of de Rham periods.

DEFINITION 3.7. There is a homomorphism *single-valued period*

$$\mathbf{s}^{-,+} : \mathcal{P}_{\mathcal{T}_\infty}^{\mathrm{dR}^-,\mathrm{dR}^+} \longrightarrow \mathbb{C} \otimes_{k^{\mathrm{dR}}} \overline{\mathbb{C}}$$

defined by the composite

$$V_{\mathrm{dR}}^+ \otimes_{k_{\mathrm{dR}}} \mathbb{C} \xrightarrow{c_{\mathrm{B},\mathrm{dR}}^+} V_{\mathrm{B}}^+ \otimes_{\mathbb{Q}_{\mathrm{B}}} \mathbb{C} \xrightarrow{F_\infty \otimes \mathrm{id}} V_{\overline{\mathrm{B}}}^- \otimes_{\mathbb{Q}_{\mathrm{B}}} \mathbb{C} \xrightarrow{(c_{\overline{\mathrm{B}},\mathrm{dR}}^-)^{-1}} V_{\mathrm{dR}}^- \otimes_{k_{\mathrm{dR}}} \overline{\mathbb{C}}\ .$$

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Since $\mathbb{C}$ and $\overline{\mathbb{C}}$ have different k_{dR}-structures, this map defines a point on the torsor of isomorphisms from ω^+_{dR} to ω^-_{dR} only after extending scalars to $\mathbb{C}\otimes_{k_{\mathrm{dR}}}\overline{\mathbb{C}}$. The single-valued map therefore sends $[V,f,\omega]^{\mathfrak{dr}}$ to $\langle f,(c^-_{\overline{\mathrm{B}},\mathrm{dR}})^{-1}F_\infty c^+_{\mathrm{B},\mathrm{dR}}\omega\rangle$.

However, in the case when $k^{\mathrm{dR}}\subset\mathbb{R}$, which is the case that we shall mostly consider, we can compose with the multiplication homomorphism $\mathbb{C}\otimes_{k^{\mathrm{dR}}}\overline{\mathbb{C}}\to\mathbb{C}$ to obtain a homomorphism

$$\mathbf{s}^{-,+}:\mathcal{P}^{\mathrm{dR}^-,\mathrm{dR}^+}_{\mathcal{T}_\infty}\longrightarrow\mathbb{C}.$$

There is a variant $\mathbf{s}^{+,-}$obtained by interchanging all $+$'s and $-$'s in the above. When it is clear from the context, we drop the superscripts and simply write $\mathbf{s}$.

Remark 3.8. Alternatively, one can view $\mathbf{s}^{-,+}$ as a morphism of the $(k_{\mathrm{dR}},k_{\mathrm{dR}})$-bimodule $\mathcal{P}^{\mathrm{dR}^-,\mathrm{dR}^+}_{\mathcal{T}_\infty}$ to $\mathbb{C}$, with the bimodule structure on the latter given by $(k_{\mathrm{dR}},\overline{k}_{\mathrm{dR}})$. In other words, for $\lambda_1,\lambda_2\in k_{\mathrm{dR}}$, one has $\mathbf{s}^{-,+}(\lambda_1\xi\lambda_2)=\lambda_1\mathbf{s}^{-,+}(\xi)\overline{\lambda_2}$. When k is real, these bimodule structures coincide and we obtain a genuine linear map.

The composition of torsors between fiber functors defines several coaction morphisms, including

$$\Delta:\mathcal{P}^{\mathrm{B}^+,\mathrm{dR}^+}_{\mathcal{T}_\infty}\longrightarrow\mathcal{P}^{\mathrm{B}^+,\mathrm{dR}^-}_{\mathcal{T}_\infty}\otimes_{k_{\mathrm{dR}}}\mathcal{P}^{\mathrm{dR}^-,\mathrm{dR}^+}_{\mathcal{T}_\infty},\tag{34}$$

and likewise with $+,-$ interchanged. The period homomorphisms defined earlier are compatible with composition of torsors between fiber functions. In particular, one has

$$\mathrm{per}\,(\xi)=(\mathrm{per}\,F_\infty\otimes\mathbf{s})\,(\Delta\xi)\qquad\text{for all}\quad\xi\in\mathcal{P}^{\mathrm{B}^+,\mathrm{dR}^+}_{\mathcal{T}_\infty}.\tag{35}$$

Definition 3.9. For s_i real and generic (13), let $k_{\mathrm{dR}}=\mathbb{Q}^{\mathrm{dR}}_{\underline{s}}$ and $\mathbb{Q}_{\mathrm{B}}=\mathbb{Q}^{\mathrm{B}}_{\underline{s}}$, and define an object of rank n in $\mathcal{T}_\infty$, denoted by $\widetilde{M}_\Sigma$, whose underlying vector spaces are given by

$$V^+_{\mathrm{dR}}=H^1_\varpi(X_\Sigma,\nabla_{+\underline{s}})\quad,\quad V^+_{\mathrm{B}}=H^1(\mathbb{C}\backslash\Sigma,\mathcal{L}_{+\underline{s}})\quad,\quad V^+_{\overline{\mathrm{B}}}=H^1(\mathbb{C}\backslash\overline{\Sigma},\mathcal{L}_{+\underline{s}})$$

and similarly with all $+$'s replaced with $-$'s. The Frobenius maps F_∞ are induced on Betti cohomology by complex conjugation conj* and its inverse. The comparison isomorphisms $c^+_{\mathrm{B},\mathrm{dR}},c^+_{\overline{\mathrm{B}},\mathrm{dR}}$ are defined by

$$\mathrm{comp}_{\mathrm{B},\varpi}(+\underline{s}):H^1_\varpi(X_\Sigma,\nabla_{+\underline{s}})\otimes_{k_{\mathrm{dR}}}\mathbb{C}\longrightarrow H^1(\mathbb{C}\backslash\Sigma,\mathcal{L}_{+\underline{s}})\otimes_{\mathbb{Q}_{\mathrm{B}}}\mathbb{C},$$

$$\mathrm{comp}_{\overline{\mathrm{B}},\varpi}(+\underline{s}):H^1_\varpi(X_\Sigma,\nabla_{+\underline{s}})\otimes_{k^{\mathrm{dR}}}\overline{\mathbb{C}}\longrightarrow H^1(\mathbb{C}\backslash\overline{\Sigma},\mathcal{L}_{+\underline{s}})\otimes_{\mathbb{Q}_{\mathrm{B}}}\mathbb{C},$$

and $c^-_{\mathrm{B},\mathrm{dR}},c^-_{\overline{\mathrm{B}},\mathrm{dR}}$are similarly defined by replacing all $+$'s with $-$'s. Since $k_{\mathrm{dR}}=\mathbb{Q}^{\mathrm{dR}}_{\underline{s}}\subset\mathbb{R}$, the complex conjugate $\overline{\mathbb{C}}$ in the left-hand side of the previous equation can, in fact, be replaced with $\mathbb{C}$.

The image of $\widetilde{M}_\Sigma$ under the functor $\mathcal{T}_\infty\to\mathcal{T}$ which forgets all data except for $(V^+_{\mathrm{B}},V^+_{\mathrm{dR}},c_{\mathrm{B},\mathrm{dR}})$ is the object M_Σ, which was defined in Definition 3.1.

Let us define an $(n\times n)$ matrix

$$\widetilde{L}^{\mathfrak{m}}_\Sigma\quad\in\quad M_{n\times n}(\mathcal{P}^{\mathrm{B}^+,\mathrm{dR}^+}_{\mathcal{T}_\infty})$$

whose entries are

$$\Big(\widetilde{L}^{\mathfrak{m}}_\Sigma\Big)_{ij}=\Big[\widetilde{M}_\Sigma\ ,\ \delta_i\otimes x^{s_0}\prod_{k=1}^{n}(1-x\sigma_k^{-1})^{s_k}\ ,\ -s_j\omega_j\Big]^{\mathfrak{m}},$$

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where the Betti homology class lies in $H^1_{\mathrm{B}}(X_\Sigma, \mathcal{L}_{\underline{s}})^\vee = H_1(\mathbb{C}\setminus\Sigma, \mathcal{L}^\vee_{\underline{s}})$ and the de Rham class lies in $H^1_{\varpi}(X_\Sigma, \nabla_{\underline{s}})$. We will use the notation $\widetilde{M}_\Sigma(\underline{s})$ and $\widetilde{L}^{\mathfrak{m}}_\Sigma(\underline{s})$ when we want to make the dependence on $\underline{s}$ explicit. The image of $\widetilde{L}^{\mathfrak{m}}_\Sigma$ under the natural map $\mathcal{P}^{\mathrm{B}^+,\mathrm{dR}^+}_{\mathcal{T}_\infty} \to \mathcal{P}^{\mathfrak{m}}_{\mathcal{T}}$ is $L^{\mathfrak{m}}_\Sigma$, and its period is the Lauricella matrix

$$\operatorname{per} \widetilde{L}^{\mathfrak{m}}_\Sigma = L_\Sigma. \tag{36}$$

3.3 de Rham motivic version and single-valued periods

Now, let us define a de Rham motivic Lauricella function

$$\left(\widetilde{L}^{\mathfrak{dr}}_\Sigma\right)_{ij} = \left[\widetilde{M}_\Sigma \ , \ \nu_i \ , \ -s_j\omega_j\right]^{\mathfrak{dr}} \quad \in \quad \mathcal{P}^{\mathrm{dR}^-,\mathrm{dR}^+}_{\mathcal{T}_\infty},$$

where the de Rham class ν_i is to be viewed in $H^1_{\varpi}(X_\Sigma, \nabla_{\underline{s}}) = H^1_{\varpi}(X_\Sigma, \nabla_{-\underline{s}})^\vee$ and the second is to be viewed in $H^1_{\varpi}(X_\Sigma, \nabla_{\underline{s}})$. Note that this differs from the earlier definition of the de Rham Lauricella matrix in the ring of de Rham periods of the category $\mathcal{T}$ because the de Rham classes reside in different cohomology groups. This is required so that we may speak of its single-valued period. By Corollary 2.12, the matrix of single-valued periods consists of the single-valued Lauricella hypergeometric functions (3):

$$\mathbf{s}(\widetilde{L}^{\mathfrak{dr}}_\Sigma) = L^{\mathbf{s}}_\Sigma,$$

whenever all the s_i are real and generic and satisfy $s_i > 0$ for all $0 \le i \le n$, and $s_0 + \cdots + s_n < 1/2$.

3.4 Coaction

The coaction (34), applied to the motivic Lauricella function, will give rise to a third but closely related quantity, given by the matrix

$$\left(\widetilde{L}^{\mathfrak{m},-}_\Sigma(\underline{s})\right)_{ij} = \left[\widetilde{M}_\Sigma(\underline{s}) \ , \ \delta_i \otimes x^{s_0} \prod_{k=1}^{n} (1 - x\sigma_k^{-1})^{s_k} \ , \ s_j\omega_j\right]^{\mathfrak{m}} \quad \in \quad \mathcal{P}^{\mathrm{B}^+,\mathrm{dR}^-}_{\mathcal{T}_\infty},$$

where the de Rham class $s_j\omega_j$ is in $H^1_{\varpi}(X_\Sigma, \nabla_{-\underline{s}})$. It is the image under F_∞ of

$$\left(\widetilde{L}^{\mathfrak{m}}_{\overline{\Sigma}}(-\underline{s})\right)_{ij} = \left[\widetilde{M}_\Sigma(\underline{s}) \ , \ \overline{\delta}_i \otimes x^{-s_0} \prod_{k=1}^{n} (1 - x\sigma_k^{-1})^{-s_k} \ , \ s_j\omega_j\right]^{\mathfrak{m}} \quad \in \quad \mathcal{P}^{\overline{\mathrm{B}}^-,\mathrm{dR}^-}_{\mathcal{T}_\infty},$$

where the Betti class is viewed in $H_1(\mathbb{C}\setminus\overline{\Sigma}, \mathcal{L}^\vee_{-\underline{s}})$ (see Remark 2.7) and the de Rham class $s_j\omega_j$ is viewed in $H^1_{\varpi}(X_\Sigma, \nabla_{-\underline{s}})$ as before. More precisely, we have

$$\widetilde{L}^{\mathfrak{m},-}_\Sigma(\underline{s}) = F_\infty \widetilde{L}^{\mathfrak{m}}_{\overline{\Sigma}}(-\underline{s}),$$

and hence, by definition of the period homomorphism $\operatorname{per} F_\infty : \mathcal{P}^{\mathrm{B}^+,\mathrm{dR}^-}_{\mathcal{T}_\infty} \to \mathbb{C}$, the period is

$$\operatorname{per} F_\infty \left(\widetilde{L}^{\mathfrak{m},-}_\Sigma(\underline{s})\right) = L_{\overline{\Sigma}}(-\underline{s}).$$

COROLLARY 3.10. *The coaction (34) satisfies*

$$\Delta \widetilde{L}^{\mathfrak{m}}_\Sigma(\underline{s}) = \widetilde{L}^{\mathfrak{m},-}_\Sigma(\underline{s}) \otimes \widetilde{L}^{\mathfrak{dr}}_\Sigma(\underline{s}).$$

Proof. This is an immediate consequence of the formula for the coaction on matrix coefficients in a Tannakian category. □

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Equation (35), applied to the matrix $\widetilde{L}^{\mathfrak{m}}_{\Sigma}(\underline{s})$, reduces to the equation

$$L_{\Sigma}(\underline{s}) = L_{\overline{\Sigma}}(-\underline{s})\, L^{\mathbf{s}}_{\Sigma}(\underline{s}),$$

which is another way of writing formula (32). Consider the following refinement of the period map:

$$\mathcal{P}^{\mathrm{B}^+,\mathrm{dR}^+}_{\mathcal{T}_\infty} \xrightarrow{\quad\Delta\quad} \mathcal{P}^{\mathrm{B}^+,\mathrm{dR}^-}_{\mathcal{T}_\infty} \otimes_{k_{\mathrm{dR}}} \mathcal{P}^{\mathrm{dR}^-,\mathrm{dR}^+}_{\mathcal{T}_\infty} \xrightarrow{\mathrm{per}\, F_\infty \otimes \mathbf{s}^{-,+}} \mathbb{C} \otimes_{\mathbb{Q}} \mathbb{C}\ .$$

Indeed, by composing it with the multiplication map $\mathbb{C}\otimes_{\mathbb{Q}}\mathbb{C}\to\mathbb{C}$, one recovers the usual period map, by (35). Then, under this map, the elements $(\widetilde{L}^{\mathfrak{m}}_{\Sigma}(\underline{s}))_{ij}$, for $1\leq i,j\leq n$, map to

$$\sum_{\ell=1}^{n}\left(s_\ell\int_{\delta_i} x^{-s_0}\prod_{k=1}^{n}(1-x\overline{\sigma_k}^{-1})^{-s_k}\frac{dx}{x-\overline{\sigma_\ell}}\right)\otimes \\ \frac{s_j}{2\pi i}\left(\iint_{\mathbb{C}}|z|^{2s_0}\prod_{k=1}^{n}|1-z\sigma_k^{-1}|^{2s_k}\left(\frac{d\overline{z}}{\overline{z}-\overline{\sigma_\ell}}-\frac{d\overline{z}}{\overline{z}}\right)\wedge\frac{dz}{z-\sigma_j}\right). \tag{37}$$

The point of this formula is that both sides of the tensor product admit a Taylor expansion in the s_i, which is the subject of the next section.

3.5 Variant

We had to assume that the parameters s_i are real in order to obtain a Tannakian interpretation of the single-valued period homomorphism. This was so that the subfield $\overline{k}(s_0,\ldots,s_n)\subset\mathbb{C}$ is isomorphic to $k(s_0,\ldots,s_n)$, which ensures that there is a comparison map for de Rham cohomology associated with the complex conjugate embedding of k:

$$H^1_{\mathrm{dR}}(X_\Sigma,\nabla_{\underline{s}})\otimes_{k^{\mathrm{dR}}_{\underline{s}}}\overline{\mathbb{C}}\longrightarrow H^1_{\mathrm{B}}(\mathbb{C}\backslash\overline{\Sigma},\mathcal{L}_{\underline{s}})\otimes_{\mathbb{Q}^{\mathrm{dR}}_{\underline{s}}}\mathbb{C}.$$

However, such an assumption is unnatural, for instance, because formulae such as (5) are true for all $s_i\in\mathbb{C}$ for which they make sense. A way to remedy this would be to treat the s_i as formal parameters and work with modules over the polynomial ring $k[s_0,\ldots,s_n]$. This would also be needed to build a bridge between the framework of cohomology with coefficients that we discussed in this section and the Taylor expansions that we consider in the rest of this article.

§4. Laurent series expansion of periods of cohomology with coefficients

The Lauricella functions (1) are not a priori defined for $s_0,\ldots,s_n$ at the origin. We show using a renormalization procedure that they extend to a neighborhood of the origin and admit a Taylor expansion there. We prove a similar statement for their single-valued versions (3). The reason for the (ab)use of the word '*renormalization*' is explained in [BD2].

4.1 Renormalization of line integrals

Let $\Sigma=\{\sigma_0,\ldots,\sigma_n\}$ be distinct complex numbers with $\sigma_0=0$. We fix an index $i\in\{1,\ldots,n\}$ and a locally finite path δ_i in $\mathbb{C}\setminus\Sigma$ from 0 to σ_i. More precisely, $\delta_i:(0,1)\to\mathbb{C}\setminus\Sigma$ is a smooth path that can be extended to a smooth path $\overline{\delta_i}:[0,1]\to\mathbb{C}$ satisfying $\overline{\delta_i}(0)=0$ and $\overline{\delta_i}(1)=\sigma_i$. As in all this paper, we assume that δ_i does not wind infinitely around 0 or σ_i (Remark 2.3).

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Let ω be a global meromorphic form on $\mathbb{P}^1(\mathbb{C})$ with logarithmic poles at $\Sigma\cup\{\infty\}$. We define

$$\Omega = x^{s_0}\prod_{k=1}^{n}(1-x\sigma_k^{-1})^{s_k}\,\omega$$

and view it as a multivalued 1-form which depends holomorphically on the parameters $\underline{s}$, and that we wish to integrate along δ_i.

REMARK 4.1. Since we shall only consider the integral of Ω along δ_i, it suffices to define the pullback $\delta_i^*\Omega$, which is a 1-form on the contractible space $(0,1)$. It depends only on a determination of $\log(\sigma_i)$, which we choose once and for all. Hence, $\sigma_i^{s_0}=\exp(s_0\log(\sigma_i))$ is defined. This fixes a branch of $\delta_i^*\Omega$ as follows. Close to 1 (i.e., for x near σ_i), the function x^{s_0} is prescribed and so $\delta_i^*x^{s_0}$ is analytically continued to the open interval $(0,1)$. Close to 0 (i.e., for x near 0), the function $\log(1-x\sigma_k^{-1})$ is analytic and the terms $(1-x\sigma_k^{-1})^{s_k}=\exp\big(s_k\log(1-x\sigma_k^{-1})\big)$ are well-defined, and so their pullbacks along δ are analytically continued to $(0,1)$.

REMARK 4.2. Let us fix a parameter $\underline{s}=(s_0,\dots,s_n)\in\mathbb{C}^{n+1}$. The recipe given in Remark 4.1 gives rise to a determination of $x^{s_0}\prod_{k=1}^{n}(1-x\sigma_k^{-1})^{s_k}$ along the path δ_i, and thus a class in the cohomology group $H_1^{\mathrm{lf}}(X_\Sigma(\mathbb{C}),\mathcal{L}_{\underline{s}}^\vee)$ which we denote $\delta_{i,\underline{s}}$.

Without any assumption on the residues of ω at 0 and σ_i, the integral of Ω along δ_i defines a holomorphic function of the parameters $\underline{s}$ in the domain $\{\mathrm{Re}(s_0),\mathrm{Re}(s_i)>0\}$.

DEFINITION 4.3. Following [BD2], let us define the renormalized version of Ω with respect to the points $\{0,\sigma_i\}$to be

$$\Omega^{\mathrm{ren}_i}=\Omega-\mathrm{Res}_0(\omega)\,x^{s_0}\frac{dx}{x}-\mathrm{Res}_{\sigma_i}(\omega)\Big(\sigma_i^{s_0}\prod_{k\neq i}(1-\sigma_i\sigma_k^{-1})^{s_k}\Big)(1-x\sigma_i^{-1})^{s_i}\frac{dx}{x-\sigma_i}\cdot$$

PROPOSITION 4.4. *We have, for every $\underline{s}$ in the domain $\{\mathrm{Re}(s_0),\mathrm{Re}(s_i)>0\}$, the equality*

$$\int_{\delta_i}\Omega=\mathrm{Res}_0(\omega)\frac{\sigma_i^{s_0}}{s_0}-\mathrm{Res}_{\sigma_i}(\omega)\frac{\sigma_i^{s_0}\prod_{k\neq i}(1-\sigma_i\sigma_k^{-1})^{s_k}}{s_i}+\int_{\delta_i}\Omega^{\mathrm{ren}_i}.\tag{38}$$

The integral of Ω^{ren_i} along δ_i defines a holomorphic function of the parameters $\underline{s}$ in the domain $\{\mathrm{Re}(s_0),\mathrm{Re}(s_i)>-1\}$.

Proof. The equality follows from the computations:

$$\int_{\delta_i}x^{s_0}\frac{dx}{x}=\frac{\sigma_i^{s_0}}{s_0}\qquad\text{and}\qquad\int_{\delta_i}(1-x\sigma_i^{-1})^{s_i}\frac{dx}{x-\sigma_i}=-\frac{1}{s_i}\cdot$$

Since $\omega-\mathrm{Res}_0(\omega)\frac{dx}{x}$ does not have a pole at 0, the singularities of Ω^{ren_i} at 0 are at worst of the type $x^{s_0}dx$ and therefore integrable for $\mathrm{Re}(s_0)>-1$. Likewise, its singularities at σ_i are at worst of the type $(1-x\sigma_i^{-1})^{s_i}dx$ and are integrable for $\mathrm{Re}(s_i)>-1$. □

Since $\underline{s}=0$ is in the domain of convergence of the integral of Ω^{ren_i} along δ_i, we see that (38) yields a Laurent series expansion at $\underline{s}=0$ for the integral of Ω along δ_i. We record the following special case, which shows that the Lauricella functions $(L_\Sigma)_{ij}$ admit Taylor expansions around $\underline{s}=0$. Here, the integrand of $(L_\Sigma)_{ij}$ is understood according to the conventions of Remark 4.1.

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PROPOSITION 4.5. *We have the following equality:*

$$(L_\Sigma)_{ij} = \mathbf{1}_{i=j}\Big(\sigma_i^{s_0}\prod_{k\neq i}(1-\sigma_i\sigma_k^{-1})^{s_k}\Big) - s_j\int_{\delta_i}\Omega_j^{\mathrm{ren}_i},$$

where

$$\Omega_j^{\mathrm{ren}_i} = \Big(x^{s_0}\prod_{k\neq i}(1-x\sigma_k^{-1})^{s_k} - \mathbf{1}_{i=j}\,\sigma_i^{s_0}\prod_{k\neq i}(1-\sigma_i\sigma_k^{-1})^{s_k}\Big)(1-x\sigma_i^{-1})^{s_i}\frac{dx}{x-\sigma_j},$$

and the integral on the right-hand side defines a holomorphic function of the parameters $\underline{s}$ *in the domain* $\{\mathrm{Re}(s_0),\mathrm{Re}(s_i)>-1\}$.

Proof. This follows from applying Proposition 4.4 to the case $\omega=\omega_j=d\log(x-\sigma_j)$ and multiplying (38) by $-s_j$. □

4.2 Renormalization of complex volume integrals

Let $\Sigma=\{\sigma_0,\ldots,\sigma_n\}$ be distinct complex numbers with $\sigma_0=0$. Let ω be a global meromorphic form on $\mathbb{P}^1(\mathbb{C})$ with logarithmic poles at $\Sigma\cup\{\infty\}$. We define

$$\Omega^{\mathbf{s}} \;=\; |z|^{2s_0}\prod_{k=1}^{n}|1-z\sigma_k^{-1}|^{2s_k}\,\omega$$

and view it as a smooth 1-form on $\mathbb{P}^1(\mathbb{C})\setminus(\Sigma\cup\{\infty\})$, which depends holomorphically on the parameters $\underline{s}$. We fix an index $i\in\{1,\ldots,n\}$ and consider the integral

$$-\frac{1}{2\pi i}\iint_{\mathbb{C}}\left(\frac{d\overline{z}}{\overline{z}-\overline{\sigma_i}}-\frac{d\overline{z}}{\overline{z}}\right)\wedge\Omega^{\mathbf{s}}.$$

Without any assumption on the residues of ω, this integral defines a holomorphic function of the parameters $\underline{s}$ in the domain

$$D=\{\mathrm{Re}(s_0),\mathrm{Re}(s_i)>0\,,\ \mathrm{Re}(s_k)>-1/2\ \ (k\notin\{0,i\})\,,\ \mathrm{Re}(s_0)+\cdots+\mathrm{Re}(s_n)<1/2\}.$$

One can check this by passing to local coordinates around every point of $\Sigma\cup\{\infty\}$.

DEFINITION 4.6. Following [BD2], let us define the renormalized version of $\Omega^{\mathbf{s}}$ with respect to the points $\{0,\sigma_i\}$to be

$$\Omega^{\mathbf{s},\mathrm{ren}_i}=\Omega^s-\mathrm{Res}_0(\omega)\,|z|^{2s_0}\frac{dz}{z}-\mathrm{Res}_{\sigma_i}(\omega)\Big(|\sigma_i|^{2s_0}\prod_{k\neq i}|1-\sigma_i\sigma_k^{-1}|^{2s_k}\Big)\,|1-z\sigma_i^{-1}|^{2s_i}\frac{dz}{z-\sigma_i}\ .$$

The following lemma will be used several times in the sequel. It is a single-valued analog of the identity $\int_0^\sigma z^s\frac{dz}{z}=\frac{\sigma^s}{s}$.

LEMMA 4.7. *For any* $\sigma,s\in\mathbb{C}$ *with* $\mathrm{Re}(s)>0$, *we have*

$$-\frac{1}{2\pi i}\iint_{\mathbb{C}}\left(\frac{d\overline{z}}{\overline{z}-\overline{\sigma}}-\frac{d\overline{z}}{\overline{z}}\right)\wedge|z|^{2s}\frac{dz}{z}=\frac{|\sigma|^{2s}}{s}.$$

Proof. One verifies in local polar coordinates that the integral converges. Let

$$F=-\frac{1}{s}|z|^{2s}\left(\frac{d\overline{z}}{\overline{z}-\overline{\sigma}}-\frac{d\overline{z}}{\overline{z}}\right).$$

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Its total derivative is the integrand on the left-hand side. By Stokes's formula applied to the complement of three disks in $\mathbb{P}^1(\mathbb{C})$ centered at 0, ∞, σ, it suffices to compute

$$-\frac{1}{2\pi i}\int_{D_\varepsilon} F \;=\; \frac{1}{s}\frac{1}{2\pi i}\int_{D_\varepsilon}|z|^{2s}\left(\frac{d\overline{z}}{\overline{z}-\overline{\sigma}}-\frac{d\overline{z}}{\overline{z}}\right),$$

where D_ε is the union of three circles of radius ε winding negatively around 0, ∞, and σ. When ε goes to zero, the integral around the first and the second vanish, and the integral around the third gives $2\pi i\,|\sigma|^{2s}$. □

PROPOSITION 4.8. *We have, for every $\underline{s}$ in the domain D, the equality*

$$-\frac{1}{2\pi i}\iint_{\mathbb{C}}\left(\frac{d\overline{z}}{\overline{z}-\overline{\sigma_i}}-\frac{d\overline{z}}{\overline{z}}\right)\wedge\Omega^{\mathbf{s}} = \operatorname{Res}_0(\omega)\frac{|\sigma_i|^{2s_0}}{s_0}-\operatorname{Res}_{\sigma_i}(\omega)\frac{|\sigma_i|^{2s_0}\prod_{k\neq i}|1-\sigma_i\sigma_k^{-1}|^{2s_k}}{s_i}-\frac{1}{2\pi i}\iint_{\mathbb{C}}\left(\frac{d\overline{z}}{\overline{z}-\overline{\sigma_i}}-\frac{d\overline{z}}{\overline{z}}\right)\wedge\Omega^{\mathbf{s},\mathrm{ren}_i}. \tag{39}$$

The integral on the right-hand side defines a holomorphic function of the parameters $\underline{s}$ in the domain

$$D'=\{\operatorname{Re}(s_k)>-1/2\ \ (k\in\{0,\dots,n\})\,,\ \operatorname{Re}(s_0)+\cdots+\operatorname{Re}(s_n)<1/2\}. \tag{40}$$

Proof. The equality follows from the computations

$$-\frac{1}{2\pi i}\iint_{\mathbb{C}}\left(\frac{d\overline{z}}{\overline{z}-\overline{\sigma_i}}-\frac{d\overline{z}}{\overline{z}}\right)\wedge|z|^{2s_0}\frac{dz}{z}=\frac{|\sigma_i|^{2s_0}}{s_0},$$

and

$$-\frac{1}{2\pi i}\iint_{\mathbb{C}}\left(\frac{d\overline{z}}{\overline{z}-\overline{\sigma_i}}-\frac{d\overline{z}}{\overline{z}}\right)\wedge|1-z\sigma_i^{-1}|^{2s_i}\frac{dz}{z-\sigma_i}=-\frac{1}{s_i},$$

which follows from Lemma 4.7 (for the second, apply the change of variables $z\leftrightarrow\sigma-z$ and multiply by $|\sigma|^{-2s}$). Since $\omega-\operatorname{Res}_0(\omega)\frac{dx}{x}$ does not have a pole at 0, the singularities of $\Omega^{\mathbf{s},\mathrm{ren}_i}$ at 0 are at worst of the type $|z|^{2s_0}dz$ and therefore

$$\left(\frac{d\overline{z}}{\overline{z}-\overline{\sigma_i}}-\frac{d\overline{z}}{\overline{z}}\right)\wedge\Omega^{\mathbf{s},\mathrm{ren}_i}$$

is integrable around zero for $\operatorname{Re}(s_0)>-1/2$. Likewise, it is integrable around σ_i for $\operatorname{Re}(s_i)>-1/2$. □

Since $\underline{s}=0$ lies in the domain D', this proposition provides a Laurent series expansion at $\underline{s}=0$ for the integral on the left-hand side. We record the following special case which shows that the single-valued Lauricella functions (3) admit Taylor expansions around $\underline{s}=0$.

PROPOSITION 4.9. *We have the following equality:*

$$(L^{\mathbf{s}}_{\Sigma})_{ij}=\mathbf{1}_{i=j}\left(|\sigma_i|^{2s_0}\prod_{k\neq i}|1-\sigma_i\sigma_k^{-1}|^{2s_k}\right)+\frac{s_j}{2\pi i}\iint_{\mathbb{C}}\left(\frac{d\overline{z}}{\overline{z}-\overline{\sigma_i}}-\frac{d\overline{z}}{\overline{z}}\right)\wedge\Omega_j^{\mathbf{s},\mathrm{ren}_i},$$

where

$$\Omega_j^{\mathbf{s},\mathrm{ren}_i}=\left(|z|^{2s_0}\prod_{k\neq i}|1-z\sigma_k^{-1}|^{2s_k}-\mathbf{1}_{i=j}\,|\sigma_i|^{2s_0}\prod_{k\neq i}|1-\sigma_i\sigma_k^{-1}|^{2s_k}\right)|1-z\sigma_i^{-1}|^{2s_i}\frac{dz}{z-\sigma_j}\ .$$

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The integral on the right-hand side defines a holomorphic function of the parameters $\underline{s}$ in the domain D' described in (40).

Proof. This follows from applying Proposition 4.8 to the case $\omega = \omega_j = d\log(x - \sigma_j)$ and multiplying (39) by $-s_j$. □

§5. Notations relating to motivic fundamental groups

In this, the second part of the paper, we of the paper, we study (1) from the point of view of the motivic fundamental group of the punctured Riemann sphere. This requires a number of notations and background, mostly from [DG], [B3], which we recall here. Let $k \subset \mathbb{C}$ be a number field.

5.1 Categorical and Tannakian

1. Let $\mathcal{MT}(k)$ denote the category of mixed Tate motives over k. One can replace $\mathcal{MT}(k)$ with a category of Betti and de Rham realizations, with essentially no change to our arguments. The category $\mathcal{MT}(k)$ has a canonical fiber functor $\varpi : \mathcal{MT}(k) \to \mathrm{Vec}_{\mathbb{Q}}$. Let us set

$$G^{\varpi}_{\mathcal{MT}(k)} = \mathrm{Aut}^{\otimes}_{\varpi}\, \mathcal{MT}(k).$$

   It is an affine group scheme over $\mathbb{Q}$. Let us denote by $\omega_{\mathrm{B}} : \mathcal{MT}(k) \to \mathrm{Vec}_{\mathbb{Q}}$ the Betti realization functor with respect to the given embedding $k \subset \mathbb{C}$. The de Rham realization functor $\omega_{\mathrm{dR}} : \mathcal{MT}(k) \to \mathrm{Vec}_k$ is obtained from ϖ by extending scalars: $\omega_{\mathrm{dR}} = \varpi \otimes_{\mathbb{Q}} k$.
2. Let $\mathcal{P}^{\mathfrak{m}} = \mathcal{O}(\mathrm{Isom}^{\otimes}_{\mathcal{MT}(k)}(\varpi, \omega_{\mathrm{B}}))$ denote the $\mathbb{Q}$-algebra of motivic periods on $\mathcal{MT}(k)$. Let

$$\mathcal{P}^{\varpi} = \mathcal{O}(G^{\varpi}_{\mathcal{MT}(k)})$$

   denote the $\mathbb{Q}$-algebra of (canonical, i.e., (ϖ, ϖ))) de Rham motivic periods. The latter is a graded Hopf algebra, and the former is a graded algebra comodule over it. Denote the corresponding motivic coaction by

$$\Delta : \mathcal{P}^{\mathfrak{m}} \longrightarrow \mathcal{P}^{\mathfrak{m}} \otimes_{\mathbb{Q}} \mathcal{P}^{\varpi}.$$

   We let $\mathrm{per} : \mathcal{P}^{\mathfrak{m}} \to \mathbb{C}$ denote the period homomorphism and let $\mathbf{s} : \mathcal{P}^{\varpi} \to \mathbb{C}$ denote the single-valued period homomorphism (see [BD1, §2.6]).
3. Let $\mathbb{L}^{\mathfrak{m}} = [\mathbb{Q}(-1), 1^{\vee}_{\mathrm{B}}, 1_{\varpi}]^{\mathfrak{m}}$, respectively, $\mathbb{L}^{\varpi} = [\mathbb{Q}(-1), 1^{\vee}_{\varpi}, 1_{\varpi}]^{\varpi}$, denote the Lefschetz motivic period, whose period is $2\pi i$, respectively, its (canonical) de Rham version.
4. Let $\mathcal{P}^{\mathfrak{m},+} \subset \mathcal{P}^{\mathfrak{m}}$ denote the subspace of effective motivic periods. It is spanned by the motivic periods of objects $M \in \mathcal{MT}(k)$ with $W_{-1}M = 0$. It is a nonnegatively graded subalgebra and sub-$\mathcal{P}^{\varpi}$-comodule of $\mathcal{P}^{\mathfrak{m}}$, and contains the Lefschetz motivic period $\mathbb{L}^{\mathfrak{m}}$. There is a canonical projection homomorphism

$$\pi^{\mathfrak{m},+}_{\varpi} : \mathcal{P}^{\mathfrak{m},+} \longrightarrow \mathcal{P}^{\varpi},$$

   which, in particular, sends $\mathbb{L}^{\mathfrak{m}}$ to zero. It can be defined by composing the coaction Δ with projection onto the weight-graded zero piece $W_0\mathcal{P}^{\mathfrak{m},+} \cong \mathbb{Q}$.

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5.2 Geometric

Let $\Sigma = \{\sigma_0, \sigma_1, \ldots, \sigma_n\} \subset \mathbb{A}^1(k)$ be distinct points with $\sigma_0 = 0$. Recall that $X_\Sigma = \mathbb{A}^1_k \backslash \Sigma$.

1. Let $t_0 \in T_0\mathbb{A}^1_k$ denote the tangent vector 1 at 0. For each $i \geq 1$, let
$$t_{\sigma_i} \quad \in \quad T_{\sigma_i}\mathbb{A}^1_k$$
be the tangent vector σ_i based at the point σ_i. Let $\pi_1^{\rm top}(X_\Sigma(\mathbb{C}), t_i, -t_j)$ denote the fundamental torsor of homotopy classes of paths between the tangential basepoints t_i and $-t_j$. Concretely, an element of this set is represented by a (piecewise) smooth path $\gamma : (0,1) \to \mathbb{C} \setminus \Sigma$ that can be extended to a path $\overline{\gamma} : [0,1] \to \mathbb{C}$ which is smooth at 0 and 1 and satisfies $\overline{\gamma}(0) = \sigma_i$ and $\overline{\gamma}(1) = \sigma_j$, with prescribed velocities $\overline{\gamma}'(0) = t_i$ and $\overline{\gamma}'(1) = t_j$.
2. For all $0 \leq i, j \leq n$, denote by
$${}_i\Pi^\bullet_j = \pi_1^\bullet(X_\Sigma, t_i, -t_j) \qquad \text{where} \quad \bullet \in \{\mathrm{B}, \varpi, \mathrm{mot}\}$$
the Betti, (canonical) de Rham, or motivic fundamental torsor of paths from the tangential basepoint t_i at σ_i, to $-t_j$ at σ_j. The versions ${}_i\Pi^{\rm B}_j$ and ${}_i\Pi^{\varpi}_j$ are affine $\mathbb{Q}$-schemes; the version ${}_i\Pi^{\rm mot}_j$ is an affine scheme in the category $\mathcal{MT}(k)$, which simply means that its affine ring is isomorphic to an algebra ind-object $\mathcal{O}({}_i\Pi^{\rm mot}_j)$ in $\mathcal{MT}(k)$, whose (B, resp. ϖ) realizations are $\mathcal{O}({}_i\Pi^{\rm B}_j)$ and $\mathcal{O}({}_i\Pi^{\varpi}_j)$. There is a groupoid structure ${}_a\Pi^\bullet_b \times {}_b\Pi^\bullet_c \longrightarrow {}_a\Pi^\bullet_c$, for all a, b, c in the set of our tangential basepoints (composition of paths). There are maps
$$\gamma \mapsto \gamma^{\rm B} \ : \ \pi_1^{\rm top}(X_\Sigma(\mathbb{C}), t_i, -t_j) \longrightarrow {}_i\Pi^{\rm B}_j(\mathbb{Q}),$$
which are Zariski-dense and compatible with the groupoid structure. (They identify ${}_i\Pi^{\rm B}_j$ as the Malčev, or pro-unipotent, completion of $\pi_1^{\rm top}(X_\Sigma(\mathbb{C}), t_i, -t_j)$.)
3. The scheme ${}_i\Pi^{\varpi}_j$ does not depend on i, j, although the action of the (canonical) motivic Galois group $G^{\varpi}_{\mathcal{MT}(k)}$ upon it does depend on i, j. On X_Σ, we considered the logarithmic 1-forms ω_i for $i = 0, \ldots, n$ (14). Since they have residue 0 or ± 1 at points of Σ, they generate the canonical $\mathbb{Q}$-structure (or ϖ-structure) on the de Rham realization of $H^1(X_\Sigma) \in \mathrm{Ob}(\mathcal{MT}(k))$, which we shall denote simply by $H^1_\varpi(X_\Sigma)$. It is the $\mathbb{Q}$-vector space spanned by the classes ω_i. The affine ring of the de Rham canonical torsor of paths is
$$\mathcal{O}({}_i\Pi^{\varpi}_j) \cong \bigoplus_{n \geq 0} H^1_\varpi(X_\Sigma)^{\otimes n}.$$
It is isomorphic to the graded tensor coalgebra on $H^1_\varpi(X_\Sigma)$, equipped with the shuffle product ш and deconcatenation coproduct. For any commutative unital $\mathbb{Q}$-algebra R, the R-points of ${}_i\Pi^{\varpi}_j$,
$${}_i\Pi^{\varpi}_j(R) \ \subset \ R\langle\langle e_0, \ldots, e_n \rangle\rangle,$$
are the set of group-like formal power series with respect to the continuous coproduct for which the e_i are primitive. They are formal power series
$$S = \sum_{w \in \{e_0, e_1, \ldots, e_n\}^\times} S(w)\, w \quad \in \quad R\langle\langle e_0, \ldots, e_n \rangle\rangle,$$

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where the linear extension of the map $w \mapsto S(w)$ is a homomorphism with respect to the shuffle product. The letters e_i, for $1 \leq i \leq n$, are dual to the ω_i.

4. For all $0 \leq i \leq n$, let ${}_0 1_i \in {}_0\Pi_i^{\varpi}(\mathbb{Q})$ denote the canonical ϖ-path. It is defined by the augmentation map $\mathcal{O}({}_0\Pi_i^{\varpi}) \to \mathbb{Q}$ onto the degree zero component (or by quotienting by the Hodge filtration F^1). It is the formal power series $1 \in \mathbb{Q}\langle\langle e_0, \ldots, e_n \rangle\rangle$ consisting only of the empty word.
5. Since the motivic fundamental torsor of paths is the spectrum of an ind-object in the category $\mathcal{MT}(k)$, there is a canonical universal comparison isomorphism of schemes (see [B3, §4.1])

$$\mathrm{comp}^{\mathfrak{m}}_{\mathrm{B},\varpi} \quad : \quad {}_i\Pi_j^{\mathrm{B}} \times_{\mathbb{Q}} \mathcal{P}^{\mathfrak{m}} \xrightarrow{\sim} {}_i\Pi_j^{\varpi} \times_{\mathbb{Q}} \mathcal{P}^{\mathfrak{m}},$$

for all $0 \leq i, j \leq n$, compatible with the groupoid structure. Composing $\mathrm{comp}^{\mathfrak{m}}_{\mathrm{B},\varpi}$ with the period homomorphism gives back the canonical comparison isomorphism whose coefficients are regularized iterated integrals:

$$\mathrm{comp}_{\mathrm{B},\varpi} \quad : \quad {}_i\Pi_j^{\mathrm{B}} \times_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} {}_i\Pi_j^{\varpi} \times_{\mathbb{Q}} \mathbb{C}.$$

§6. Generalized associators and their beta quotients

We now study the generalized associators $\mathcal{Z}^i$, which are formal power series in non-commuting variables $e_0, \ldots, e_n$ whose coefficients are regularized iterated integrals on X_Σ. We compute their beta quotients, which are formal power series in commuting variables $s_0, \ldots, s_n$, and use them to define a matrix FL_Σ of power series which we prove to be the matrix of Taylor series of the Lauricella functions L_Σ (1) (Theorem 6.18, which is Theorem 1.1(i) from the introduction).

6.1 Generalized (motivic) associators

Let us fix an index $1 \leq i \leq n$. The generalized associator $\mathcal{Z}^i$, that we will soon define, is a formal power series that records *all* the iterated integrals that can be computed on X_Σ on a given (homotopy class of) path γ_i between the tangential basepoints t_0 and $-t_i$. For this, we will need to fix a class $\gamma_i \in \pi_1^{\mathrm{top}}(X_\Sigma(\mathbb{C}), t_0, -t_i)$ as in §5.2(1). We will need to impose an extra condition on γ_i that can be stated as follows. Consider the following maps between fundamental torsors:

$$\pi_1^{\mathrm{top}}(X_\Sigma(\mathbb{C}), t_0, -t_i) \longrightarrow \pi_1^{\mathrm{top}}(\mathbb{C} \setminus \{0, \sigma_i\}, t_0, -t_i) \longrightarrow \pi_1^{\mathrm{top}}(\mathbb{C} \setminus \{\sigma_i\}, 0, -t_i) \simeq \mathbb{Z}. \tag{41}$$

The first map is induced by the inclusion $X_\Sigma(\mathbb{C}) \hookrightarrow \mathbb{C} \setminus \{0, \sigma_i\}$. The second map is also induced by the inclusion $\mathbb{C} \setminus \{0, \sigma_i\} \hookrightarrow \mathbb{C} \setminus \{\sigma_i\}$, where the tangential basepoint t_0 becomes the usual basepoint 0. There is a canonical element in $\pi_1^{\mathrm{top}}(\mathbb{C} \setminus \{\sigma_i\}, 0, -t_i)$, namely the homotopy class of the straight path $t \mapsto \sigma_i t$, which provides the identification of the torsor $\pi_1^{\mathrm{top}}(\mathbb{C} \setminus \{\sigma_i\}, 0, -t_i)$ with the fundamental group $\pi_1^{\mathrm{top}}(\mathbb{C} \setminus \{\sigma_i\}, 0) \simeq \mathbb{Z}$.

Definition 6.1. The class $\gamma_i \in \pi_1^{\mathrm{top}}(X_\Sigma(\mathbb{C}), t_0, -t_i)$ is said to be *admissible* if its image under the composite (41) is the class of the straight path $t \mapsto \sigma_i t$.

Roughly speaking, it means that γ_i '*does not wind around*' σ_i. A representative of an admissible class in $\pi_1^{\mathrm{top}}(X_\Sigma(\mathbb{C}), t_0, -t_i)$ can be constructed as follows: first, travel via a small arc from the tangential basepoint t_0 to $\sigma_i \varepsilon$ for a small $\varepsilon > 0$, and then travel via the straight path $t \mapsto \sigma_i t$ toward the tangential basepoint $-t_i$ (avoiding if necessary the points σ_j that are on the line segment between $\sigma_i \varepsilon$ and σ_i via small arcs). We see in Remark 6.12 that

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the admissibility of γ_i is equivalent to the vanishing of the following regularized iterated integral:

$$\int_{\gamma_i} \omega_i = 0.$$

We fix an admissible class γ_i for every $1 \leq i \leq n$ for the remainder of this article.

DEFINITION 6.2. Define a formal power series

$$\mathcal{Z}^{i,\mathfrak{m}} \in {}_0\Pi_i^{\varpi}(\mathcal{P}^{\mathfrak{m}}) \subset \mathcal{P}^{\mathfrak{m}}\langle\langle e_0, \ldots, e_n\rangle\rangle$$

by $\mathcal{Z}^{i,\mathfrak{m}} = \mathrm{comp}^{\mathfrak{m}}_{\mathrm{B},\varpi}(\gamma_i^{\mathrm{B}})$ where $\gamma_i^{\mathrm{B}} \in {}_0\Pi_i^{\mathrm{B}}(\mathbb{Q})$ is the image of γ_i as in §5.2(2). It is called a *generalized motivic associator*.

Since $\mathcal{O}({}_0\Pi_i^{\varpi})$ has weights ≥ 0, it follows that $\mathcal{Z}^{i,\mathfrak{m}}$ actually lies in ${}_0\Pi_i(\mathcal{P}^{\mathfrak{m},+})$. Explicitly,

$$\mathcal{Z}^{i,\mathfrak{m}} = \sum_{w \in \{e_0,\ldots,e_n\}^{\times}} \left[\mathcal{O}(\pi_1^{\mathrm{mot}}(X_\Sigma, t_0, -t_i)), \gamma_i^{\mathrm{B}}, w\right]^{\mathfrak{m}} w,$$

where the sum is over all words w in e_i, which are in turn dual to words in the ω_i (14) and hence define an element $w \in \mathcal{O}({}_0\Pi_i^{\varpi})$. The path γ_i^{B} is viewed as an element in $\mathcal{O}({}_0\Pi_i^{\mathrm{B}})^{\vee}$.

DEFINITION 6.3. The image $\mathcal{Z}^i = \mathrm{per}\left(\mathcal{Z}^{i,\mathfrak{m}}\right)$ of $\mathcal{Z}^{i,\mathfrak{m}}$ under the period homomorphism is called a *generalized associator*.

Explicitly, it is the group-like formal power series

$$\mathcal{Z}^i = \sum_{w \in \{e_0,\ldots,e_n\}^{\times}} \left(\int_{\gamma_i} w\right) w \quad \in \quad \mathbb{C}\langle\langle e_0, \ldots, e_n\rangle\rangle,$$

where the sum is over all words w in $\{e_0, \ldots, e_n\}$, and the integral is the regularized iterated integral (from left to right) of the corresponding word in $\{\omega_0, \ldots, \omega_n\}$.

When we wish to emphasize the dependence on the variables e_i, we shall write $\mathcal{Z}^{i,\mathfrak{m}}(e_0, \ldots, e_n)$ for $\mathcal{Z}^{i,\mathfrak{m}}$, and so on.

EXAMPLE 6.4. Let $\Sigma = \{0, 1\}$ and $k = \mathbb{Q}$. Then $\mathcal{Z}^{1,\mathfrak{m}} = \mathcal{Z}^{\mathfrak{m}}$, where

$$\mathcal{Z}^{\mathfrak{m}} = \sum_{w \in \{e_0,e_1\}^{\times}} \zeta^{\mathfrak{m}}(w) w \quad \in \quad \mathcal{P}^{\mathfrak{m}}_{\mathcal{MT}(\mathbb{Q})}\langle\langle e_0, e_1\rangle\rangle \tag{42}$$

is the motivic Drinfeld associator. Drinfeld's associator is $\mathcal{Z} = \mathrm{per}\,(\mathcal{Z}^{\mathfrak{m}}) \in \mathbb{R}\langle\langle e_0, e_1\rangle\rangle$.

6.2 Beta quotients

Let R be any commutative unital $\mathbb{Q}$-algebra.

DEFINITION 6.5. Consider the abelianization map

$$F \mapsto \overline{F} \quad : \quad R\langle\langle e_0, \ldots, e_n\rangle\rangle \longrightarrow R[[s_0, \ldots, s_n]],$$

which sends e_i to s_i, where the s_i are commuting variables. We shall call $\overline{F}$ the *abelianization* of F.

Computing the abelianization of a group-like series is easy, as the next lemma shows.

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LEMMA 6.6. *For a group-like series F, we have*

$$\overline{F} = \exp\left(F(e_0)s_0 + \cdots + F(e_n)s_n\right) = \prod_{k=0}^{n} \exp(F(e_k)s_k).$$

Proof. Since F is group-like, $F = \exp(\log F))$, where $\log(F)$ is a Lie series of the form

$$\log(F) = \sum_{k=0}^{n} F(e_k)e_k + \text{commutators}.$$

The exponential and the logarithm are taken with respect to the concatenation product, and commute with the abelianization map. Since abelianization sends all commutators to zero, we have $\overline{\log F} = \sum_{k=0}^{n} F(e_k)s_k$ and the result follows. ∎

DEFINITION 6.7. For any series $F \in R\langle\langle e_0, \ldots, e_n\rangle\rangle$, let us write

$$F = F_\varnothing + F_0 e_0 + \cdots + F_n e_n, \tag{43}$$

where $F_\varnothing \in R$ denotes the coefficient of the empty word (constant coefficient) and F_j is obtained from F by deleting the last letter from all words ending in e_j. We call the abelianization $\overline{F_j}$ the *jth beta quotient* of F.

REMARK 6.8. The $\overline{F_j}$ are very closely related to the image of F in what is known as the metabelian quotient.

For any two series $A, B \in R\langle\langle e_0, \ldots, e_n\rangle\rangle$, their product satisfies

$$(AB)_j = A\,(B_j) + A_j B_\varnothing. \tag{44}$$

We verify that, if F is invertible, then

$$\overline{(F^{-1})_j} = -\frac{1}{F_\varnothing}\frac{\overline{F_j}}{\overline{F}}. \tag{45}$$

This follows from applying (44) to $FF^{-1} = 1$, which gives $F(F^{-1})_j + F_j F_\varnothing^{-1} = 0$, and then applying the abelianization map. All series F that we shall consider are group-like (for the continuous coproduct on formal power series for which all letters e_k are primitive) and therefore have constant term $F_\varnothing = 1$.

Recall that $F(w)$ denotes (a linear combination of) coefficients of words w in F.

LEMMA 6.9. *For any series $F \in R\langle\langle e_0, \ldots, e_n\rangle\rangle$, we have*

$$\overline{F_j} = \sum_{m_0,\ldots,m_n \geq 0} F\left(\left[e_0^{\text{ш} m_0} \text{ш} \cdots \text{ш} e_n^{\text{ш} m_n}\right] e_j\right) \frac{s_0^{m_0}}{m_0!} \cdots \frac{s_n^{m_n}}{m_n!},$$

where $[w]e_j$ denotes the right concatenation of e_j to any linear combination w of words in the letters $e_0, \ldots, e_n$. The previous expression can also be written

$$\overline{F_j} = \sum_{m_0,\ldots,m_n \geq 0} F\left(\left[e_0^{m_0} \text{ш} \cdots \text{ш} e_n^{m_n}\right] e_j\right) s_0^{m_0} \cdots s_n^{m_n}.$$

Proof. Notice that

$$\overline{F_j} = \sum_w \overline{w}\, F(we_j) = \sum_{m_0,\ldots,m_n \geq 0} s_0^{m_0} \cdots s_n^{m_n} \left(\sum_{\overline{w} = s_0^{m_0} \cdots s_n^{m_n}} F(we_j) \right)$$

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and substitute in the expression

$$\left(\sum_{\overline{w}=s_0^{m_0}\cdots s_n^{m_n}} w\right) = \frac{e_0^{\text{ш}\, m_0}}{m_0!} \text{ш} \cdots \text{ш} \frac{e_n^{\text{ш}\, m_n}}{m_n!} = e_0^{m_0} \text{ш} \cdots \text{ш}\, e_n^{m_n}.$$

□

6.3 Beta quotients of generalized associators

DEFINITION 6.10. We define the following $n \times n$ matrices of power series:

$$(FL_\Sigma^{\mathfrak{m}})_{ij} = \mathbf{1}_{i=j}\overline{\mathcal{Z}^{i,\mathfrak{m}}} - s_j\overline{\mathcal{Z}_j^{i,\mathfrak{m}}} \quad \in \mathcal{P}^{\mathfrak{m}}[[s_0,\dots,s_n]].$$

$$(FL_\Sigma)_{ij} = \mathbf{1}_{i=j}\overline{\mathcal{Z}^{i}} - s_j\overline{\mathcal{Z}_j^{i}} \quad \in \mathbb{C}[[s_0,\dots,s_n]].$$

We clearly have $\mathrm{per}(FL_\Sigma^{\mathfrak{m}}) = FL_\Sigma$, where $\mathrm{per}: \mathcal{P}^{\mathfrak{m}} \to \mathbb{C}$ denotes the period map of $\mathcal{MT}(k)$ applied coefficientwise. Our next goal (Theorem 6.18) is to prove that FL_Σ equals the matrix of Taylor series of the Lauricella functions L_Σ (1). (This justifies the notation FL_Σ where the letter F stands for *formal*.) We now compute the abelianization and beta quotients of the generalized associators $\mathcal{Z}^i$.

We first need a lemma that clarifies the role of the choice of tangential basepoints. For a point $x = \gamma_i(t)$, for $t \in (0,1)$, we let γ_i^x denote the restriction of γ_i to the interval $(0,t)$. By abuse of notation, we manipulate functions and forms on $\mathbb{C}\setminus\Sigma$ when we actually mean their pullbacks to $(0,1)$ via the path γ_i.

LEMMA 6.11. *For $0 \le k \le n$, we have the equalities, for $x = \gamma_i(t)$ with $t \in (0,1)$:*

$$\int_{\gamma_i^x} \omega_k = \begin{cases} \log(x), & \text{if } k = 0\ , \\ \log(1 - x\sigma_k^{-1}), & \text{if } 1 \le k \le n. \end{cases}$$

Furthermore, for $x = \sigma_i$, that is, for $t = 1$, we have

$$\int_{\gamma_i} \omega_k = \begin{cases} \log(\sigma_i), & \text{if } k = 0\ , \\ \log(1 - \sigma_i\sigma_k^{-1}), & \text{if } 1 \le k \ne i \le n\ , \\ 0, & \text{if } k = i. \end{cases}$$

Proof. For the first claim, the $1 \le k \le n$ case is clear, since the differential form ω_j does not have a singularity at 0. For $j = 0$, we compute, by definition,

$$\int_{\gamma_i^x} \omega_0 = \mathrm{Reg}_{\varepsilon\to 0} \int_\varepsilon^t \gamma_i^*\omega_0 = \mathrm{Reg}_{\varepsilon\to 0}(\log(x) - \log(\gamma_i(\varepsilon))),$$

where applying $\mathrm{Reg}_{\varepsilon\to 0}$ amounts to formally setting $\log(\varepsilon) = 0$ and $\varepsilon = 0$ in the logarithmic asymptotic development (see [BGF, Def. 3.237]). Since γ_i extends to a smooth function on $[0,1]$ whose derivative at 0 is 1, we have $\gamma_i(\varepsilon) = \varepsilon + O(\varepsilon^2)$ and thus $\log(\gamma_i(\varepsilon)) = \log(\varepsilon) + O(\varepsilon)$. We thus have $\mathrm{Reg}_{\varepsilon\to 0}(\log(\gamma_i(\varepsilon))) = 0$ and the claim follows.

For the second claim, only the $k = i$ case requires a comment since in the other cases the form ω_j does not have a singularity at σ_i and we can simply pass to the limit $x \to \sigma_i$ in the first statement. We will use the admissibility condition (Definition 6.1). Since ω_i only has poles at σ_i and ∞, its iterated integral along γ_i only depends on the image of γ_i in $\pi_1^{\mathrm{top}}(\mathbb{C}\setminus\{\sigma_i\}, 0, -t_i)$. By the admissibility condition, this image is the class of $\widetilde{\gamma_i}(t) = \sigma_i t$

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and we can compute

$$\int_{\gamma_i} \omega_i = \int_{\widetilde{\gamma}_i} \omega_i = \mathrm{Reg}_{\varepsilon\to 0} \int_{\varepsilon}^{1-\varepsilon} \widetilde{\gamma}_i{}^* \omega_i = \mathrm{Reg}_{\varepsilon\to 0} \int_{\varepsilon}^{1-\varepsilon} \frac{dt}{t-1} = \mathrm{Reg}_{\varepsilon\to 0} \log\left(\frac{\varepsilon}{1-\varepsilon}\right) = 0.$$

The claim follows. □

REMARK 6.12. From the proof of Lemma 6.11, one sees that the admissibility condition on γ_i (Definition 6.1) is equivalent to the vanishing of the regularized iterated integral $\int_{\gamma_i} \omega_i$. In general, that iterated integral could be any integer multiple of $2\pi i$.

PROPOSITION 6.13. *The abelianization of the generalized associator $\mathcal{Z}^i$ is*

$$\overline{\mathcal{Z}^i} = \sigma_i^{s_0} \prod_{k\neq i} (1-\sigma_i\sigma_k^{-1})^{s_k}. \tag{46}$$

For every $0 \leq j \leq n$, the jth beta quotient of $\mathcal{Z}^i$ is

$$\overline{\mathcal{Z}^i_j} = \int_{\gamma_i} x^{s_0} \prod_{k=1}^{n} \left(1 - x\,\sigma_k^{-1}\right)^{s_k} \frac{dx}{x-\sigma_j}. \tag{47}$$

These expressions are formal power series in the s_i obtained by expanding the exponentials as power series and interpreting the various logarithms which appear as coefficients as regularized iterated integrals via Lemma 6.11.

Proof. Since $\mathcal{Z}^i$ is a group-like series, the first claim follows from Lemma 6.6 and the computations of $\mathcal{Z}^i(e_j)$ performed in the second part of Lemma 6.11. The second claim follows from Lemma 6.9 and the equality

$$\int_{\gamma_i} [e_0^{\text{ш} m_0} \text{ш} \cdots \text{ш} e_n^{\text{ш} m_n}] e_j = \int_{\gamma_i} \Big(\int_{\gamma_i^x} \omega_0\Big)^{m_0} \cdots \Big(\int_{\gamma_i^x} \omega_n\Big)^{m_n} \omega_j,$$

which by the first part of Lemma 6.11 equals

$$\int_{\gamma_i} \log^{m_0}(x) \prod_{k=1}^{n} \log^{m_k}(1-x\sigma_k^{-1}) \frac{dx}{x-\sigma_j}\ .$$

The claim follows. □

Equation (47) in the case $\Sigma = \{0,1\}$ reduces to Drinfeld's computation of the metabelian quotient of his associator in terms of the usual beta function [D2]. See also [E], [L] for further developments.

The following lemma will be useful.

LEMMA 6.14. *For all $m \geq 0$,*

$$\int_{\gamma_i} \log^m(1-x\sigma_i^{-1}) \frac{dx}{x-\sigma_i} = 0. \tag{48}$$

Proof. The integral is proportional to the $(m+1)$-fold iterated integral of $\frac{dx}{x-\sigma_i}$ along γ_i, which, by the shuffle product formula, is in turn proportional to the $(m+1)$th power of the integral of $\frac{dx}{x-\sigma_i}$ along γ_i, which vanishes by Lemma 6.11. □

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6.4 Comparing L_Σ and FL_Σ

Up until now, we have considered two types of integrals: convergent line integrals of smooth 1-forms along a smooth (locally finite) path δ_i from 0 to σ_i, and regularized iterated integrals of words in the ω_j's along a path γ_i from 0 to σ_i with prescribed tangent directions at 0 and σ_i. In order to compare these two different objects, we have to impose some compatibility between γ_i and δ_i. Let us first fix parameters $\underline{s} = (s_0, s_1, \dots, s_n)$. There is a natural map

$$h_i : \pi_1^{\mathrm{top}}(X_\Sigma(\mathbb{C}), t_0, -t_i) \longrightarrow H_1^{\mathrm{lf}}(X_\Sigma(\mathbb{C}), \mathcal{L}_{\underline{s}}^\vee)$$

defined in the following way. For a class in $\pi_1^{\mathrm{top}}(X_\Sigma(\mathbb{C}), t_0, -t_i)$ represented by a continuous map $\gamma : (0,1) \to \mathbb{C} \setminus \Sigma$, its image under h_i has γ as its underlying (locally finite) path, together with the section of $\mathcal{L}_{\underline{s}}^\vee$ given by

$$x^{s_0} \prod_{k=1}^{n} (1 - x\sigma_k^{-1})^{s_k} = \prod_{k=0}^{n} \exp\Big(s_k \int_{\gamma^x} \omega_k\Big),$$

which makes sense thanks to Lemma 6.11. One checks that this is well defined.

Let $\delta_i : (0,1) \to \mathbb{C} \setminus \Sigma$ be a smooth map that can be extended to a smooth map $\overline{\delta_i} : [0,1] \to \mathbb{C}$, and which does not wind infinitely around 0 or σ_i (see Remark 2.3). Let us fix a determination of $\log(\sigma_i)$ for every $1 \leq i \leq n$. In Remark 4.2, we defined out of this data a class $\delta_{i,\underline{s}} \in H_1^{\mathrm{lf}}(X_\Sigma(\mathbb{C}), \mathcal{L}_{\underline{s}}^\vee)$ for every parameter $\underline{s} = (s_0, \dots, s_n) \in \mathbb{C}^{n+1}$. Let γ_i be a class in $\pi_1^{\mathrm{top}}(X_\Sigma(\mathbb{C}), t_0, -t_i)$.

Definition 6.15. We say that δ_i and γ_i are compatible if for every $\underline{s} \in \mathbb{C}^{n+1}$ we have

$$h_i(\gamma_i) = \delta_{i,\underline{s}}.$$

Note that this implies that the determinations of $\log(\sigma_i)$ are given by the regularized iterated integrals $\log(\sigma_i) = \int_{\gamma_i} \omega_0$.

For all $1 \leq j \leq n$, we define

$$\Omega_j = x^{s_0} \prod_{k=1}^{n} (1 - x\sigma_k^{-1})^{s_k} \frac{dx}{x - \sigma_j}\ , \tag{49}$$

which we interpret as a 1-form on δ_i as in Remark 4.1. Let $\Omega_j^{\mathrm{ren}_i}$ denote its renormalized versions (Definition 4.3) with respect to $\{0, \sigma_i\}$, given by

$$\Omega_j^{\mathrm{ren},i} = \Big(x^{s_0} \prod_{k \neq i} (1 - x\sigma_k^{-1})^{s_k} - \mathbf{1}_{i=j}\, \sigma_i^{s_0} \prod_{k \neq i} (1 - \sigma_i \sigma_k^{-1})^{s_k}\Big)(1 - x\sigma_i^{-1})^{s_i} \frac{dx}{x - \sigma_j}\ .$$

According to Proposition 4.5, the integral $\int_{\delta_i} \Omega_j^{\mathrm{ren}_i}$ defines a holomorphic function of the parameters $\underline{s}$ around $\underline{s} = 0$ and thus has a Taylor expansion. In the next proposition, we identify this Taylor expansion with the jth beta quotient $\overline{\mathcal{Z}_j^i}$ of the generalized associator, defined via regularized iterated integrals along γ_i.

Proposition 6.16. *For all $1 \leq j \leq n$, the Taylor series at $\underline{s} = 0$ of the integral $\int_{\delta_i} \Omega_j^{\mathrm{ren}_i}$ is $\overline{\mathcal{Z}_j^i}$.*

Before proving the proposition, we need a technical lemma that allows us to exchange summation and integration.

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LEMMA 6.17. *Let us write $\Omega_j^{\mathrm{ren}_i}$ as a formal power series:*

$$\Omega_j^{\mathrm{ren}_i} = \sum_{m_0,\dots,m_n\geq 0} \Omega_j^{\mathrm{ren}_i}(\underline{m})\,\frac{s_0^{m_0}\cdots s_n^{m_n}}{m_0!\cdots m_n!}\ .$$

Then we have the equality of formal power series:

$$\int_{\delta_i} \Omega_j^{\mathrm{ren}_i} = \sum_{m_0,\dots,m_n\geq 0} \int_{\delta_i} \Omega_j^{\mathrm{ren}_i}(\underline{m})\,\frac{s_0^{m_0}\cdots s_n^{m_n}}{m_0!\cdots m_n!}\ .$$

Proof. Let us write $\delta_i^*\Omega_j^{\mathrm{ren}_i}(\underline{m}) = f_{\underline{m}}(t)dt$. By Fubini, it is enough to prove that the multiple series

$$\sum_{m_0,\dots,m_n\geq 0} \int_0^1 |f_{\underline{m}}(t)|dt\,\frac{s_0^{m_0}\cdots s_n^{m_n}}{m_0!\cdots m_n!} \tag{50}$$

is absolutely convergent for $|\underline{s}|$ small enough, and therefore we need to estimate the integral of $|f_{\underline{m}}(t)|$. We do so in the case $j=i$; the case $j\neq i$ is even simpler. For simplicity, we only treat the case $n=1$. The general case is similar and is left to the reader. We write $\sigma=\sigma_1$ and $\delta=\delta_1$. For indices $m_0,m_1\geq 0$, we have, according to Definition 4.3,

$$f_{m_0,m_1}(t) = \frac{\log^{m_0}(\delta(t))-\log^{m_0}(\sigma)}{\delta(t)-\sigma}\log^{m_1}(1-\delta(t)\sigma^{-1})\,\delta'(t).$$

Note that $\delta'(t)$ is bounded for $t\in(0,1)$. To prove convergence, we can clearly assume that there exist small closed disks D_0 and D_σ around 0 and σ, respectively, and constants $0<\alpha<\beta<1$, such that $\delta(t)\notin D_\sigma$ for all $t\in(0,\beta)$, and $\delta(t)\notin D_0$ for all $t\in(\alpha,1)$. We treat separately the cases $t\in(0,\beta)$ and $t\in(\alpha,1)$, and assume that $m_0,m_1\geq 1$.

– For $t\in(0,\beta)$, $\log(1-\delta(t)\sigma^{-1})$ is bounded and $\delta(t)-\sigma$ is bounded below in absolute value, which gives an estimate

$$|f_{m_0,m_1}(t)| < (|\log^{m_0}(\delta(t))|+|\log^{m_0}(\sigma)|)\,A^{m_1}$$

for some positive constant A. Now, since by assumption (Remark 2.3) the argument of $\delta(t)$ is bounded as t approaches zero, and since $|\log|\delta(t)||$ tends to infinity when t goes to zero, we have $|\log(\delta(t))|<B|\log|\delta(t)||$ for some positive constant B, and therefore since δ is smooth at 0, we have $|\log(\delta(t))|<B'|\log(t)|$ for all $t\in(0,1/2)$, for some positive constant B'. We deduce the estimate $|\log^{m_0}(\delta(t))|+|\log^{m_0}(\sigma)|<(B'')^{m_0}|\log(t)|^{m_0}$ for all $t\in(0,\beta)$, for some positive constant B''. Therefore,

$$|f_{m_0,m_1}(t)| < C^{m_0+m_1}|\log(t)|^{m_0},$$

for all $t\in(0,\beta)$, for some positive constant C.

– For $t\in(\alpha,1)$, the quotient $\frac{\log^{m_0}(\delta(t))-\log^{m_0}(\sigma)}{\delta(t)-\sigma}$ is bounded by D^{m_0} for some positive constant D, and by the same argument as before (using the assumption of Remark 2.3) we have an estimate $|\log^{m_1}(1-\delta(t)\sigma^{-1})|<E^{m_1}|\log(1-t)|^{m_1}$ for all $t\in(\alpha,1)$, for some positive constant E. Therefore,

$$|f_{m_0,m_1}(t)| < F^{m_0+m_1}|\log(1-t)|^{m_1},$$

for all $t\in(\alpha,1)$, for some positive constant F.

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Putting those two contributions together, we get

$$\int_0^1 |f_{m_0,m_1}(t)|dt < G^{m_0+m_1}\int_0^1 (|\log(t)|^{m_0}+|\log(t)|^{m_1})dt$$

for some positive constant G. Since $\int_0^1 |\log(t)|^m dt = m!$ for all m, we therefore get the estimate

$$\int_0^1 |f_{m_0,m_1}(t)|dt < G^{m_0+m_1}(m_0!+m_1!)$$

for some positive constant G. Therefore, the series (50) converges for $|s_0|,|s_1| < G^{-1}$, and the claim follows. ∎

We now give two proofs of Proposition 6.16, since they are instructive.

Proof (First proof). We consider only the case $\overline{\mathcal{Z}_i^i}$ since the argument for $\overline{\mathcal{Z}_j^i}$ with $j \neq i$ is even simpler. According to Definition 4.3, we have, with the notation of Lemma 6.17,

$$\Omega_i^{\mathrm{ren}_i}(\underline{m}) = \log^{m_i}(1-x\sigma_i^{-1})\frac{dx}{x-\sigma_i} \times \left(\log^{m_0}(x)\prod_{1\leq k\neq i}\log^{m_k}(1-x\sigma_k^{-1}) - \log^{m_0}(\sigma_i)\prod_{1\leq k\neq i}\log^{m_k}(1-\sigma_i\sigma_k^{-1})\right).$$

According to Lemma 6.17, the Taylor series of $\int_{\delta_i}\Omega_i^{\mathrm{ren}_i}$ has coefficients $\int_{\delta_i}\Omega_i^{\mathrm{ren}_i}(\underline{m})$, and we claim that they equal

$$\int_{\delta}\Omega_i^{\mathrm{ren}_i}(\underline{m}) \;=\; \int_{\gamma_i}\Omega_i^{\mathrm{ren}_i}(\underline{m}), \tag{51}$$

where the integral on the left is an ordinary, convergent integral, and the one on the right is regularized along the path γ_i between tangential basepoints (see §5.2). To see this, use the fact that regularization with respect to the tangential basepoint $-t_i$ is equivalent to taking a primitive of $\Omega_i^{\mathrm{ren}_i}(\underline{m})$ in the ring $\mathbb{C}[[x-\sigma_i]][\log(x-\sigma_i)]$, and formally setting all $\log(x-\sigma_i)$ terms to zero, before in turn setting x to σ_i. Since the term in brackets in the above expression for $\Omega_i^{\mathrm{ren}_i}(\underline{m})$ vanishes at $x=\sigma_i$, it actually has a primitive in the subspace $(x-\sigma_i)\mathbb{C}[[x-\sigma_i]][\log(x-\sigma_i)]$, and one can simply take its limit as $x\to\sigma_i$, which is the procedure for computing an ordinary integral (without tangential basepoint regularization). A simpler argument applies at $x=0$ and proves (51).

The formula for $\overline{\mathcal{Z}_i^i}$ follows by applying Lemma 6.14 and implies that

$$\int_{\gamma_i}\Omega_i^{\mathrm{ren}_i}(\underline{m}) \;=\; \int_{\gamma_i}\log^{m_0}(x)\prod_{k=1}^{n}\log^{m_k}(1-x\sigma_k^{-1})\frac{dx}{x-\sigma_i}.$$

This is precisely the coefficient of $\frac{s_0^{m_0}}{m_0!}\cdots\frac{s_n^{m_n}}{m_n!}$ in the Taylor expansion of (47). ∎

Proof (Second proof). For γ a path between (tangential) basepoints $x,y\in X_\Sigma$, and ω a closed formal 1-form taking values in the Lie algebra of the ring of formal non-commutative power series $\mathbb{C}\langle\langle e_0,\ldots,e_n\rangle\rangle$, consider the formal power series defined by iterated integration:

$$I_\gamma(\omega) = 1+\int_\gamma\omega+\int_\gamma\omega\omega+\cdots.$$

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It is known as the transport of (the connection associated with) ω along γ, and satisfies the composition of paths formula $I_{\gamma\gamma'}(\omega) = I_\gamma(\omega) I_{\gamma'}(\omega)$. By applying the prescription for computing iterated integrals with respect to tangential basepoints, we find that the transport of the formal 1-form $\omega_\Sigma = e_0\omega_0 + \cdots + e_n\omega_n$ along the path γ_i defined in §5.2 equals

$$\begin{aligned}\mathcal{Z}^i = I_{\gamma_i}(\omega_\Sigma) &= I_{\gamma_i^x}(\omega_\Sigma) I_{\nu_x}(\omega_\Sigma) \\ &= \lim_{x\to\sigma_i} \left(I_{\gamma_i^x}(\omega_\Sigma) I_{\nu_x}(\omega_\Sigma) \right) \\ &= \lim_{x\to\sigma_i} \left(I_{\gamma_i^x}(\omega_\Sigma) I_{\nu_x}(e_i\omega_i) \right),\end{aligned}$$

where $x = \gamma_i(t)$ for some $0 < t < 1$; γ_i^x is the restriction of γ_i to $[0,t]$; and ν_x is a path from x to $-t_i$. The first equation is simply the composition of paths formula. Since the left-hand side does not depend on the choice of point x, we may take a limit as $x \to \sigma_i$, which implies the second equation. In the third equation, we view the path ν_x as a path in the tangent space at σ_i, which we identify with $\mathbb{P}^1\backslash\{\sigma_i, \infty\}$. The form $e_i\omega_i$ is the localisation of ω_Σ to the punctured tangent space and captures the divergent iterated integrals terminating in the letter e_i.

It follows from (44) that

$$\overline{\mathcal{Z}^i_j} = \lim_{x\to\sigma_i} \left(\overline{I_{\gamma_i^x}(\omega_\Sigma)}\, \overline{I_{\nu_x}(e_i\omega_i)}_j + \overline{I_{\gamma_i^x}(\omega_\Sigma)}_j \right). \tag{52}$$

The tangential integral $I_{\nu_x}(e_i\omega_i)$ is simply an exponential:

$$I_{\nu_x}(e_i\omega_i) = \exp(-e_i \log(1 - x\sigma_i^{-1})), \tag{53}$$

and its abelianization is therefore obtained by replacing e_i with s_i in the previous expression. Equation (53) follows, for example, from

$$\int_x^{-t_i} \frac{dz}{z-\sigma_i} = \left(\int_x^{t_0} + \int_{t_0}^{-t_i} \right) \frac{dz}{z-\sigma_i} \overset{(48)}{=} \int_x^0 \frac{dz}{z-\sigma_i} + 0 = -\log(1 - x\sigma_i^{-1}).$$

From equation (53), the expression $\overline{I_{\nu_x}(e_i\omega_i)}_j$ vanishes if $j \neq i$, but equals

$$\overline{I_{\nu_x}(e_i\omega_i)}_i = \frac{1}{s_i}\left((1 - x\sigma_i^{-1})^{-s_i} - 1 \right)$$

otherwise. Thus, if $j \neq i$, the term $\overline{I_{\gamma_i^x}(\omega_\Sigma) I_{\nu_x}(e_i\omega_i)_j}$ in (52) vanishes and we find that

$$\overline{\mathcal{Z}^i_j} = \lim_{x\to\sigma_i} \left(\overline{I_{\gamma_i^x}(\omega_\Sigma)}_j \right) = \lim_{x\to\sigma_i} \left(\int_0^x x^{s_0} \prod_{k=1}^n \left(1 - x\,\sigma_k^{-1}\right)^{s_k} \frac{dx}{x - \sigma_j} \right),$$

using the version of (47) with the upper range of integration replaced with the point x (which follows from the computations in the second paragraph of the proof of Proposition 6.13—one needs only check that one can replace γ_i^x with an ordinary path from 0 to x, i.e., that the tangential component of γ_i^x at the origin plays no role since the integral is convergent there). This proves the formula for $j \neq i$, thanks to Lemma 6.17. In the case $j = i$, a version of (46) with upper range of integration x implies that

$$\overline{I_{\gamma_i^x}(\omega_\Sigma)} = x^{s_0} \prod_{k=1}^n \left(1 - x\,\sigma_k^{-1}\right)^{s_k}.$$

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Substituting this into (52) gives

$$\overline{\mathcal{Z}_i^i} = \lim_{x\to\sigma_i}\left(x^{s_0}\prod_{k=1}^{n}\left(1-x\,\sigma_k^{-1}\right)^{s_k} \times \frac{1}{s_i}\left((1-x\sigma_i^{-1})^{-s_i}-1\right) + \int_0^x x^{s_0}\prod_{k=1}^{n}\left(1-x\,\sigma_k^{-1}\right)^{s_k}\frac{dx}{x-\sigma_i}\right).$$

Using the identity $(1-x\sigma_i^{-1})^{s_i}\times\left((1-x\sigma_i^{-1})^{-s_i}-1\right) = 1-(1-x\sigma_i^{-1})^{s_i}$, the previous expression can be rewritten in the form

$$\lim_{x\to\sigma_i}\left(\sigma_i^{s_0}\prod_{k\neq i}(1-\sigma_i\sigma_k^{-1})^{s_k} \times \frac{1}{s_i}\left(1-(1-x\sigma_i^{-1})^{s_i}\right) + \int_0^x x^{s_0}\prod_{k=1}^{n}\left(1-x\,\sigma_k^{-1}\right)^{s_k}\frac{dx}{x-\sigma_i}\right).$$

Finally, substitute in the following identity

$$\frac{1}{s_i}\left(1-(1-x\sigma_i^{-1})^{s_i}\right) = -\int_0^x (1-x\sigma_i^{-1})^{s_i}\frac{dx}{x-\sigma_i} \tag{54}$$

to deduce the stated formula for $\overline{\mathcal{Z}_i^i}$, thanks to Lemma 6.17. □

We are now ready to prove Theorem 1.1(i) from the introduction.

THEOREM 6.18. *For all i,j, $(FL_\Sigma)_{ij}$ is the Taylor series at $\underline{s}=0$ of the Lauricella function $(L_\Sigma)_{ij}$.*

Proof. This follows from comparing the definition of $(FL_\Sigma)_{ij}$ with the expression in Proposition 4.5, and using the expressions for $\overline{\mathcal{Z}^i}$ and $\overline{\mathcal{Z}_j^i}$ of Propositions 6.13 and 6.16. □

§7. Generalized single-valued associators and their beta quotients

This section is the single-valued version of the previous one. We introduce the single-valued versions $\mathcal{Z}^{i,\mathbf{s}}$ of the generalized associators, which are non-commutative power series whose coefficients are single-valued versions of iterated integrals on X_Σ. Through their beta quotients, we define a matrix $FL_\Sigma^{\mathbf{s}}$ of power series which we prove to be the matrix of Taylor series of the single-valued Lauricella functions $L_\Sigma^{\mathbf{s}}$ (3) (Theorem 7.11, which is Theorem 1.1(ii) from the introduction).

7.1 Generalized de Rham and single-valued associators

We fix an index $1\leq i\leq n$. We start by defining de Rham and single-valued versions of the generalized associators from the previous section.

DEFINITION 7.1. We define

$$\mathcal{Z}^{i,\varpi} \;\in\; {}_0\Pi_i^{\varpi}(\mathcal{P}^{\varpi}) \subset \mathcal{P}^{\varpi}\langle\langle e_0,\dots,e_n\rangle\rangle$$

to be the canonical element in $\mathrm{Hom}\left(\mathcal{O}({}_0\Pi_i^{\varpi}),\mathcal{P}^{\varpi}\right)$ given by the morphism of schemes $G^{\varpi}_{\mathcal{MT}(k)}\to {}_0\Pi_i^{\varpi}$ induced by the action $g\mapsto g.{}_01_i$ of $G^{\varpi}_{\mathcal{MT}(k)}$ on the canonical ϖ-path ${}_01_i$. It is called a *generalized (canonical) de Rham associator*.

It is given explicitly by the group-like formal power series

$$\mathcal{Z}^{i,\varpi} = \sum_{w\in\{e_0,\dots,e_n\}^\times}\left[\mathcal{O}(\pi_1^{\mathrm{mot}}(X_\Sigma,t_0,-t_i)),{}_01_i,w\right]^{\varpi} w$$

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whose coefficients are (canonical) de Rham versions of iterated integrals. Since the empty iterated integral along γ_i is 1, it follows that $\mathcal{Z}^{i,\varpi}$ is the image of $\mathcal{Z}^{i,\mathfrak{m}}$ under the coefficientwise application of the projection $\pi_{\varpi}^{\mathfrak{m},+}$, that is,

$$\mathcal{Z}^{i,\varpi} = \pi_{\varpi}^{\mathfrak{m},+} \mathcal{Z}^{i,\mathfrak{m}}. \tag{55}$$

Definition 7.2. We define

$$\mathcal{Z}^{i,\mathbf{s}} = \mathbf{s}(\mathcal{Z}^{i,\varpi}) \ \in \mathbb{C}\langle\langle e_0, \dots, e_n\rangle\rangle,$$

where $\mathbf{s} : \mathcal{P}^{\varpi} \to \mathbb{C}$ is the single-valued period map of $\mathcal{MT}(k)$ applied coefficientwise. It is called a *generalized single-valued associator*.

The coefficients of $\mathcal{Z}^{i,\mathbf{s}}$ are single-valued versions of iterated integrals.

Remark 7.3. As should be clear from the definitions, the power series $\mathcal{Z}^{i,\varpi}$ and $\mathcal{Z}^{i,\mathbf{s}}$ do not depend on the choice of a (class of a) path γ_i as in the previous section.

Example 7.4. Let $\Sigma = \{0,1\}$ and $k = \mathbb{Q}$. Then $\varpi = \omega_{\mathrm{dR}}$ and $\mathcal{Z}^{1,\varpi} = \mathcal{Z}^{\mathfrak{dr}}$ where

$$\mathcal{Z}^{\mathfrak{dr}} = \sum_{w \in \{e_0,e_1\}^{\times}} \zeta^{\mathfrak{dr}}(w) w \quad \in \quad \mathcal{P}^{\mathfrak{dr}}_{\mathcal{MT}(\mathbb{Q})} \langle\langle e_0, e_1\rangle\rangle$$

is the de Rham Drinfeld associator. It is obtained from the motivic Drinfeld associator (42) by replacing every motivic multiple zeta value $\zeta^{\mathfrak{m}}$ with its de Rham version $\zeta^{\mathfrak{dr}}$. Its image under the single-valued period map is the Deligne associator

$$\mathcal{Z}^{\mathbf{s}} = \sum_{w \in \{e_0,e_1\}^{\times}} \zeta^{\mathbf{s}}(w) w \quad \in \quad \mathbb{R}\langle\langle e_0, e_1\rangle\rangle,$$

whose coefficients are single-valued multiple zeta values [B2].

The following definition is parallel to Definition 6.10.

Definition 7.5. We define the following $n \times n$ matrices of power series:

$$(FL_{\Sigma}^{\varpi})_{ij} = \mathbf{1}_{i=j} \overline{\mathcal{Z}^{i,\varpi}} - s_j \overline{\mathcal{Z}_j^{i,\varpi}} \ \in \mathcal{P}^{\varpi}[[s_0, \dots, s_n]],$$

$$(FL_{\Sigma}^{\mathbf{s}})_{ij} = \mathbf{1}_{i=j} \overline{\mathcal{Z}^{i,\mathbf{s}}} - s_j \overline{\mathcal{Z}_j^{i,\mathbf{s}}} \ \in \mathbb{C}[[s_0, \dots, s_n]].$$

We clearly have

$$\mathbf{s}(FL_{\Sigma}^{\varpi}) = FL_{\Sigma}^{\mathbf{s}},$$

where $\mathbf{s} : \mathcal{P}^{\varpi} \to \mathbb{C}$ denotes the single-valued period map of $\mathcal{MT}(k)$ applied coefficientwise. By (55), we also have

$$FL_{\Sigma}^{\varpi} = \pi_{\varpi}^{\mathfrak{m},+} FL_{\varpi}^{\mathfrak{m}},$$

where $\pi_{\varpi}^{\mathfrak{m},+} : \mathcal{P}^{\mathfrak{m},+} \to \mathcal{P}^{\mathfrak{dr}}$ is the de Rham projection applied coefficientwise.

Our next goal (Theorem 7.11) is to prove that $FL_{\Sigma}^{\mathbf{s}}$ equals the matrix of Taylor series of the single-valued Lauricella functions $L_{\Sigma}^{\mathbf{s}}$ (3). To this end, we first compute the abelianization and beta quotients of the generalized single-valued associators $\mathcal{Z}^{i,\mathbf{s}}$. Our techniques can be used more generally to give integral formulae for the single-valued periods of motivic torsors of paths between tangential basepoints.

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PROPOSITION 7.6. *The abelianization of the generalized single-valued associator* $\mathcal{Z}^{i,\mathbf{s}}$ *satisfies*

$$\overline{\mathcal{Z}^{i,\mathbf{s}}} = |\sigma_i|^{2s_0} \prod_{k\neq i} |1-\sigma_i\sigma_k^{-1}|^{2s_k}.$$

Proof. This follows from Lemma 6.6 and the following claim about the length one coefficients of $\mathcal{Z}^{i,\mathbf{s}}$:

$$\mathcal{Z}^{i,\mathbf{s}}(e_k) = \begin{cases} \log|\sigma_i|^2, & \text{for } k=0\ , \\ \log|1-\sigma_i\sigma_k^{-1}|^2, & \text{for } 1\leq k\neq i\leq n\ , \\ 0, & \text{for } k=i. \end{cases}$$

Let us prove this claim. By definition and since ω_k only has poles at σ_k and ∞, we have

$$\mathcal{Z}^{i,\varpi}(e_k) = \left[W_2\mathcal{O}(\pi_1^{\mathrm{mot}}(\mathbb{A}^1_k\setminus\{\sigma_k\},t_0,-t_i)),{}_0 1_i,[\omega_k]\right]^{\varpi}.$$

The object $W_2\mathcal{O}(\pi_1^{\mathrm{mot}}(\mathbb{A}^1_k\setminus\{\sigma_k\},t_0,-t_i))$ denotes the weight ≤ 2 (or length ≤ 1) subobject; it has rank 2, with de Rham basis $(1,[\omega_k])$ and Betti basis $([\gamma],[\gamma']-[\gamma])$ for some paths γ, γ' from t_0 to $-t_i$ such that the closed path $\gamma'\gamma^{-1}$ is homologous to a small positively oriented loop around σ_k. (Note that for $k\neq 0$, t_0 denotes the usual basepoint 0, and for $k\neq i$, $-t_i$ denotes the usual basepoint σ_i.) We note that, by definition, ${}_0 1_i(1)=1$ and ${}_0 1_i([\omega_k])=0$. The period matrix in these bases is

$$P = \begin{pmatrix} 1 & \int_\gamma \omega_k \\ 0 & 2\pi i \end{pmatrix}.$$

By the definition of the single-valued period homomorphism [BD1, Def. 2.5], $\mathcal{Z}^{i,\mathbf{s}}(e_k) = \mathbf{s}(\mathcal{Z}^{i,\varpi}(e_k))$ is the top-right coefficient of the single-valued period matrix $\overline{P}^{-1}P$, where $\overline{P}$ denotes the complex conjugate matrix. An easy computation shows that this equals $\mathcal{Z}^{i,\mathbf{s}}(e_k) = 2\mathrm{Re}(\int_\gamma \omega_k)$ and the claim follows from Lemma 6.11. ∎

7.2 Beta quotients of generalized single-valued associators

We start with a version of beta quotients which have finite, rather than tangential, basepoints. We extend Definition 7.1, and let

$$I^{\varpi}(x,y) \ \in\ {}_x\Pi_y^{\varpi}(\mathcal{P}^{\varpi})$$

denote the generating series of canonical de Rham periods from x to y, where x and y are either finite basepoints in $X_\Sigma(k)$ or tangential basepoints $\pm t_k$, for $0\leq k\leq n$. We let $I^{\mathbf{s}}(x,y) = \mathbf{s}(I^{\varpi}(x,y))$ denote its image by the single-valued period homomorphism. We thus

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have $\mathcal{Z}^{i,\bullet} = I^\bullet(t_0, -t_i)$ for $\bullet \in \{\varpi, \mathbf{s}\}$. Note that the same method of proof as in Proposition 7.6 gives the abelianization of $I^{\mathbf{s}}(x,y)$:

$$\overline{I^{\mathbf{s}}(x,y)} = \left|\frac{y}{x}\right|^{2s_0} \prod_{k=1}^{n} \left|\frac{1-y\sigma_k^{-1}}{1-x\sigma_k^{-1}}\right|^{2s_k} . \tag{56}$$

We will need the following lemma, which gives a single-valued version of the integral

$$\int_x^z \frac{dw}{w-\sigma} = \log\left(\frac{z-\sigma}{x-\sigma}\right).$$

LEMMA 7.7. *Suppose that $\sigma, x, z \in \mathbb{C}$ are distinct. Then*

$$-\frac{1}{2\pi i}\iint_{\mathbb{C}} \left(\frac{d\overline{w}}{\overline{w}-\overline{z}} - \frac{d\overline{w}}{\overline{w}-\overline{x}}\right) \wedge \frac{dw}{w-\sigma} \;=\; \log\left|\frac{z-\sigma}{x-\sigma}\right|^2.$$

Proof. See [BD1, §6.3]. □

PROPOSITION 7.8. *For every $0 \leq j \leq n$, the jth beta quotient of $I^{\mathbf{s}}(x,y)$ is*

$$\overline{I^{\mathbf{s}}(x,y)_j} = -\frac{1}{2\pi i}\iint_{\mathbb{C}} \left|\frac{z}{x}\right|^{2s_0} \prod_{k=1}^{n} \left|\frac{1-z\sigma_k^{-1}}{1-x\sigma_k^{-1}}\right|^{2s_k} \left(\frac{d\overline{z}}{\overline{z}-\overline{y}} - \frac{d\overline{z}}{\overline{z}-\overline{x}}\right) \wedge \frac{dz}{z-\sigma_j}. \tag{57}$$

This expression is a formal power series in the s_i obtained by expanding the exponentials as power series.

Proof. The following argument is slightly more intuitive using motivic, rather than de Rham periods, so we shall first compute $I^{\mathfrak{m}}_\gamma(x,y) \in {}_x\Pi^{\varpi}_y(\mathcal{P}^{\mathfrak{m}})$, the image under the universal comparison map (see §5.2(5)) of a path $\gamma \in \pi_1(\mathbb{C}\backslash\Sigma, x, y)$, and then use the projection

$$I^{\varpi}(x,y) = \pi^{\mathfrak{m},+}_{\varpi}\left(I^{\mathfrak{m}}_\gamma(x,y)\right)$$

to deduce a formula for $I^{\varpi}(x,y)$. Since x, y are ordinary basepoints, the motive underlying the torsor of paths is given by Beilinson's cosimplicial construction [DG, §3.3] and

$$I^{\mathfrak{m}}_\gamma(x,y) = \sum_{w \in \{e_0,\ldots,e_n\}^\times, |w|=\ell} [H^\ell(X^\ell_\Sigma, Y^\ell), [\gamma\Delta_\ell], w]^{\mathfrak{m}}\, w,$$

where $|w|$ denotes the length of a word w, and the divisor $Y^\ell \subset X^\ell_\Sigma$ is

$$Y^\ell = \{z_1 = x\} \cup \{z_1 = z_2\} \cup \cdots \cup \{z_{\ell-1} = z_\ell\} \cup \{z_\ell = y\},$$

and Δ_ℓ is the standard simplex

$$\Delta_\ell = \{t_i \in \mathbb{R} \;:\; 0 \leq t_1 \leq t_2 \leq \cdots \leq t_\ell \leq 1\} \subset \mathbb{R}^\ell.$$

The coordinates $z_1, \ldots, z_\ell$ are the coordinates on X^ℓ_Σ. By Lemma 6.9, the coefficient of $\frac{s_0^{m_0}}{m_0!}\ldots\frac{s_n^{m_n}}{m_n!}$ in $\overline{I^{\mathfrak{m}}_\gamma(x,y)_j}$ is

$$\left[H^{m+1}(X^{m+1}_\Sigma, Y^{m+1}) \;,\; [\gamma\Delta_{m+1}] \;,\; e_0^{\text{ш}\, m_0} \,\text{ш}\, \ldots \,\text{ш}\, e_n^{\text{ш}\, m_n} e_j\right]^{\mathfrak{m}},$$

where $m = m_0 + \cdots + m_n$. By expanding out the shuffle products, we get a sum of $m!$ terms indexed by permutations $\sigma \in \mathfrak{S}_m$. After permuting the coordinates, the sum can be rewritten as

$$\xi = \left[H^{m+1}(X^{m+1}_\Sigma \;,\; \widetilde{Y}^{m+1}) \;,\; [\gamma C_{m+1}] \;,\; e_0^{m_0}\ldots e_n^{m_n} e_j\right]^{\mathfrak{m}}, \tag{58}$$

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where

$$\widetilde{Y}^{m+1} = \bigcup_{\sigma\in\mathfrak{S}_m} \sigma Y^{m+1} \qquad \text{and} \qquad C_{m+1} = \bigcup_{\sigma\in\mathfrak{S}_m} \sigma\,\Delta_{m+1}$$

and $\sigma \in \mathfrak{S}_m$ ranges over permutations of all but the last coordinate, that is,

$$\sigma(z_1,\dots,z_m,z_{m+1}) = (z_{\sigma^{-1}(1)},\dots,z_{\sigma^{-1}(m)},z_{m+1}).$$

The union of the $m!$ simplices $\sigma\Delta_{m+1}$ glue together to form a cone

$$C_{m+1} = \{t_i \in \mathbb{R} : 0 \le t_1,\dots,t_m \le t_{m+1} \le 1\} \subset \mathbb{R}^{m+1}.$$

The boundary of $\gamma(C_{m+1})$ is contained in the complex points of the divisor $V \subset X_\Sigma^{m+1}$ defined by the union of $\{z_i = x\}$, $\{z_i = z_{m+1}\}$ for $1 \le i \le m$, and $\{z_{m+1} = y\}$. In (58), therefore, we can replace $H^{m+1}(X_\Sigma^{m+1},\widetilde{Y}^{m+1})$ with $H^{m+1}(X_\Sigma^{m+1},V)$. Now, take the image of (58) under the projection $\pi_\varpi^{\mathfrak{m},+}$. By [BD1, §4.4], the image of the homology framing under the rational period map $c_0^\vee$ studied in [BD1] is the differential form (writing $z = z_{m+1}$):

$$\nu = (-1)^{\frac{m(m+1)}{2}} \bigwedge_{i=1}^{m} \left(\frac{dz_i}{z_i - z} - \frac{dz_i}{z_i - x}\right) \wedge \left(\frac{dz}{z-y} - \frac{dz}{z-x}\right).$$

It follows, then, from Theorem 3.17 in [BD1] that the single-valued period of $\pi_\varpi^{\mathfrak{m},+}\xi$ is

$$\frac{(-1)^{\frac{m(m+1)}{2}}}{(-2\pi i)^{m+1}} \iint_{\mathbb{C}^{m+1}} \overline{\nu} \wedge \frac{dz_1}{z_1-\beta_1} \wedge \cdots \wedge \frac{dz_m}{z_m-\beta_m} \wedge \frac{dz}{z-\sigma_j},$$

where $(\beta_1,\dots,\beta_m)$ is $(0^{m_0},\sigma_1^{m_1},\dots,\sigma_n^{m_n})$ (a sequence of m_0 0's followed by m_1 σ_1's, and so on), corresponding to the differential form associated with the word $e_0^{m_0}\dots e_n^{m_n}e_j$ in the affine ring of the de Rham fundamental groupoid. Now, rearrange the integrand and apply Lemma 7.7 repeatedly to perform the m integrals:

$$\frac{-1}{2\pi i} \iint_{\mathbb{C}} \left(\frac{d\overline{z_i}}{\overline{z_i}-\overline{z}} - \frac{d\overline{z_i}}{\overline{z_i}-\overline{x}}\right) \wedge \frac{dz_i}{z_i-\beta_i} = \log\left(\left|\frac{z-\beta_i}{x-\beta_i}\right|^2\right)$$

for $1 \le i \le m$ to obtain

$$-\frac{1}{2\pi i} \iint_{\mathbb{C}} \log^{m_0}\left(\left|\frac{z}{x}\right|^2\right) \prod_{k=1}^{n} \log^{m_k}\left(\left|\frac{1-z\sigma_k^{-1}}{1-x\sigma_k^{-1}}\right|^2\right) \left(\frac{d\overline{z}}{\overline{z}-\overline{y}} - \frac{d\overline{z}}{\overline{z}-\overline{x}}\right) \wedge \frac{dz}{z-\sigma_j}.$$

This yields (57) after expanding the exponential factors as power series in the s_i. □

Our next step is to replace x by t_0 and y by $-t_i$ in (57). This is slightly subtle because Beilinsons's description of the motivic fundamental group with finite basepoints that was used in the proof of Proposition 7.8 is not available in the case of tangential basepoints. Thus, we will proceed as in [DG, §4] and use the composition of paths to travel between two tangential basepoints by using finite basepoints as intermediate steps. We will thus need to understand the behaviour of single-valued versions of iterated integrals between a tangential basepoint and an infinitesimally close finite basepoint.

We will consider non-commutative power series $F(\tau) \in \mathbb{C}\langle\langle e_0,e_1,\dots,e_n\rangle\rangle$ depending on a '*small*' rational point $\tau \in k^\times$ and having constant coefficient 1. We say that such a series $F(\tau)$ has *logarithmic growth* if each of its coefficients $a(\tau)$ satisfies $a(\tau) = O(\log^r|\tau|)$ for

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some integer r (that may depend on the coefficient) when $\tau \to 0$. We say that such a series $F(\tau)$ is *asymptotic to* 1 and write $F(\tau) \sim_{\tau \to 0} 1$ if each of its nonconstant coefficients $a(\tau)$ satisfies $a(\tau) = O(|\tau|^{1-\varepsilon})$ for every $\varepsilon > 0$ when $\tau \to 0$. The class of series with logarithmic growth and the subclass of series asymptotic to 1 are stable under products and inversion. This implies that the following relation is an equivalence relation that is compatible with products:

$$F(\tau) \underset{\tau \to 0}{\sim} G(\tau) \qquad \Leftrightarrow \qquad F(\tau) G(\tau)^{-1} \underset{\tau \to 0}{\sim} 1.$$

LEMMA 7.9. *We have, for $\tau \in X_\Sigma(k)$,*

$$I^{\mathbf{s}}(t_0, \tau) \underset{\tau \to 0}{\sim} \exp(e_0 \log |\tau|^2).$$

Proof. By a slight generalization of [B2, (5.4)] to the case of the projective line minus several points Σ (which follows, e.g., by the argument in [B2, 6.3]), we have an expression of the form

$$I^{\mathbf{s}}(t_0, \tau) = I(t_0, \tau) \widetilde{I}(t_0, \tau),$$

where $\widetilde{I}$ is the complex conjugate of the series $I(t_0, \tau)$, in which the letters e_i, for $i \geq 1$, are replaced with certain power series, the letter e_0 is unchanged, and all words are reversed. Note that in that paper, the order is reversed because iterated integrals were computed from left to right, but it makes no difference to the conclusion of the lemma. Since

$$I(t_0, \tau) \underset{\tau \to 0}{\sim} \exp(e_0 \log \tau),$$

it follows also that $\widetilde{I}(t_0, \tau) \underset{\tau \to 0}{\sim} \exp(e_0 \log \overline{\tau})$ and hence $I^{\mathbf{s}}(t_0, \tau) \underset{\tau \to 0}{\sim} \exp(e_0 \log |\tau|^2)$. □

For all $1 \leq j \leq n$, we define

$$\Omega_j^{\mathbf{s}} = |z|^{2s_0} \prod_{k=1}^{n} |1 - z\sigma_k^{-1}|^{2s_k} \frac{dz}{z - \sigma_j}. \tag{59}$$

Let $\Omega_j^{\mathbf{s}, \mathrm{ren}_i}$ denote its renormalized versions (Definition 4.6) with respect to $\{0, \sigma_i\}$, given by

$$\Omega_j^{\mathbf{s}, \mathrm{ren}, i} = \Big(|z|^{2s_0} \prod_{k \neq i} |1 - z\sigma_k^{-1}|^{2s_k} - \mathbf{1}_{i=j} |\sigma_i|^{2s_0} \prod_{k \neq i} |1 - \sigma_i \sigma_k^{-1}|^{2s_k} \Big) |1 - z\sigma_i^{-1}|^{2s_i} \frac{dz}{z - \sigma_j} \cdot$$

According to Proposition 4.9, the integral $-\frac{1}{2\pi i} \iint_{\mathbb{C}} \left(\frac{d\overline{z}}{\overline{z} - \overline{\sigma_i}} - \frac{d\overline{z}}{\overline{z}} \right) \wedge \Omega_j^{\mathbf{s}, \mathrm{ren}_i}$ defines a holomorphic function of the parameters $\underline{s}$ around $\underline{s} = 0$ and thus has a Taylor expansion. We now identify this Taylor expansion with the jth beta quotient $\overline{\mathcal{Z}_j^{i, \mathbf{s}}}$ of the generalized single-valued associator $\mathcal{Z}^{i, \mathbf{s}}$.

PROPOSITION 7.10. *For all $1 \leq j \leq n$, the jth beta quotient of $\overline{\mathcal{Z}_j^{i, \mathbf{s}}}$ is the Taylor series at $\underline{s} = 0$ of the integral*

$$-\frac{1}{2\pi i} \iint_{\mathbb{C}} \left(\frac{d\overline{z}}{\overline{z} - \overline{\sigma_i}} - \frac{d\overline{z}}{\overline{z}} \right) \wedge \Omega_j^{\mathbf{s}, \mathrm{ren}_i}. \tag{60}$$

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Proof. By the composition of paths formula, we have, for any small enough $\tau \in X_\Sigma(k)$, the equality

$$\mathcal{Z}^{i,\mathbf{s}} = I^{\mathbf{s}}(t_0,\tau)\, I^{\mathbf{s}}(\tau,\sigma_i-\tau)\, I^{\mathbf{s}}(\sigma_i-\tau,-t_i). \tag{61}$$

Now, Lemma 7.9 implies that we have

$$I^{\mathbf{s}}(t_0,\tau) \underset{\tau\to 0}{\sim} \exp(e_0 \log|\tau|^2) \qquad \text{and} \qquad I^{\mathbf{s}}(\sigma_i-\tau,-t_i) \underset{\tau\to 0}{\sim} \exp(-e_i\log|\tau|^2 + e_i \log|\sigma_i|^2).$$

(For the second claim, one needs to apply the change of variables $x \mapsto 1-x\sigma_i^{-1}$.) Since $\mathcal{Z}^{i,\mathbf{s}}$ is independent of τ, we conclude that

$$\overline{\mathcal{Z}^{i,\mathbf{s}}_j} = \mathrm{Reg}_{\tau\to 0}\, \overline{(I^{\mathbf{s}}(\tau,\sigma_i-\tau)\exp(e_i\log|\sigma_i|^2))_j},$$

where $\mathrm{Reg}_{\tau\to 0}$ means the following: formally set $\log|\tau|^2$ to zero, and then take the limit as $\tau\to 0$. Since the coefficients in the formal power series $I^{\mathbf{s}}(\tau,\sigma_i-\tau)$ can be expressed as elements in $\mathbb{C}[[\tau,\overline{\tau}]][\log|\tau|^2]$, this operation is well defined. By (44), we thus get

$$\overline{\mathcal{Z}^{i,\mathbf{s}}_j} = \mathrm{Reg}_{\tau\to 0}\left(\overline{I^{\mathbf{s}}(\tau,\sigma_i-\tau)}\,\overline{(|\sigma_i|^{2e_i})_j} + \overline{I^{\mathbf{s}}(\tau,\sigma_i-\tau)_j}\right). \tag{62}$$

Note that by (56) we have

$$\overline{I^{\mathbf{s}}(\tau,\sigma_i-\tau)} = \left|\frac{\sigma_i-\tau}{\tau}\right|^{2s_0} \prod_{k=1}^{n} \left|\frac{1-(\sigma_i-\tau)\sigma_k^{-1}}{1-\tau\sigma_k^{-1}}\right|^{2s_k},$$

so that

$$\mathrm{Reg}_{\tau\to 0}(\overline{I^{\mathbf{s}}(\tau,\sigma_i-\tau)}) = |\sigma_i|^{2s_0}\,|\sigma_i|^{-2s_i} \prod_{k\neq i} |1-\sigma_i\sigma_k^{-1}|^{2s_k}.$$

Indeed, the factors $|\tau|^{2s_0}$ and $|\tau|^{2s_i}$ are sent to 1 by the regularization map. We also have $\overline{(|\sigma_i|^{2e_i})_j} = \mathbf{1}_{i=j}\frac{1}{s_i}(|\sigma_i|^{2s_i}-1)$, which yields

$$\mathrm{Reg}_{\tau\to 0}(\overline{I^{\mathbf{s}}(\tau,\sigma_i-\tau)}\,\overline{(|\sigma_i|^{2e_i})_j}) = \mathbf{1}_{i=j}|\sigma_i|^{2s_0}\left(\prod_{k\neq i}|1-\sigma_i\sigma_k^{-1}|^{2s_k}\right)\frac{1}{s_i}(1-|\sigma_i|^{-2s_i})\cdot \tag{63}$$

Using a variant of Lemma 4.7, we interpret the right-most factor as

$$\frac{1}{s_i}\left(1-|\sigma_i|^{-2s_i}\right) = \mathrm{Reg}_{\tau\to 0}\left(\frac{1}{2\pi i}\iint_{\mathbb{C}} \left|1-z\sigma_i^{-1}\right|^{2s_i}\left(\frac{d\overline{z}}{\overline{z}-\overline{\sigma_i}+\overline{\tau}} - \frac{d\overline{z}}{\overline{z}}\right)\wedge\frac{dz}{z-\sigma_i}\right).$$

This allows us to rewrite (63) as

$$\mathrm{Reg}_{\tau\to 0}\left(\frac{1}{2\pi i}\iint_{\mathbb{C}} \mathbf{1}_{i=j}|\sigma_i|^{2s_0}\prod_{k\neq i}|1-\sigma_i\sigma_k^{-1}|^{2s_k}\times|1-z\sigma_i^{-1}|^{2s_i}\left(\frac{d\overline{z}}{\overline{z}-\overline{\sigma_i}+\overline{\tau}}-\frac{d\overline{z}}{\overline{z}}\right)\wedge\frac{dz}{z-\sigma_i}\right). \tag{64}$$

By Proposition 7.8,

$$\overline{I^{\mathbf{s}}(\tau,\sigma_i-\tau)_j} = -\frac{1}{2\pi i}\iint_{\mathbb{C}}\left(\left|\frac{z}{\tau}\right|^{2s_0}\prod_{k=1}^{n}\left|\frac{1-z\sigma_k^{-1}}{1-\tau\sigma_k^{-1}}\right|^{2s_k}\right)\left(\frac{d\overline{z}}{\overline{z}-\overline{\sigma_i}+\overline{\tau}}-\frac{d\overline{z}}{\overline{z}-\overline{\tau}}\right)\wedge\frac{dz}{z-\sigma_j},$$

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which implies that

$$\mathrm{Reg}_{\tau\to 0}(\overline{I^{\mathbf{s}}(\tau,\sigma_i-\tau)_j}) = \mathrm{Reg}_{\tau\to 0}\left(-\frac{1}{2\pi i}\iint_{\mathbb{C}}\left(\frac{d\overline{z}}{\overline{z}-\overline{\sigma_i}+\overline{\tau}}-\frac{d\overline{z}}{\overline{z}}\right)\wedge\Omega_j^{\mathbf{s}}\right)\ , \tag{65}$$

since one has

$$\mathrm{Reg}_{\tau\to 0}\left(\left|\frac{z}{\tau}\right|^{2s_0}\prod_{k=1}^{n}\left|\frac{1-z\sigma_k^{-1}}{1-\tau\sigma_k^{-1}}\right|^{2s_k}\right)\frac{dz}{z-\sigma_j} = \Omega_j^{\mathbf{s}}$$

and the left-hand side can be expanded out in terms in $\tau,\overline{\tau}$ and $\log|\tau|^2$, which factor out of the integral. Resubstituting (64) and (65) into (62) yields

$$\overline{\mathcal{Z}_j^{i,\mathbf{s}}} = \mathrm{Reg}_{\tau\to 0}\left(-\frac{1}{2\pi i}\iint_{\mathbb{C}}\left(\frac{d\overline{z}}{\overline{z}-\overline{\sigma_i}+\overline{\tau}}-\frac{d\overline{z}}{\overline{z}}\right)\wedge\Omega_j^{\mathrm{ren},i}\right) = -\frac{1}{2\pi i}\iint_{\mathbb{C}}\left(\frac{d\overline{z}}{\overline{z}-\overline{\sigma_i}}-\frac{d\overline{z}}{\overline{z}}\right)\wedge\Omega_j^{\mathrm{ren},i}.$$

Here, $\mathrm{Reg}_{\tau\to 0}$ is simply the limit when $\tau\to 0$ since the last integral is convergent by Proposition 4.9. The claim follows. □

7.3 Comparing $L^{\mathbf{s}}_{\Sigma}$ and $FL^{\mathbf{s}}_{\Sigma}$

We are now ready to prove Theorem 1.1(ii) from the introduction.

THEOREM 7.11. *For every i,j, $(FL^{\mathbf{s}}_{\Sigma})_{ij}$ is the Taylor series at $\underline{s}=0$ of the single-valued Lauricella function $(L^{\mathbf{s}}_{\Sigma})_{ij}$.*

Proof. This follows from comparing the definition of $(FL^{\mathbf{s}}_{\Sigma})_{ij}$ with the expression in Proposition 4.9 and using the expressions for $\overline{\mathcal{Z}^{i,\mathbf{s}}}$ and $\overline{\mathcal{Z}_j^{i,\mathbf{s}}}$ from Propositions 7.6 and 7.10, respectively. □

§8. *Local* motivic coaction

We compute the action of the motivic Galois group (or equivalently, the motivic coaction) on the full motivic torsor of paths, and use it to deduce a formula for the *local* coaction on the beta quotients and on the Lauricella functions viewed as formal power series in their parameters.

8.1 Formula for the motivic Galois action

Since the ${}_i\Pi_j^{\varpi}$ are (dual to) realizations of ind-objects in the Tannakian category $\mathcal{MT}(k)$, they admit an action of the motivic Galois group. More precisely, the Galois group $G^{\varpi}_{\mathcal{MT}(k)}$ acts *on the left* on the $\mathbb{Q}$-algebra $\mathcal{O}({}_0\Pi_i^{\varpi})$ and thus naturally acts *on the right* on the set of points ${}_0\Pi_i^{\varpi}(R)$ for every $\mathbb{Q}$-algebra R. Let

$$\lambda : G^{\varpi}_{\mathcal{MT}(k)} \to \mathbb{G}_m\ ,\ g\mapsto\lambda_g$$

denote the homomorphism given by the action of $G^{\varpi}_{\mathcal{MT}(k)}$ on $\varpi(\mathbb{Q}(-1))=\mathbb{Q}$.

PROPOSITION 8.1. *Let R be any $\mathbb{Q}$-algebra. For every element $g\in G^{\varpi}_{\mathcal{MT}(k)}(R)$, its right action on any $F\in{}_0\Pi_i^{\varpi}(R)$ is given by a version of Ihara's formula:*

$$(F\cdot g)(e_0,e_1,\dots,e_n) = F\left(\lambda_g e_0,\lambda_g G_1 e_1 G_1^{-1},\dots,\lambda_g G_n e_n G_n^{-1}\right)G_i, \tag{66}$$

where $G_k\in\mathbb{Q}\langle\langle e_0,\dots,e_n\rangle\rangle$ is the group-like formal power series $G_k={}_01_k\cdot g$ for all $1\le k\le n$.

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Proof. The argument is a very mild generalization of the argument given in [DG, §5] (with the reverse conventions) or [B1, Prop. 2.5], so we shall be brief. One first computes the action of g on ${}_0\Pi_0^{\varpi}$. It acts on the element $\exp(e_0) \in {}_0\Pi_0^{\varpi}$ by scaling

$$\exp(e_0) \cdot g = \exp(\lambda_g e_0),$$

since $\exp(e_0)$ is in the image of the local monodromy $\pi_1^{\mathrm{mot}}(\mathbb{G}_m, 1)$, which is isomorphic to $\mathbb{Q}(1) = \mathbb{Q}(-1)^{\vee}$. The fact that we get a λ_g and not a λ_g^{-1} is because g acts on the right. Another way to see this is that g acts on the element *coefficient of* e_0^n, which lies in $\varpi(\mathbb{Q}(-n))$, by λ_g^n. For all $1 \leq i \leq n$, the element $\exp(e_i) \in {}_i\Pi_i^{\varpi}$ is in the image of the local monodromy

$$x \mapsto \sigma_i x : (\mathbb{G}_m, 1) \longrightarrow ((T_{\sigma_i}\mathbb{A}^1_k)^{\times}, t_i),$$

and hence, by a similar argument, is also acted upon by g by scaling $\exp(e_i) \cdot g = \exp(\lambda_g e_i)$. We transport this action back to ${}_0\Pi_0^{\varpi}$ via

$${}_i\Pi_i^{\varpi} \longrightarrow {}_0\Pi_0^{\varpi} \quad , \quad {}_iF_i \mapsto ({}_01_i)\,{}_iF_i\,({}_01_i)^{-1},$$

where ${}_xF_y \in {}_x\Pi_y^{\varpi}$ denotes the element defined by a power series $F \in \mathbb{Q}\langle\langle e_0, \ldots, e_n\rangle\rangle$. Since the action of the motivic Galois group is compatible with the composition of paths, we deduce that g acts on $\exp(e_i) \in {}_0\Pi_0^{\varpi}$ via $\exp(e_i) \mapsto G_i \exp(\lambda_g e_i) G_i^{-1} = \exp(\lambda_g G_i e_i G_i^{-1})$ for all $1 \leq i \leq n$ since G_i is by definition ${}_01_i \cdot g$. Finally, use the torsor structure

$${}_0\Pi_0^{\varpi} \longrightarrow {}_0\Pi_i^{\varpi} \quad , \quad {}_0F_0 \mapsto {}_0F_0\,{}_01_i$$

to conclude that the action of g on any $F \in {}_0\Pi_i^{\varpi}$ is indeed as claimed. □

The motivic Galois group acts in (at least) two different ways on the set ${}_0\Pi_i^{\varpi}(\mathcal{P}^{\mathfrak{m}})$:

1. *on the right* via the *Ihara action* (66) for $R = \mathcal{P}^{\mathfrak{m}}$, described in 66;
2. *on the left* via its action *on the coefficients* $\mathcal{P}^{\mathfrak{m}}$, that is, term by term on the coefficients of formal power series in $\mathcal{P}^{\mathfrak{m}}\langle\langle e_0, \ldots, e_n\rangle\rangle$.

We are interested in computing the action (2) on the generalized motivic associators $\mathcal{Z}^{i,\mathfrak{m}}$. The next lemma shows that this action is equivalent to the action (1) on these elements.

LEMMA 8.2. *The Ihara action (1) and the action on the coefficients (2) coincide on the motivic associator* $\mathcal{Z}^{i,\mathfrak{m}}$*, that is, we have, for every* $g \in G^{\varpi}_{\mathcal{MT}(k)}(\mathbb{Q})$*,*

$$\mathcal{Z}^{i,\mathfrak{m}} \cdot g = g \cdot \mathcal{Z}^{i,\mathfrak{m}}.$$

Proof. We have a map $\mathcal{O}({}_0\Pi_i^{\varpi}) \to \mathcal{P}^{\mathfrak{m}}$ that sends a word w to the motivic period $[\mathcal{O}({}_0\Pi_i^{\mathrm{mot}}), \gamma_i, w]^{\mathfrak{m}}$. It is left $G^{\varpi}_{\mathcal{MT}(k)}$-equivariant by definition, and induces a map

$${}_0\Pi_i^{\varpi}(\mathcal{O}({}_0\Pi_i^{\varpi})) \longrightarrow {}_0\Pi_i^{\varpi}(\mathcal{P}^{\mathfrak{m}}) \tag{67}$$

that is also left $G^{\varpi}_{\mathcal{MT}(k)}$-equivariant, where $G^{\varpi}_{\mathcal{MT}(k)}$ acts on the coefficients, that is, by (2). It is also obviously right $G^{\varpi}_{\mathcal{MT}(k)}$-equivariant for the action (1). By definition, $\mathcal{Z}^{i,\mathfrak{m}}$ is the image under (67) of the element $\Phi^i \in {}_0\Pi_i^{\varpi}(\mathcal{O}({}_0\Pi_i^{\varpi}))$ which corresponds to the map $\mathrm{id} : \mathcal{O}({}_0\Pi_i^{\varpi}) \to \mathcal{O}({}_0\Pi_i^{\varpi})$. It is thus enough to prove the claim that $\Phi^i \cdot g = g \cdot \Phi^i$, where the right and left actions of $G^{\varpi}_{\mathcal{MT}(k)}$ on ${}_0\Pi_i^{\varpi}(\mathcal{O}({}_0\Pi_i^{\varpi})) = \mathrm{Hom}_{\mathrm{Alg}}(\mathcal{O}({}_0\Pi_i^{\varpi}), \mathcal{O}({}_0\Pi_i^{\varpi}))$ are on the source and on the target, respectively. It is obvious that $\mathrm{id} \cdot g = g \cdot \mathrm{id}$, from which the claim follows. □

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Applying 66 to the series $F = \mathcal{Z}^{i,\mathfrak{m}}$ and using Lemma 8.2, we deduce that the action (2) of $G^{\varpi}_{\mathcal{MT}(k)}$ term by term on the coefficients of $\mathcal{Z}^{i,\mathfrak{m}}$ is given by the formula

$$g \cdot \mathcal{Z}^{i,\mathfrak{m}}(e_0, e_1, \dots, e_n) = \mathcal{Z}^{i,\mathfrak{m}}\left(\lambda_g e_0, \lambda_g G_1 e_1 G_1^{-1}, \dots, \lambda_g G_n e_n G_n^{-1}\right) G_i \ , \tag{68}$$

where $G_k = {}_0 1_i \cdot g$. The same formula holds with $\mathfrak{m}$ replaced by ϖ. This formula can be re-expressed as a universal coaction formula for

$$\Delta \mathcal{Z}^{i,\mathfrak{m}} \in (\mathcal{P}^{\mathfrak{m}} \otimes_{\mathbb{Q}} \mathcal{P}^{\varpi}) \langle\langle e_0, \dots, e_n \rangle\rangle,$$

where Δ is applied term by term to each coefficient and acts trivially on the e_i. The action of g is retrieved by the usual formula $g \cdot \mathcal{Z}^{i,\mathfrak{m}} = (\mathrm{id} \otimes g)\, \Delta \mathcal{Z}^{i,\mathfrak{m}}$.

Proposition 8.3. *The coaction* $\Delta : \mathcal{P}^{\mathfrak{m}} \langle\langle e_0, e_1, \dots, e_n \rangle\rangle \to (\mathcal{P}^{\mathfrak{m}} \otimes \mathcal{P}^{\varpi}) \langle\langle e_0, e_1, \dots, e_n \rangle\rangle$ *applied to the generalized motivic associator* $\mathcal{Z}^{i,\mathfrak{m}}$ *is*

$$\Delta \mathcal{Z}^{i,\mathfrak{m}} = \mathcal{Z}^{i,\mathfrak{m}}\left(\mathbb{L}^{\varpi} e_0, \mathbb{L}^{\varpi} e'_1, \dots, \mathbb{L}^{\varpi} e'_n\right) \mathcal{Z}^{i,\varpi}, \tag{69}$$

where $e'_1, \dots, e'_n$ *are defined by*

$$e'_k = \left(\mathcal{Z}^{k,\varpi}\right) e_k \left(\mathcal{Z}^{k,\varpi}\right)^{-1} \qquad \text{for all} \quad 1 \leq k \leq n. \tag{70}$$

In the right-hand side of (69), we view $\mathcal{Z}^{i,\mathfrak{m}}$ inside $(\mathcal{P}^{\mathfrak{m}} \otimes \mathcal{P}^{\varpi}) \langle\langle e_0, e_1, \dots, e_n \rangle\rangle$ via the natural inclusion $\mathcal{P}^{\mathfrak{m}} \langle\langle e_0, e_1, \dots, e_n \rangle\rangle \subset (\mathcal{P}^{\mathfrak{m}} \otimes \mathcal{P}^{\varpi}) \langle\langle e_0, e_1, \dots, e_n \rangle\rangle$ which replaces every coefficient $a^{\mathfrak{m}}$ with $a^{\mathfrak{m}} \otimes 1$. Similarly, the terms $\mathbb{L}^{\varpi} e_0$, $\mathbb{L}^{\varpi} e'_k$, and $\mathcal{Z}^{i,\varpi}$ are viewed in $(\mathcal{P}^{\mathfrak{m}} \otimes \mathcal{P}^{\varpi}) \langle\langle e_0, e_1, \dots, e_n \rangle\rangle$ via the natural inclusion $\mathcal{P}^{\varpi} \langle\langle e_0, \dots, e_n \rangle\rangle \subset (\mathcal{P}^{\mathfrak{m}} \otimes \mathcal{P}^{\varpi}) \langle\langle e_0, e_1, \dots, e_n \rangle\rangle$ which replaces every coefficient b^{ϖ} with $1 \otimes b^{\varpi}$. Thus, we interpret the term $\mathcal{Z}^{i,\mathfrak{m}}\left(\mathbb{L}^{\varpi} e_0, \mathbb{L}^{\varpi} e'_1, \dots, \mathbb{L}^{\varpi} e'_n\right)$ as a composition of noncommutative formal power series with coefficients in the ring $\mathcal{P}^{\mathfrak{m}} \otimes \mathcal{P}^{\varpi}$. This composition makes sense since e_0 and the e'_k have vanishing constant coefficient. All products in the formulae in the proposition are given by concatenation of non-commutative formal power series.

Proof. For all $g \in G^{\varpi}_{\mathcal{MT}(k)}$, $G_k = {}_0 1_k \cdot g$ is obtained by applying to g the function (more precisely the power series of functions) $\mathcal{Z}^{k,\varpi} \in {}_0\Pi^{\varpi}_k(\mathcal{O}(G^{\varpi}_{\mathcal{MT}(k)}))$, and λ_g is obtained by applying to g the function $\mathbb{L}^{\varpi}$. The claim follows from equation (68). ∎

Example 8.4. In the setting of Examples 6.4 and 7.4, Proposition 8.3 yields

$$\Delta \mathcal{Z}^{\mathfrak{m}}(e_0, e_1) = \mathcal{Z}^{\mathfrak{m}}\left(\mathbb{L}^{\mathfrak{dr}} e_0, \mathbb{L}^{\mathfrak{dr}} \mathcal{Z}^{\mathfrak{dr}} e_1 \left(\mathcal{Z}^{\mathfrak{dr}}\right)^{-1}\right) \mathcal{Z}^{\mathfrak{dr}}(e_0, e_1),$$

which is a motivic version of Ihara's formula, and expresses the coaction on motivic multiple zeta values. For example, reading off the coefficient of $-e_1 e_0^{n-1}$ yields

$$\Delta \zeta^{\mathfrak{m}}(n) = \zeta^{\mathfrak{m}}(n) \otimes (\mathbb{L}^{\mathfrak{dr}})^n + 1 \otimes \zeta^{\mathfrak{dr}}(n).$$

Note that this provides a very efficient method of computing the coaction.

8.2 Motivic coaction on the beta quotients

We want to use Proposition 8.3 to derive a formula for the motivic coaction on the series $FL^{\mathfrak{m}}_{\Sigma}(s_0, \dots, s_n)$. We now make a slight modification and consider the *normalized coaction*

$$\Delta_{\mathrm{nor}} : \mathcal{P}^{\mathfrak{m}}[[s_0, \dots, s_n]] \to (\mathcal{P}^{\mathfrak{m}} \otimes \mathcal{P}^{\varpi})[[s_0, \dots, s_n]]$$

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obtained by acting via Δ on each coefficient and on the formal variables s_k by

$$\Delta_{\mathrm{nor}}(s_k) = (1 \otimes (\mathbb{L}^{\varpi})^{-1})\, s_k.$$

This is equivalent to viewing the variables s_k as spanning a copy of $\mathbb{Q}(1)_{\mathrm{dR}}$. In particular, they have motivic weight -2. This is the correct normalisation which makes the factors

$$y^{s_k} = \exp(s_k \log(y)) = \sum_{n\geq 0} \frac{\log^n(y)}{n!} s^n$$

have total weight 0, and all the coefficients in the expansion of $FL^{\mathfrak{m}}_{\Sigma}$ have total weight 0. Another effect of this normalization is that it makes the coaction formulae below land in the subspace

$$\mathcal{P}^{\mathfrak{m}}[[s_0,\dots,s_n]] \otimes_{\mathbb{Q}[[s_0,\dots,s_n]]} \mathcal{P}^{\varpi}[[s_0,\dots,s_n]] \;\subset\; (\mathcal{P}^{\mathfrak{m}} \otimes_{\mathbb{Q}} \mathcal{P}^{\varpi})[[s_0,\dots,s_n]].$$

Note that the tensor product on the left-hand side is an ordinary, not a completed, tensor product. It is a highly restrictive condition for an element in the space on the right-hand side to lie in the subspace defined by the left-hand side. It is a crucial, and nontrivial fact, that this condition is satisfied for the image of the motivic coaction on generalized associators. Indeed, this can already be seen from the abelianization of (69):

$$\Delta_{\mathrm{nor}} \overline{\mathcal{Z}^{i,\mathfrak{m}}}(s_0,\dots,s_n) = \overline{\mathcal{Z}^{i,\mathfrak{m}}}(s_0,\dots,s_n) \otimes \overline{\mathcal{Z}^{i,\varpi}}((\mathbb{L}^{\varpi})^{-1}s_0,\dots,(\mathbb{L}^{\varpi})^{-1}s_n). \tag{71}$$

THEOREM 8.5. *The (normalized) motivic coaction, applied to the entries of $FL^{\mathfrak{m}}_{\Sigma}$, satisfies*

$$\Delta_{\mathrm{nor}} FL^{\mathfrak{m}}_{\Sigma}(s_0,\dots,s_n) = FL^{\mathfrak{m}}_{\Sigma}(s_0,\dots,s_n) \otimes FL^{\varpi}_{\Sigma}((\mathbb{L}^{\varpi})^{-1}s_0,\dots,(\mathbb{L}^{\varpi})^{-1}s_n). \tag{72}$$

Proof. It is convenient to compute modulo $\mathbb{L}^{\varpi} = 1$ and restore all powers of $\mathbb{L}^{\varpi}$ at the end, since they are uniquely determined by the weight grading. Using formula (69), we have

$$\Delta_{\mathrm{nor}} \overline{\mathcal{Z}^{i,\mathfrak{m}}_j} = \overline{(\mathcal{Z}^{i,\mathfrak{m}}(e_0,e'_1,\dots,e'_n)\,\mathcal{Z}^{i,\varpi})_j}.$$

The right-hand side reduces via (44) to

$$\overline{\mathcal{Z}^{i,\mathfrak{m}}}(s_0,s_1,\dots,s_n) \otimes \overline{\mathcal{Z}^{i,\varpi}_j}(s_0,s_1,\dots,s_n) + \overline{\mathcal{Z}^{i,\mathfrak{m}}(e_0,e'_1,\dots,e'_n)_j},$$

since e'_k is conjugate to e_k via equation (70) and so they have the same image $\overline{e_k} = \overline{e'_k} = s_k$ under abelianization. The previous expression can in turn be written as

$$\overline{\mathcal{Z}^{i,\mathfrak{m}}} \otimes \overline{\mathcal{Z}^{i,\varpi}_j} + \sum_{k=1}^{n} \overline{\mathcal{Z}^{i,\mathfrak{m}}_k} \otimes \overline{(e'_k)_j} \;=\; \overline{\mathcal{Z}^{i,\mathfrak{m}}} \otimes \overline{\mathcal{Z}^{i,\varpi}_j} + \sum_{k=1}^{n} \overline{\mathcal{Z}^{i,\mathfrak{m}}_k} \otimes \overline{\left(\mathcal{Z}^{k,\varpi} e_k (\mathcal{Z}^{k,\varpi})^{-1}\right)_j},$$

by applying definition (43) to $\overline{\mathcal{Z}^{i,\mathfrak{m}}(e'_0,e'_1,\dots,e'_n)_j}$ and using (70). We have

$$\overline{\left(\mathcal{Z}^{k,\varpi} e_k (\mathcal{Z}^{k,\varpi})^{-1}\right)_j} \overset{(44)}{=} \overline{\mathcal{Z}^{k,\varpi} e_k} \overline{\left((\mathcal{Z}^{k,\varpi})^{-1}\right)_j} + \overline{\mathcal{Z}^{k,\varpi}} \mathbf{1}_{j=k} \overset{(45)}{=} -s_k \overline{\mathcal{Z}^{k,\varpi}_j} + \mathbf{1}_{j=k} \overline{\mathcal{Z}^{j,\varpi}}.$$

Putting the pieces together and multiplying by $-s_j$, we get

$$\Delta_{\mathrm{nor}} \left(-s_j \overline{\mathcal{Z}^{i,\mathfrak{m}}_j}\right) = -s_j \overline{\mathcal{Z}^{i,\mathfrak{m}}} \otimes \overline{\mathcal{Z}^{i,\varpi}_j} - s_j \overline{\mathcal{Z}^{i,\mathfrak{m}}_j} \otimes \overline{\mathcal{Z}^{j,\varpi}} + \sum_{k=1}^{n} s_k \overline{\mathcal{Z}^{i,\mathfrak{m}}_k} \otimes s_j \overline{\mathcal{Z}^{k,\varpi}_j}. \tag{73}$$

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Now, by (71), we have

$$\Delta_{\mathrm{nor}}\overline{\mathcal{Z}^{i,\mathfrak{m}}} = \overline{\mathcal{Z}^{i,\mathfrak{m}}} \otimes \overline{\mathcal{Z}^{i,\varpi}}. \tag{74}$$

Substituting (73) and (74) into the definitions $(FL^{\bullet}_{\Sigma})_{ij} = \mathbf{1}_{i=j}\overline{\mathcal{Z}^{i,\bullet}} - s_j\overline{\mathcal{Z}^{i,\bullet}_j}$, we get

$$\Delta_{\mathrm{nor}} FL^{\mathfrak{m}}_{\Sigma} = FL^{\mathfrak{m}}_{\Sigma} \otimes FL^{\varpi}_{\Sigma}.$$

On the other hand, homogeneity in the weight forces the right-hand side of the coaction to have weight equal to the degree in the s_i. This determines the powers of $\mathbb{L}^{\varpi}$ as in equation (72). □

Remark 8.6. The (normalized) coproduct in $\mathcal{P}^{\varpi}[[s_0,\dots,s_n]]$ is given on the elements FL^{ϖ}_{Σ} by a formula similar to (72): $\Delta_{\mathrm{nor}} FL^{\varpi}_{\Sigma}(s_0,\dots,s_n) = FL^{\varpi}_{\Sigma}(s_0,\dots,s_n) \otimes FL^{\varpi}_{\Sigma}((\mathbb{L}^{\varpi})^{-1}s_0,\dots,(\mathbb{L}^{\varpi})^{-1}s_n)$.

§9. Example: single-valued version of ${}_2F_1$ and double copy formula

A large family of special functions commonly found in the mathematical literature can be derived as special cases or limits of the Gauss hypergeometric function. As a result, one can derive single-valued versions for a range of special functions from the single-valued versions of the hypergeometric functions (106) and (107), which we shall prove here.

9.1 The Gauss hypergeometric function

We denote it by $F = {}_2F_1$. It is defined for $y \in \mathbb{C}$, $|y| < 1$, by the power series

$$F(a,b,c;y) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{y^n}{n!}, \tag{75}$$

where $(x)_n = \prod_{i=1}^{n}(x+i-1)$ is the rising Pochhammer symbol. It is a solution of the famous hypergeometric differential equation

$$\left(y(1-y)\frac{d^2}{dy^2} + (c-(a+b+1)y)\frac{d}{dy} - ab\right) F(a,b,c;y) = 0. \tag{76}$$

9.1.1. Integral representation

Traditionally, F is viewed as a function of y for fixed values of the exponents a,b,c. In this case, it admits an analytic continuation to a multivalued function on $\mathbb{C}\backslash\{0,1\}$ via the following integral representation which is valid for $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$:

$$F(a,b,c;y) = \frac{1}{\beta(b,c-b)} \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-yx)^{-a} dx \ , \tag{77}$$

where β denotes Euler's beta function.

To fix branches, it is convenient to assume that $y \notin \mathbb{R}_{>0}$. The path of integration in (77) can then be chosen to be the line segment $(0,1)$, and the branch of $(1-yx)^{-a} = \exp(-a\log(1-yx))$ is determined by $\log(1-yx) = \int_0^x d\log(1-yu)$ for $x \in (0,1)$.

We multiply through by the β factor and set

$$\mathcal{F}(a,b,c;y) = \beta(b,c-b)\,F(a,b,c;y) = \int_0^1 x^b(1-x)^{c-b}(1-yx)^{-a}\frac{dx}{x(1-x)} \ . \tag{78}$$

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It can be expressed in terms of the Lauricella functions (1) for $\Sigma = \{\sigma_0, \sigma_1, \sigma_2\}$ with

$$\sigma_0 = 0\,, \quad \sigma_1 = 1\,, \quad \sigma_2 = y^{-1} \qquad \text{and} \qquad s_0 = b\,, \quad s_1 = c-b\,, \quad s_2 = -a.$$

Indeed, we find that

$$\mathcal{F}(a,b,c;y) = \frac{c}{b(c-b)} L_{11} + \frac{1}{b} L_{12}, \tag{79}$$

where we use the shorthand notation

$$L_{ij} = \big(L_{\{0,1,y^{-1}\}}(b, c-b, -a)\big)_{ij}\,. \tag{80}$$

Equation (79) can be proved by writing $\frac{dx}{x(1-x)} = \frac{dx}{x} + \frac{dx}{1-x}$, eliminating $\frac{dx}{x}$ using

$$b\,\frac{dx}{x} - (c-b)\,\frac{dx}{1-x} + a\,\frac{y\,dx}{1-xy} = d\log\left(x^b (1-x)^{c-b} (1-yx)^{-a}\right),$$

and integrating by parts.

9.1.2. Contiguity relations

By integrating by parts in the integral representation (78), one proves the following *contiguity relations* for $\mathcal{F}(a,b,c;y)$, which are valid when $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$.

$$\begin{cases} \mathcal{F}(a,b,c;y) = \dfrac{c}{b}\mathcal{F}(a,b+1,c+1;y) - \dfrac{a}{b} y\,\mathcal{F}(a+1,b+1,c+2;y), \\ \mathcal{F}(a,b,c;y) = \dfrac{c}{c-b}\mathcal{F}(a,b,c+1;y) + \dfrac{a}{c-b} y\,\mathcal{F}(a+1,b+1,c+2;y). \end{cases} \tag{81}$$

One can use these relations to prove that the integral (78) can be analytically continued as a holomorphic function of a, b, c in the domain $b, c-b \notin \mathbb{Z}_{\leq 0}$. (This also follows from the expression (75) and the properties of the beta function.) For instance, the first relation extends $\mathcal{F}(a,b,c;y)$ to $\mathrm{Re}(c-b) > 0, \mathrm{Re}(b) > -1$ and the second extends it to $\mathrm{Re}(c-b) > -1, \mathrm{Re}(b) > 0$.

9.1.3. Laurent series expansion

When viewed as a function of a, b, c, the function $\mathcal{F}(a,b,c)$ can be renormalized around zero as in Proposition 4.4:

$$\mathcal{F}(a,b,c;y) = \frac{1}{b} + \frac{(1-y)^{-a}}{c-b} + \int_0^1 \Omega_{a,b,c}\,,$$

where the following form, denoted by $\Omega^{\mathrm{ren},1}$ in Definition 4.3,

$$\Omega_{a,b,c} = x^b (1-x)^{c-b} (1-yx)^{-a} \frac{dx}{x(1-x)} - x^b \frac{dx}{x} - (1-x)^{c-b} (1-y)^{-a} \frac{dx}{1-x}$$

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is absolutely integrable on $(0,1)$ for $\mathrm{Re}(c) > \mathrm{Re}(b) > -1$. After expanding it, one obtains a Laurent series in the variables a, b, $c-b$:

$$\begin{aligned}&\mathcal{F}(a,b,c;y)\\&= \frac{1}{b} + \frac{(1-y)^{-a}}{c-b} + \sum_{\substack{i,j,k\geq 0\\(j,k)\neq(0,0)}} \frac{b^i}{i!}\frac{(c-b)^j}{j!}\frac{(-a)^k}{k!}\int_0^1 \log^i(x)\log^j(1-x)\log^k(1-yx)\frac{dx}{x}\\&+ \sum_{\substack{i,j,k\geq 0\\(i,k)\neq(0,0)}} \frac{b^i}{i!}\frac{(c-b)^j}{j!}\frac{(-a)^k}{k!}\int_0^1 \log^i(x)\log^j(1-x)\left(\log^k(1-yx) - \mathbf{1}_{i=0}\log^k(1-y)\right)\frac{dx}{1-x}\,.\end{aligned} \tag{82}$$

The first integral converges at $x=0$ since at least one of the terms $\log^j(1-x)$ or $\log^k(1-yx)$ vanishes at $x=0$; the second integral converges at $x=1$ for similar reasons.

9.1.4. The companion functions G and $\mathcal{G}$

The formulae involving F and $\mathcal{F}$ frequently involve companion functions G and $\mathcal{G}$ which we now introduce. We first define

$$\mathcal{G}(a,b,c;y) = \int_{\infty}^{y^{-1}} x^b(1-x)^{c-b}(1-yx)^{-a}\,\frac{dx}{x(1-x)} \tag{83}$$

for all a,b,c such that $\mathrm{Re}(c) < \mathrm{Re}(a)+1 < 2$. When discussing specific branches, under the assumption that $y \notin \mathbb{R}_{>0}$, we adopt the convention that the above integral is computed along the path given by $x = y^{-1}/t$ for $t \in (0,1)$. Let us set $\log(-1) = \pi i$ for our fixed choice of i. This determines a branch of the complex logarithm on $\mathbb{C}\setminus[0,\infty)$, and in particular a value of $\log(y^{-1})$. (Beware that this branch satisfies $\log(y^{-1}) = -\log(y) + 2\pi i$.) We fix a branch of $x^b(1-x)^{c-b}(1-yx)^{-a}$ along (∞, y^{-1}) by fixing branches of $\log(y^{-1}/t)$, $\log(1-y^{-1}/t)$, and $\log(1-yy^{-1}/t) = \log(1-1/t)$ on $(0,1)$. We set $\log(y^{-1}/t) = \log(y^{-1}) - \log(t)$; $\log(1-y^{-1}/t) = \log(-1) + \log(y^{-1}) - \log(t) + \log(1-yt)$, where $\log(1-yt)$ equals 0 if $t=0$; and finally $\log(1-1/t) = \log(-1) + \log(1-t) - \log(t)$.

We have the following expression for $\mathcal{G}$ in terms of $\mathcal{F}$:

$$\mathcal{G}(a,b,c;y) = e^{\pi i(c-a-b)}y(y^{-1})^c\mathcal{F}(1+b-c, 1+a-c, 2-c; y),$$

which follows by making the change of variables $x = y^{-1}/t$ in (83) and comparing with (78). This proves that $\mathcal{G}(a,b,c;y)$ extends to a holomorphic function of the variables a,b,c in the domain $a, c-a \notin \mathbb{Z}_{\geq 1}$ and a multivalued holomorphic function of $y \in \mathbb{C}\setminus\{0,1\}$. It is a solution of the hypergeometric differential equation (76). Note that $\mathcal{G}(a,b,c;y)$ is not meromorphic at $y=0$ because of the prefactor $(y^{-1})^c$.

The following expression of $\mathcal{G}$ in terms of the Lauricella functions (80) will be proved below (see §9.2.4) using homological intersection pairings:

$$\begin{aligned}&\mathcal{G}(a,b,c;y)\\&= \frac{ce^{2\pi i(c-b)}}{b(e^{2\pi ia}-e^{2\pi ic})}\left(\frac{e^{2\pi ic}-e^{2\pi ib}}{c-b}L_{11} + \frac{e^{2\pi ic}-e^{2\pi ib}}{c}L_{12} - \frac{e^{2\pi ic}-1}{c-b}L_{21} - \frac{e^{2\pi ic}-1}{c}L_{22}\right).\end{aligned} \tag{84}$$

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Here, we choose the path δ_2 to be the straight path between 0 and y^{-1}. The branch of $\log(x)$ on this path is our chosen branch of the complex logarithm on $\mathbb{C}\setminus[0,\infty)$; the branches of $\log(1-x)$ and $\log(1-yx)$ are chosen so that they vanish when $x=0$.

Since the integral (83) is convergent near $a=b=c=0$, it has the Taylor expansion:

$$\mathcal{G}(a,b,c;y)=\sum_{i,j,k\geq 0}\frac{b^i}{i!}\frac{(c-b)^j}{j!}\frac{(-a)^k}{k!}\int_{\infty}^{y^{-1}}\log^i(x)\log^j(1-x)\log^k(1-yx)\frac{dx}{x(1-x)}. \tag{85}$$

We introduce the following normalization of the function $\mathcal{G}(a,b,c;y)$:

$$G(a,b,c;y)=\frac{\sin(\pi a)\sin(\pi(c-a))}{\pi\sin(\pi c)}\,\beta(b,c-b)^{-1}\,\mathcal{G}(a,b,c;y). \tag{86}$$

The prefactor is chosen so that $G(a,b,c;y)$ is symmetric in a and b. Indeed, it can be expressed in terms of F as

$$G(a,b,c;y)=e^{\pi i(c-a-b)}\frac{y(y^{-1})^c}{1-c}\,\frac{F(1+b-c,1+a-c,2-c;y)}{\beta(a,c-a)\,\beta(b,c-b)}. \tag{87}$$

This expression can be derived from the identity

$$\beta(1+a-c,1-a)=(1-c)^{-1}\,\beta(a,c-a)^{-1}\frac{\pi\sin(\pi c)}{\sin(\pi a)\sin(\pi(c-a))}\ , \tag{88}$$

which easily follows from the functional equations of the gamma function.

9.2 Cohomology with coefficients

As in §2, we can view $\mathcal{F}(a,b,c;y)$ as a '*period*' of cohomology with coefficients. We work with parameters $a,b,c\in\mathbb{C}$ that are generic in the sense that

$$a,b,c,c-a,c-b\notin\mathbb{Z}. \tag{89}$$

This is the genericity condition (13) plus the requirement that $c\notin\mathbb{Z}$ which we add in order to be able to work with preferred bases of cohomology with coefficients. Let $k\subset\mathbb{C}$ be a subfield, and let us fix $y\in k\setminus\{0,1\}$. We consider the coefficient fields $k_{\mathrm{dR}}=k(a,b,c)$ and $\mathbb{Q}_{\mathrm{B}}=\mathbb{Q}(e^{2\pi ia},e^{2\pi ib},e^{2\pi ic})$ and work in the corresponding category $\mathcal{T}$ as in §3.1. (We could work in the more refined setting of the category $\mathcal{T}_\infty$ as in §3.2 without any substantial changes.) We write $M_{a,b,c}(y)$ for the object $M_{\{0,1,y^{-1}\}}(b,c-b,-a)$ in $\mathcal{T}$; it has rank 2 and we now describe its de Rham and Betti components along with the comparison between the two, and the intersection pairings.

9.2.1. de Rham

The de Rham component $M_{a,b,c}(y)_{\mathrm{dR}}$ is the first hypercohomology group (in the Zariski topology) of $X=\mathbb{A}^1_{k_{\mathrm{dR}}}\setminus\{0,1,y^{-1}\}$ with coefficients in the twisted de Rham complex $(\Omega^\bullet_X,\nabla_{a,b,c})$ where

$$\nabla_{a,b,c}(1)=b\,d\log(x)+(c-b)\,d\log(1-x)-a\,d\log(1-yx).$$

Concretely, it is spanned by the (classes of) $d\log(x)$, $d\log(1-x)$, and $d\log(1-yx)$, modulo the relation $\nabla_{a,b,c}(1)=0$. Since we are interested in (78), we choose to work with the ad

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hoc basis of $M_{a,b,c}(y)_{\mathrm{dR}}$ consisting of (the classes of) the logarithmic forms

$$\eta_1 = \frac{dx}{x(1-x)} \quad \text{and} \quad \eta_2 = \frac{dx}{1-yx} \cdot \tag{90}$$

They are expressed in terms of the basis (17) via the change of basis matrix

$$\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \frac{c}{b(c-b)} & \frac{1}{b} \\ 0 & -\frac{1}{ay} \end{pmatrix} \begin{pmatrix} (b-c)\omega_1 \\ a\omega_2 \end{pmatrix}, \tag{91}$$

where $\omega_1 = d\log(1-x)$ and $\omega_2 = d\log(1-yx)$. This matrix is invertible because $c \neq 0$ by (89) and therefore (η_1, η_2) indeed forms a basis of $M_{a,b,c}(y)_{\mathrm{dR}}$.

9.2.2. Betti

The Betti component $M_{a,b,c}(y)_{\mathrm{B}}$ is the first singular cohomology group of $\mathbb{C} \setminus \{0, 1, y^{-1}\}$ with coefficients in the $\mathbb{Q}_{\mathrm{B}}$-local system $\mathcal{L}_{a,b,c}$ whose local sections are branches of the function $x^{-b}(1-x)^{-(c-b)}(1-yx)^{+a}$. It is convenient to work with locally finite homology via the isomorphism

$$M_{a,b,c}(y)_{\mathrm{B}}^{\vee} \xrightarrow{\sim} H_1^{\mathrm{lf}}(\mathbb{C} \setminus \{0, 1, y^{-1}\}, \mathcal{L}_{a,b,c}^{\vee}). \tag{92}$$

We choose to work with a basis of $M_{a,b,c}(y)_{\mathrm{B}}^{\vee}$ which has the simplest possible intersection pairing and consider the two locally finite paths defined by the following parametrizations:

$$p_1 : (0,1) \to \mathbb{C} \setminus \{0, 1, y^{-1}\} \ , \ t \mapsto t \qquad \text{and} \qquad p_2 : (0,1) \to \mathbb{C} \setminus \{0, 1, y^{-1}\} \ , \ t \mapsto \frac{y^{-1}}{t} \cdot$$

Note that for this to make sense, we have to assume as before that $y \notin (1, +\infty)$ so that $p_1(t) \neq y^{-1}$ and $p_2(t) \neq 1$ for all $t \in (0,1)$. Note that p_1 is a locally finite path from 0 to 1, p_2 is a locally finite path from ∞ to y^{-1}, and the images of p_1 and p_2 have disjoint closures. We denote by φ_1 the class in $M_{a,b,c}(y)_{\mathrm{B}}^{\vee}$ corresponding via (92) to the path p_1 with the canonical branch of $x^b(1-x)^{c-b}(1-yx)^{-a}$ defined in §9.1.1. We denote by φ_2 the class in $M_{a,b,c}(y)_{\mathrm{B}}^{\vee}$ corresponding via (92) to the path p_2, together with the branch of $x^b(1-x)^{c-b}(1-yx)^{-a}$ defined in §9.1.4.

In order to justify the fact that (φ_1, φ_2) indeed forms a basis of $M_{a,b,c}(y)_{\mathrm{B}}^{\vee}$, we use the homological Betti intersection pairing

$$\langle \, , \rangle_{\mathrm{B}} : M_{-a,-b,-c}(y)_{\mathrm{B}}^{\vee} \otimes_{\mathbb{Q}_{\mathrm{B}}} M_{a,b,c}(y)_{\mathrm{B}}^{\vee} \longrightarrow \mathbb{Q}_{\mathrm{B}},$$

discussed in §9.2.4. We denote by φ_i^- the class φ_i viewed in $M_{-a,-b,-c}(y)_{\mathrm{B}}^{\vee}$, and by φ_i^+ the class φ_i viewed in $M_{a,b,c}(y)_{\mathrm{B}}^{\vee}$, for $i = 1, 2$. We compute in the proof of Lemma 9.3

$$\langle \varphi_1^-, \varphi_1^+ \rangle_{\mathrm{B}} = \frac{1}{2i} \frac{\sin(\pi c)}{\sin(\pi b)\sin(\pi(c-b))} \quad \text{and} \quad \langle \varphi_2^-, \varphi_2^+ \rangle_{\mathrm{B}} = -\frac{1}{2i} \frac{\sin(\pi c)}{\sin(\pi a)\sin(\pi(c-a))},$$

which are nonzero because $c \notin \mathbb{Z}$ by assumption (89), and $\langle \varphi_1^-, \varphi_2^+ \rangle_{\mathrm{B}} = 0$. It follows that the classes φ_1^+ and φ_2^+ are linearly independent in $M_{a,b,c}(y)_{\mathrm{B}}^{\vee}$.

9.2.3. Period matrix

We let $P_{a,b,c}(y)$ denote the matrix of the natural comparison isomorphism

$$\mathrm{comp}_{\mathrm{B,dR}} : M_{a,b,c}(y)_{\mathrm{dR}} \otimes_{k_{\mathrm{dR}}} \mathbb{C} \longrightarrow M_{a,b,c}(y)_{\mathrm{B}} \otimes_{\mathbb{Q}_{\mathrm{B}}} \mathbb{C}$$

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in the bases (η_1,η_2) and (φ_1,φ_2). It is naturally described in terms of the functions $\mathcal{F}$ and $\mathcal{G}$.

PROPOSITION 9.1. *For all generic values of a,b,c, the period matrix of $M_{a,b,c}(y)$ reads*

$$P_{a,b,c}(y) = \begin{pmatrix} \mathcal{F}(a,b,c;y) & \mathcal{F}(1+a,1+b,2+c;y) \\ \mathcal{G}(a,b,c;y) & \mathcal{G}(1+a,1+b,2+c;y) \end{pmatrix}.$$

Proof. By the argument in Lemma 2.4, the (i,j)th entry of the period matrix $P_{a,b,c}(y)_{i,j}$ is given explicitly by an integral of the form

$$\langle \varphi_i, \mathrm{comp}_{\mathrm{B,dR}}(\eta_j)\rangle = \int_{p_i} x^b(1-x)^{c-b}(1-yx)^{-a}\,\eta_j$$

whenever a,b,c lie in the region where the integral converges. Note that there exist no values of a,b,c for which all four entries of the period matrix are simultaneously convergent. We first verify, therefore, that each individual entry of the period matrix is as stated for a restricted range of values of a,b,c. We then explain how to extend this for all generic values of a,b,c.

First, the top-left entry $(i=j=1)$ is defined for all generic a,b,c with $\mathrm{Re}(c)>\mathrm{Re}(b)>0$ by $\mathcal{F}(a,b,c;y)$ by definition (78). For the bottom-left entry $(i=2,j=1)$, use the integral formula (83), valid in the region $\mathrm{Re}(c)<\mathrm{Re}(a)+1<2$. In order to deduce formulae for the entries in the right-hand column $j=2$, use the fact that

$$\eta_2 = \frac{dx}{1-yx} = x(1-x)(1-xy)^{-1}\frac{dx}{x(1-x)},$$

which amounts to shifting $(b,c-b,-a)\mapsto(b+1,c-b+1,-a-1)$ in the arguments of $\mathcal{F}$ or $\mathcal{G}$, respectively, that is, $(a,b,c)\mapsto(a+1,b+1,c+2)$. Thus, the top-right entry $(i=1,j=2)$ is valid for $\mathrm{Re}(c)+1>\mathrm{Re}(b)>-1$, and the bottom right $(i=j=2)$ for $\mathrm{Re}(c)<\mathrm{Re}(a)<0$.

Finally, to show that these formulae remain valid for generic values of a,b,c, we use the fact that the entries of $P_{a,b,c}(y)$, as well as the functions $\mathcal{F}(a,b,c;y)$ and $\mathcal{G}(a,b,c;y)$, satisfy contiguity relations. This follows from the fact that multiplication by x, $(1-x)$ and $(1-xy)$ can be expressed as cohomology relations. For example, using the fact that

$$\nabla_{a,b,c}(1) = b\frac{dx}{x} + (b-c)\frac{dx}{1-x} + a\frac{y\,dx}{1-xy},$$

we find that the following relation holds in $M_{a,b,c}(y)_{\mathrm{dR}}$:

$$[x\eta_1] = \left[\frac{dx}{1-x}\right] = \frac{b}{c}[\eta_1] + \frac{ay}{c}[\eta_2],$$

since both sides differ by $\frac{1}{c}\nabla_{a,b,c}(1)$. Therefore,

$$P_{a,b+1,c+1}(y)_{1,1} = \frac{b}{c}P_{a,b,c}(y)_{1,1} + \frac{ay}{c}P_{a,b,c}(y)_{1,2}.$$

From this and similar relations for the other entries, we deduce that all entries of $P_{a,b,c}(y)$ extend to a holomorphic function of (a,b,c) on the domain where they are generic. Since the same is true for $\mathcal{F},\mathcal{G}$, we conclude by analytic continuation. ∎

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9.2.4. Intersection pairings and twisted period relations

The de Rham intersection pairing

$$\langle\,,\rangle^{\mathrm{dR}} : M_{-a,-b,-c}(y)_{\mathrm{dR}} \otimes_{k_{\mathrm{dR}}} M_{a,b,c}(y)_{\mathrm{dR}} \longrightarrow k_{\mathrm{dR}} \tag{93}$$

is easily computed. Let us denote by η_i^- the class η_i viewed in $M_{-a,-b,-c}(y)_{\mathrm{dR}}$ and by η_i^+ the class η_i viewed in $M_{a,b,c}(y)_{\mathrm{dR}}$, for $i=1,2$.

LEMMA 9.2. *The matrix of the de Rham intersection pairing (93) in the bases* (η_1^-,η_2^-) *and* (η_1^+,η_2^+) *is as follows:*

$$I^{\mathrm{dR}}_{a,b,c}(y) = \begin{pmatrix} -\frac{c}{b(c-b)} & 0 \\ 0 & \frac{1}{y^2}\frac{c}{a(c-a)} \end{pmatrix}.$$

Proof. We check that

$$\begin{pmatrix}\eta_1^-\\ \eta_2^-\end{pmatrix} = \begin{pmatrix} -1 & 0 \\ \frac{(b-c)}{y(a-c)} & \frac{c}{y(a-c)} \end{pmatrix}\begin{pmatrix}\nu_1\\ \nu_2\end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix}\eta_1^+\\ \eta_2^+\end{pmatrix} = \begin{pmatrix} \frac{c}{b(c-b)} & \frac{1}{b} \\ 0 & -\frac{1}{ay} \end{pmatrix}\begin{pmatrix}(b-c)\omega_1\\ a\omega_2\end{pmatrix},$$

where the second equation is (91). The result follows from Lemma 2.5, which states in this case that $\langle\nu_1,\omega_1\rangle^{\mathrm{dR}} = (b-c)^{-1}$, $\langle\nu_1,\omega_2\rangle^{\mathrm{dR}} = \langle\nu_2,\omega_1\rangle^{\mathrm{dR}} = 0$, and $\langle\nu_2,\omega_2\rangle^{\mathrm{dR}} = a^{-1}$. □

We now turn to the (cohomological) Betti intersection pairing

$$\langle\,,\rangle^{\mathrm{B}} : M_{-a,-b,-c}(y)_{\mathrm{B}} \otimes_{\mathbb{Q}_{\mathrm{B}}} M_{a,b,c}(y)_{\mathrm{B}} \longrightarrow \mathbb{Q}_{\mathrm{B}}. \tag{94}$$

Let us denote by φ_i^- the class φ_i viewed in $M_{-a,-b,-c}(y)^\vee_{\mathrm{B}}$, and by φ_i^+ the class φ_i viewed in $M_{a,b,c}(y)^\vee_{\mathrm{B}}$, for $i=1,2$.

LEMMA 9.3. *The matrix of the Betti intersection pairing (94) in the bases* $(\varphi_1^-,\varphi_2^-)$ *and* $(\varphi_1^+,\varphi_2^+)$ *is as follows:*

$$I^{\mathrm{B}}_{a,b,c}(y) = \begin{pmatrix} 2i\frac{\sin(\pi b)\sin(\pi(c-b))}{\sin(\pi c)} & 0 \\ 0 & -2i\frac{\sin(\pi a)\sin(\pi(c-a))}{\sin(\pi c)} \end{pmatrix}.$$

Proof. By the same computation as in [KY1], one easily computes the homological Betti pairing:

$$\langle\varphi_1^-,\varphi_1^+\rangle_{\mathrm{B}} = \frac{1}{2i}\frac{\sin(\pi c)}{\sin(\pi b)\sin(\pi(c-b))} \quad , \quad \langle\varphi_2^-,\varphi_2^+\rangle_{\mathrm{B}} = -\frac{1}{2i}\frac{\sin(\pi c)}{\sin(\pi a)\sin(\pi(c-a))},$$

and $\langle\varphi_i^-,\varphi_j^+\rangle_{\mathrm{B}} = 0$ for $i\neq j$ since the closures of p_1 and p_2 do not intersect. In fact, the top-left entry reduces to a statement equivalent to [MY, (7)], and the bottom right can be deduced from it by applying $z\mapsto z/y$. One deduces the matrix of the cohomological Betti pairing by inverting the matrix $(\langle\varphi_i^-,\varphi_j^+\rangle_{\mathrm{B}})$, and the claim follows. □

The compatibility between the intersection pairings and the comparison isomorphism gives rise to the *twisted period relations*:

$$^tP_{-a,-b,-c}(y)\, I^{\mathrm{B}}_{a,b,c}(y)\, P_{a,b,c}(y) = 2\pi i\, I^{\mathrm{dR}}_{a,b,c}(y). \tag{95}$$

Reinserting the beta factor (77) leads to quadratic relations for the hypergeometric function F as in [CM, §4]. For example, the bottom-left entry is equivalent to Gauss's relation

$$F(a,b,c;y)\,F(1-a,1-b,2-c;y) = F(-a+c,-b+c,c;y)\,F(1+a-c,1+b-c,2-c;y).$$

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We end this paragraph with a proof of the expression (84) of $\mathcal{G}$ in terms of Lauricella functions.

Proof of(84). We denote by $\delta_1^{\pm}$ (resp. $\delta_2^{\pm}$) the class in $M_{\pm a,\pm b,\pm c}(y)^{\vee}_{\mathrm{B}}$ of the path $(0,1)$ (resp. of the path $(0,y^{-1})$) together with the branch of $x^b(1-x)^{c-b}(1-yx)^{-a}$ defined in §9.1.1 (resp. in §9.1.4). By the same computation as in [KY1], we see that the matrix of the Betti intersection pairing (94) in the bases (δ_1^-,δ_2^-) and (δ_1^+,δ_2^+) is as follows:

$$J=\frac{1}{e^{2\pi ib}-1}\begin{pmatrix}\frac{e^{2\pi ic}-1}{e^{2\pi i(c-b)}-1} & e^{2\pi ib}\\ 1 & \frac{e^{2\pi i(b-a)}-1}{e^{-2\pi ia}-1}\end{pmatrix}.$$

Let us denote by ψ the class in $M_{a,b,c}(y)^{\vee}_{\mathrm{B}}$ of the path from ∞ to y^{-1} together with the branch of $x^b(1-x)^{c-b}(1-yx)^{-a}$ defined in §9.1.4. By the same computation as in [KY1], we have $\langle\delta_1^-,\psi\rangle_{\mathrm{B}}=0$ because $(0,1)$ and (∞,y^{-1}) do not intersect, and

$$\langle\delta_2^-,\psi\rangle_{\mathrm{B}}=\frac{e^{2\pi i(c-b-a)}}{e^{-2\pi ia}-1}\ .$$

Therefore, we have $\psi=u_1\delta_1^+ + u_2\delta_2^+$ where

$$\begin{pmatrix}u_1\\ u_2\end{pmatrix}=J^{-1}\begin{pmatrix}0\\ \frac{e^{2\pi i(c-b-a)}}{e^{-2\pi ia}-1}\end{pmatrix}=\frac{e^{2\pi i(c-b-a)}}{e^{2\pi ic}-e^{2\pi ia}}\begin{pmatrix}e^{2\pi ib}-e^{2\pi ic}\\ e^{2\pi ic}-1\end{pmatrix}.$$

On the other hand, we have the identity in $M_{a,b,c}(y)_{\mathrm{dR}}$:

$$\eta_1=\frac{c}{b(c-b)}(b-c)\omega_1+\frac{1}{b}a\omega_2,$$

which follows from (91). This leads to an expression of $\mathcal{G}(a,b,c;y)=\langle\psi,\mathrm{comp}_{\mathrm{B,dR}}(\eta_1)\rangle$ in terms of the $L_{ij}=\langle\delta_i,\mathrm{comp}_{\mathrm{B,dR}}(-s_j\omega_j)\rangle$ for $1\leq i,j\leq 2$. One checks that it is given by (84).

9.3 The single-valued period matrix and the double copy formula

We have the *single-valued period map*

$$\mathbf{s}: M_{a,b,c}(y)_{\mathrm{dR}}\otimes_{k_{\mathrm{dR}}}\mathbb{C}\longrightarrow M_{-a,-b,-c}(y)_{\mathrm{dR}}\otimes_{k_{\mathrm{dR}}}\overline{\mathbb{C}},$$

where $\overline{\mathbb{C}}$ denotes $\mathbb{C}$ with the conjugate structure of a k_{dR}-algebra (it amounts to conjugating y). As discussed in §2.3.1, strictly speaking, this only makes sense if a,b,c are real. In general, one would need to replace the target of $\mathbf{s}$ with the analytic de Rham cohomology of $\mathbb{C}\setminus\overline{\Sigma}$ with differential $\nabla_{-a,-b,-c}$, which does not have a natural k_{dR}-structure. However, all the formulae below make sense in this setting and we will continue to treat a,b,c as complex numbers.

Let us denote by $S_{a,b,c}(y)$ the single-valued period matrix of $M_{a,b,c}(y)$, that is, the matrix of $\mathbf{s}$, in the bases (η_1^+,η_2^+) in the source and (η_1^-,η_2^-) in the target. From the recipe given in Remark 2.7, one sees that the matrix of the real Frobenius in the bases $(\varphi_1^+,\varphi_2^+)$ and $(\varphi_1^-,\varphi_2^-)$ is the identity matrix. Therefore, the definition of $\mathbf{s}$ reads, in matrix form:

$$S_{a,b,c}(y)=P_{-a,-b,-c}(\overline{y})^{-1}\,P_{a,b,c}(y). \tag{96}$$

This leads directly to formulae for the entries of the single-valued period matrix $S_{a,b,c}(y)$, using the fact that the determinant of the period matrix can be computed explicitly (see,

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e.g., [LS]). In more detail, we find that in the bases used here we have

$$\det\left(P_{a,b,c}(y)\right) = e^{\pi i(c-a-b)}\, y^{-c-1}(1-y)^{c-a-b}\,\beta(b,c-b)\beta(-a,-c+a). \tag{97}$$

It follows then from (96) that the top-left entry of $S_{a,b,c}(y)$ is the single-valued function:

$$S_{a,b,c}(y)_{1,1} = e^{-\pi i(c-a-b)}\overline{y}^{1-c}(1-\overline{y})^{c-a-b}\,\beta(a,c-a)^{-1}\,\beta(-b,-c+b)^{-1}$$
$$\times\left(\mathcal{F}(a,b,c;y)\,\mathcal{G}(1-a,1-b,2-c;\overline{y}) - \mathcal{G}(a,b,c;y)\,\mathcal{F}(1-a,1-b,2-c;\overline{y})\right).$$

However, a more symmetric looking formula, called *double copy formula*, is obtained by using the twisted period relations, which is the approach which we shall adopt henceforth.

9.3.1. The single-valued period matrix

We provide formulae (see (105)) for the single-valued periods of $M_{a,b,c}(y)$ after composing $\mathbf{s}$ with the isomorphism

$$M_{-a,-b,-c}(y)_{\mathrm{dR}} \otimes_{k_{\mathrm{dR}}} \overline{\mathbb{C}} \longrightarrow M_{a,b,c}(y)^{\vee}_{\mathrm{dR}} \otimes_{k_{\mathrm{dR}}} \overline{\mathbb{C}} \tag{98}$$

obtained from the de Rham intersection pairing (93). Note that the matrix of (98) is

$$I^{\mathrm{dR}}_{-a,-b,-c}(\overline{y}) = -\,{}^{t}I^{\mathrm{dR}}_{a,b,c}(\overline{y}). \tag{99}$$

Proposition 9.4. *The matrix $I^{\mathrm{dR}}_{-a,-b,-c}(\overline{y})\,S_{a,b,c}(y)$ has entries given by the complex integrals*

$$\langle \eta_i, \mathbf{s}\eta_j\rangle^{\mathrm{dR}} = -\frac{1}{2\pi i}\iint_{\mathbb{C}} |z|^{2b}|1-z|^{2(c-b)}|1-yz|^{-2a}\,\overline{\eta_i}\wedge\eta_j\ , \tag{100}$$

for $1\leq i,j\leq 2$, which converge for a,b,c in the domains

$$\begin{cases} 0<\mathrm{Re}(b)<\mathrm{Re}(c)<\mathrm{Re}(a)+1<2, & \text{for } (i,j)=(1,1)\ ,\\ -\frac{1}{2}<\mathrm{Re}(b)<\mathrm{Re}(c)+\frac{1}{2}<\mathrm{Re}(a)+1<\frac{3}{2}, & \text{for } (i,j)=(1,2) \text{ and } (i,j)=(2,1)\ ,\\ -1<\mathrm{Re}(b)<\mathrm{Re}(c)+1<\mathrm{Re}(a)+1<1, & \text{for } (i,j)=(2,2).\end{cases}$$

Proof. This follows from Proposition 2.11. □

Remark 9.5. By Proposition 9.1, the entries of the period matrix $P_{a,b,c}(y)$ are holomorphic functions of generic arguments a,b,c. This implies that the entries of the single-valued period matrix $S_{a,b,c}(y)$ have the same property and that (100) extends to a holomorphic function of generic a,b,c. One could also prove this directly by using variants of the contiguity relations (81).

Definition 9.6. For generic values of a,b,c, the single-valued versions of $\mathcal{F}(a,b,c;y)$ and $\mathcal{G}(a,b,c;y)$ are given by

$$\mathcal{F}^{\mathbf{s}}(a,b,c;y) = \langle -\eta_1, \mathbf{s}\eta_1\rangle^{\mathrm{dR}} \qquad \text{and} \qquad \mathcal{G}^{\mathbf{s}}(a,b,c;y) = \langle -y\eta_2, \mathbf{s}\eta_1\rangle^{\mathrm{dR}}.$$

The heuristic for these formulae (see Remark 3.2) is that the form $-\eta_1$ has simple poles and residues -1 at 0 and 1 at 1, and is the image under the map $c_0^{\vee}$ of [BD1] of the class of a path from 0 to 1. Similarly, the form $-y\eta_2 = d\log(1-yx)$ has simple poles and residues -1 at ∞ and 1 at y^{-1}, and is thus the image under $c_0^{\vee}$ of the class of a path from ∞ to y^{-1}.

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REMARK 9.7. By expressing η_1 in the basis $((b-c)\omega_1, a\omega_2)$ as in (91) and noting that $-\eta_1$ equals the form ν_1 defined in (24), one gets the following expression:

$$\mathcal{F}^{\mathbf{s}}(a,b,c;y) = \frac{c}{b(c-b)} L^{\mathbf{s}}_{11} + \frac{1}{b} L^{\mathbf{s}}_{12} \ , \tag{101}$$

where we have used the shorthand notation

$$L^{\mathbf{s}}_{ij} = \left(L^{\mathbf{s}}_{\{0,1,y^{-1}\}}(b, c-b, -a) \right)_{ij} .$$

In view of the similar expression (79), Theorem 1.1 implies that $\mathcal{F}^{\mathbf{s}}(a,b,c;y)$ has a Laurent expansion in the variables $a, b, c-b$, whose coefficients are obtained by applying the de Rham projection and the single-valued period map to (motivic lifts of) the coefficients of the Laurent expansion (82) of $\mathcal{F}(a,b,c;y)$. This justifies the fact that we call $\mathcal{F}^{\mathbf{s}}$ a *single-valued version* of $\mathcal{F}$. A similar computation, using the identity

$$-y\eta_2 = -\frac{c-b}{c-a}\nu_1 + \frac{c}{c-a}\nu_2$$

in $M_{a,b,c}(y)_{\mathrm{dR}}$, leads to the expression

$$\mathcal{G}^{\mathbf{s}}(a,b,c;y) = \frac{c}{b(a-c)} \left(L^{\mathbf{s}}_{11} + \frac{c-b}{c} L^{\mathbf{s}}_{12} - \frac{c}{c-b} L^{\mathbf{s}}_{21} - L^{\mathbf{s}}_{22} \right). \tag{102}$$

By comparing it with (84) and applying Theorem 1.1, one deduces that $\mathcal{G}^{\mathbf{s}}(a,b,c;y)$ has a Laurent expansion in the variables $a, b, c-b$ whose coefficients are obtained by applying the de Rham projection and the single-valued period map to (motivic lifts of) the coefficients of the Laurent expansion of $\mathcal{G}(a,b,c;y)$. This is because the de Rham projection sends the Lefschetz element $\mathbb{L}^{\mathfrak{m}}$ of $\mathcal{MT}(k)$, which is a motivic lift of $2\pi i$, to zero.

REMARK 9.8. Note that even though $\mathcal{G}(a,b,c;y)$ is essentially $\mathcal{F}(1+b-c, 1+a-c, 2-c; y)$, there does not seem to be a natural way to express $\mathcal{G}^{\mathbf{s}}$ in terms of $\mathcal{F}^{\mathbf{s}}$. This should not be surprising since the single-valued versions of the functions $\mathcal{F}$ and $\mathcal{G}$ are related to the original functions by looking at Laurent expansions around $a=b=c=0$, and the Laurent expansion of $\mathcal{F}(1+b-c, 1+a-c, 2-c; y)$ is not directly related to that of $\mathcal{F}(a,b,c;y)$.

Proposition 9.4 gives the integral formulae

$$\mathcal{F}^{\mathbf{s}}(a,b,c;y) = \frac{1}{2\pi i} \iint_{\mathbb{C}} |z|^{2b} |1-z|^{2(c-b)} |1-zy|^{-2a} \frac{d\overline{z} \wedge dz}{|z|^2 |1-z|^2} \tag{103}$$

and

$$\mathcal{G}^{\mathbf{s}}(a,b,c;y) = \frac{1}{2\pi i} \iint_{\mathbb{C}} |z|^{2b} |1-z|^{2(c-b)} |1-zy|^{-2a} \frac{\overline{y}\, d\overline{z} \wedge dz}{(1-\overline{y}\,\overline{z}) z (1-z)} \ , \tag{104}$$

that are valid in the domains $0 < \mathrm{Re}(b) < \mathrm{Re}(c) < \mathrm{Re}(a)+1 < 2$ and $-\frac{1}{2} < \mathrm{Re}(b) < \mathrm{Re}(c) + \frac{1}{2} < \mathrm{Re}(a)+1 < \frac{3}{2}$, respectively.

In view of Remark 9.5, $\mathcal{F}^{\mathbf{s}}$ and $\mathcal{G}^{\mathbf{s}}$ are holomorphic functions of the generic complex parameters a, b, c. This will also be apparent in the double copy formulae (106) and (107).

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The single-valued period matrix $S_{a,b,c}(y)$ can now be written entirely in terms of $\mathcal{F}^{\mathbf{s}}$ and $\mathcal{G}^{\mathbf{s}}$ via

$$-I^{\mathrm{dR}}_{-a,-b,-c}(\overline{y})\, S_{a,b,c}(y) = \begin{pmatrix} \mathcal{F}^{\mathbf{s}}(a,b,c;y) & y^{-1}\,\mathcal{G}^{\mathbf{s}}(a,b,c;\overline{y}) \\ \overline{y}^{-1}\,\mathcal{G}^{\mathbf{s}}(a,b,c;y) & \mathcal{F}^{\mathbf{s}}(a+1,b+1,c+2;y) \end{pmatrix}, \tag{105}$$

where the de Rham intersection matrix I^{dR} can be found in Lemma 9.2.

9.3.2. Double copy formula

PROPOSITION 9.9. *We have the equality, for all generic values of a,b,c:*

$$I^{\mathrm{dR}}_{-a,-b,-c}(\overline{y})\, S_{a,b,c}(y) = \frac{1}{2\pi i}\, {}^{t}P_{a,b,c}(\overline{y})\, I^{\mathrm{B}}_{-a,-b,-c}(\overline{y})\, P_{a,b,c}(y).$$

Proof. This formula follows directly from (95) and (96). □

The top-left entry of the double copy formula reads

$$\begin{aligned} \mathcal{F}^{\mathbf{s}}(a,b,c;y) = {} & \frac{\sin(\pi b)\sin(\pi(c-b))}{\pi\sin(\pi c)}\,\mathcal{F}(a,b,c;y)\,\mathcal{F}(a,b,c;\overline{y}) \\ & - \frac{\sin(\pi a)\sin(\pi(c-a))}{\pi\sin(\pi c)}\,\mathcal{G}(a,b,c;y)\,\mathcal{G}(a,b,c;\overline{y}). \end{aligned} \tag{106}$$

The bottom-left entry is

$$\begin{aligned} \overline{y}^{-1}\,\mathcal{G}^{\mathbf{s}}(a,b,c;y) = {} & \frac{\sin(\pi b)\sin(\pi(c-b))}{\pi\sin(\pi c))}\,\mathcal{F}(a,b,c;y)\,\mathcal{F}(a+1,b+1,c+2;\overline{y}) \\ & - \frac{\sin(\pi a)\sin(\pi(c-a))}{\pi\sin(\pi c))}\mathcal{G}(a,b,c;y)\,\mathcal{G}(a+1,b+1,c+2;\overline{y}). \end{aligned} \tag{107}$$

The other entries are easily deduced from these two.

REMARK 9.10. Certain special cases of entries of single-valued period matrix $S_{a,b,c}(y)$ were previously considered in [Mi 1], [MY], along the lines of Remark 2.8. The presentation above seems to be the first systematic approach to constructing all the single-valued periods associated with the hypergeometric function.

9.4 The single-valued hypergeometric function

Recall the formula

$$\beta^{\mathbf{s}}(s_0,s_1) = \frac{1}{2\pi i}\iint_{\mathbb{C}} |z|^{2s_0}|1-z|^{2s_1}\frac{d\overline{z}\wedge dz}{|z|^2|1-z|^2} = \frac{\Gamma(s_0)\Gamma(s_1)\Gamma(1-s_0-s_1)}{\Gamma(s_0+s_1)\Gamma(1-s_0)\Gamma(1-s_1)}$$

for the single-valued (or '*complex*') version of the beta function [BD2, §1.1]. In view of (77), we propose the following definition of a *single-valued hypergeometric function*. We also define a single-valued of the function $G(a,b,c;y)$.

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DEFINITION 9.11. For generic values of a,b,c, the single-valued versions of F and G are given by

$$F^{\mathbf{s}}(a,b,c;y) = \beta^{\mathbf{s}}(b,c-b)^{-1}\,\mathcal{F}^{\mathbf{s}}(a,b,c;y) \quad \text{and}$$
$$G^{\mathbf{s}}(a,b,c;y) = \frac{a(c-a)}{c}\,\beta^{\mathbf{s}}(b,c-b)^{-1}\,\mathcal{G}^{\mathbf{s}}(a,b,c;y).$$

As explained in Remark 9.7, the coefficients of the Laurent expansion of $F^{\mathbf{s}}(a,b,c;y)$ around $a=b=c=0$ are obtained by applying the de Rham projection and the single-valued period map to (motivic lifts of) the coefficients of the Laurent expansion of $F(a,b,c;y)$. The same statement is true for $G^{\mathbf{s}}$ and G in view of (86) (again because the Lefschetz element $\mathbb{L}^{\mathfrak{m}}$ of $\mathcal{MT}(k)$, which is a motivic lift of $2\pi i$, is sent to zero by the de Rham projection.) There is no natural expression for $G^{\mathbf{s}}$ in terms of $F^{\mathbf{s}}$ for the same reasons as in Remark 9.8.

The next proposition expresses single-valued hypergeometric functions as a double copy of the classical hypergeometric functions. It shows in particular that $F^{\mathbf{s}}$ and $G^{\mathbf{s}}$ are holomorphic functions of generic complex numbers a,b,c and (single-valued) analytic functions of $y\in\mathbb{C}\setminus\{0,1\}$.

PROPOSITION 9.12. *For all generic a,b,c, we have the double copy formulae*

$$F^{\mathbf{s}}(a,b,c;y) \;=\; F(a,b,c;y)\,F(a,b,c;\overline{y}) - w_{a,c-a}\,w_{b,c-b}\,G(a,b,c;y)\,G(a,b,c;\overline{y})$$

and

$$G^{\mathbf{s}}(a,b,c;y) = \frac{a(c-a)b(c-b)\,\overline{y}}{c^2(1+c)}\Big(F(a,b,c;y)\,F(b+1,a+1,c+2;\overline{y}) \\ - w_{a,c-a}w_{b,c-b}G(a,b,c;y)\,G(b+1,a+1,c+2;\overline{y})\Big),$$

where we have set

$$w_{s,t} = \frac{\pi\sin(\pi(s+t))}{\sin(\pi s)\sin(\pi t)}\ .$$

Proof. This follows from multiplying (106) by $\beta^{\mathbf{s}}(b,c-b)^{-1}$ and using the identities (88) and

$$\beta^{\mathbf{s}}(b,c-b) = \frac{\sin(\pi b)\sin(\pi(c-b))}{\pi\sin(\pi c)}\beta(b,c-b)^2.$$

□

REMARK 9.13. It is obvious from the double copy formula that both $F^{\mathbf{s}}$ and $G^{\mathbf{s}}$ satisfy the *holomorphic part* of the hypergeometric differential equation (76), namely

$$\left(y(1-y)\frac{\partial^2}{\partial y^2} + (c-(a+b+1)y)\frac{\partial}{\partial y} - ab\right) f(y) = 0,$$

for $f(y)=F^{\mathbf{s}}(a,b,c;y)$ or $f(y)=G^{\mathbf{s}}(a,b,c;y)$. This can also be derived from first principles.

§10. Example: motivic coaction of ${}_2F_1$

We derive formulae for the motivic coaction of the hypergeometric function $F={}_2F_1$ both in the global and in the local setting (Proposition 10.5 and Theorem 10.9, respectively). As

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in the case of Lauricella functions, these formulae turn out to be formally identical, even though the contexts in which they appear are very different.

10.1 Motivic coaction of the hypergeometric function: the global point of view

As in the previous section, let $k \subset \mathbb{C}$ be a subfield and let us fix $y \in k \setminus \{0,1\}$. We consider the coefficient fields $k_{\mathrm{dR}} = k(a,b,c)$ and $\mathbb{Q}_{\mathrm{B}} = \mathbb{Q}(e^{2\pi i a}, e^{2\pi i b}, e^{2\pi i c})$ and work in the Tannakian category $\mathcal{T}$ defined in §3.1. We define the (global) motivic and canonical de Rham variants of $\mathcal{F}(a,b,c;y)$:

$$\mathcal{F}^{\mathfrak{m}}(a,b,c;y) = [M_{a,b,c}(y), \varphi_1, \eta_1]^{\mathfrak{m}} \ \in \mathcal{P}^{\mathfrak{m}}_{\mathcal{T}},$$

$$\mathcal{F}^{\mathfrak{dr}}(a,b,c;y) = [M_{a,b,c}(y), -\eta_1, \eta_1]^{\mathfrak{dr}} \ \in \mathcal{P}^{\mathfrak{dr}}_{\mathcal{T}}.$$

We will also need the (global) de Rham variant of $\mathcal{G}(a,b,c;y)$:

$$\mathcal{G}^{\mathfrak{dr}}(a,b,c;y) = [M_{a,b,c}(y), -y\eta_2, \eta_1]^{\mathfrak{dr}} \ \in \mathcal{P}^{\mathfrak{dr}}_{\mathcal{T}},$$

where η_i and φ_i were defined in §9.2. Applying the period map $\mathrm{per} : \mathcal{P}^{\mathfrak{m}}_{\mathcal{T}} \to \mathbb{C}$ to $\mathcal{F}^{\mathfrak{m}}(a,b,c;y)$ gives back $\mathcal{F}(a,b,c;y)$. If one works in the more refined Tannakian formalism of §3.2, one can replace $\mathcal{F}^{\mathfrak{dr}}(a,b,c;y)$ and $\mathcal{G}^{\mathfrak{dr}}(a,b,c,;y)$ with elements in the ring $\mathcal{P}^{\mathrm{dR}^+,\mathrm{dR}^-}_{\mathcal{T}_\infty}$ whose single-valued periods are $\mathcal{F}^{\mathfrak{s}}(a,b,c;y)$ and $\mathcal{G}^{\mathfrak{s}}(a,b,c;y)$, respectively. We now compute the (global) motivic coaction $\Delta : \mathcal{P}^{\mathfrak{m}}_{\mathcal{T}} \to \mathcal{P}^{\mathfrak{m}}_{\mathcal{T}} \otimes_{k_{\mathrm{dR}}} \mathcal{P}^{\mathfrak{dr}}_{\mathcal{T}}$.

Proposition 10.1. *We have the following (global) motivic coaction formula:*

$$\begin{aligned}\Delta\mathcal{F}^{\mathfrak{m}}(a,b,c;y) = {} & \frac{b(c-b)}{c}\,\mathcal{F}^{\mathfrak{m}}(a,b,c;y) \otimes \mathcal{F}^{\mathfrak{dr}}(a,b,c;y) \\ & - y\,\frac{a(c-a)}{c}\,\mathcal{F}^{\mathfrak{m}}(1+a,1+b,2+c;y) \otimes \mathcal{G}^{\mathfrak{dr}}(a,b,c;y).\end{aligned}$$

Proof. By the general formula for the motivic coaction, we have, writing M for $M_{a,b,c}(y)$,

$$\Delta\mathcal{F}^{\mathfrak{m}}(a,b,c;y) = [M,\varphi_1,\eta_1]^{\mathfrak{m}} \otimes [M,\eta_1^\vee,\eta_1]^{\mathfrak{dr}} + [M,\varphi_1,\eta_2]^{\mathfrak{m}} \otimes [M,\eta_2^\vee,\eta_1]^{\mathfrak{dr}}.$$

By Lemma 9.2, the dual basis elements $\eta_1^\vee$ and $\eta_2^\vee$ are represented in $M_{-a,-b,-c}(y)_{\mathrm{dR}}$ by the elements

$$\eta_1^\vee = -\frac{b(c-b)}{c}\,\eta_1 \quad \text{and} \quad \eta_2^\vee = y^2\,\frac{a(c-a)}{c}\,\eta_2.$$

Thus, what remains is to prove the equality $[M,\varphi_1,\eta_2]^{\mathfrak{m}} = \mathcal{F}^{\mathfrak{m}}(1+a,1+b,2+c)$. We describe an isomorphism

$$\alpha : M_{1+a,1+b,2+c}(y) \xrightarrow{\sim} M$$

in the category $\mathcal{T}$. At the level of de Rham components, it is induced by multiplication of (smooth) differential forms by $x(1-x)(1-yx)^{-1}$. The equality

$$\nabla_{a,b,c}(x(1-x)(1-yx)^{-1} f) = x(1-x)(1-yx)^{-1}\nabla_{1+a,1+b,2+c}(f)$$

proves that it induces an isomorphism of de Rham complexes. On the level of Betti components, it is induced by multiplication of sections by $x(1-x)(1-yx)^{-1}$. On easily

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checks that this gives rise to an isomorphism α in the category $\mathcal{T}$. It satisfies $\alpha_{\mathrm{B}}^{\vee}(\varphi_1)=\varphi_1$ and $\alpha_{\mathrm{dR}}(\eta_1)=\eta_2$, and therefore induces the desired equality of motivic periods. ∎

We now define a motivic version of the function F and compute its motivic coaction. We first need to record a few facts about the motivic and de Rham versions of the beta function. For complex numbers $s_0,s_1\in k_{\mathrm{dR}}$ such that $s_0,s_1,s_0+s_1\notin\mathbb{Z}$, we let $M_{s_0,s_1}=M_{\{0,1\}}(s_0,s_1)\in\mathcal{T}$ and let φ denote the class of $(0,1)\otimes x^{s_0}(1-x)^{s_1}\in(M_{s_0,s_1})_{\mathrm{B}}^{\vee}$ and η denote the class of $\frac{dx}{x(1-x)}$ in $(M_{s_0,s_1})_{\mathrm{dR}}$. We thus have the (global) motivic and de Rham beta functions

$$\beta^{\mathfrak{m}}(s_0,s_1)=[M_{s_0,s_1},\varphi,\eta]^{\mathfrak{m}}\ \in\mathcal{P}_{\mathcal{T}}^{\mathfrak{m}}\qquad\text{and}\qquad\beta^{\mathfrak{dr}}(s_0,s_1)=[M_{s_0,s_1},-\eta,\eta]^{\mathfrak{dr}}\ \in\mathcal{P}_{\mathcal{T}}^{\mathfrak{dr}}.$$

They are related to the Lauricella function via

$$\beta^{\bullet}(s_0,s_1)=\frac{s_0+s_1}{s_0s_1}L^{\bullet}_{\{0,1\}}\qquad(\bullet\in\{\mathfrak{m},\mathfrak{dr}\}),$$

and thus we have the coaction formula

$$\Delta\,\beta^{\mathfrak{m}}(s_0,s_1)=\frac{s_0s_1}{s_0+s_1}\,\beta^{\mathfrak{m}}(s_0,s_1)\otimes\beta^{\mathfrak{dr}}(s_0,s_1).\tag{108}$$

LEMMA 10.2. *The matrix coefficients $\beta^{\mathfrak{m}}(s_0,s_1)\in\mathcal{P}_{\mathcal{T}}^{\mathfrak{m}}$ and $\beta^{\mathfrak{dr}}(s_0,s_1)\in\mathcal{P}_{\mathcal{T}}^{\mathfrak{dr}}$ are invertible.*

Proof. The object $M=M_{s_0,s_1}\in\mathcal{T}$ has rank one. Thus, the evaluation map $M^{\vee}\otimes M\xrightarrow{\sim}\mathbb{1}$ is an isomorphism, where $\mathbb{1}$ denotes the unit object $(\mathbb{Q}_{\mathrm{B}},k_{\mathrm{dR}},\mathrm{id}_{\mathbb{C}})$ in the Tannakian category $\mathcal{T}$. Since the matrix coefficients $\beta^{\mathfrak{m}}(s_0,s_1)\in\mathcal{P}_{\mathcal{T}}^{\mathfrak{m}}$ and $\beta^{\mathfrak{dr}}(s_0,s_1)\in\mathcal{P}_{\mathcal{T}}^{\mathfrak{dr}}$ are defined by nonzero classes, they are invertible and their inverses are matrix coefficients of $M^{\vee}$. ∎

We can thus mimic (78) to define (global) motivic and de Rham lifts of $F(a,b,c;y)$.

DEFINITION 10.3. We define the (global) motivic and de Rham hypergeometric functions

$$F^{\bullet}(a,b,c;y)=\beta^{\bullet}(b,c-b)^{-1}\mathcal{F}^{\bullet}(a,b,c;y)\ \in\mathcal{P}_{\mathcal{T}}^{\bullet}\qquad(\bullet\in\{\mathfrak{m},\mathfrak{dr}\}).$$

In view of Definition 9.11, we also define

$$G^{\mathfrak{dr}}(a,b,c;y)=\frac{a(c-a)}{c}\,\beta^{\mathfrak{dr}}(b,c-b)^{-1}\mathcal{G}^{\mathfrak{dr}}(a,b,c;y)\ \in\mathcal{P}_{\mathcal{T}}^{\mathfrak{dr}}.$$

Before turning to the motivic coaction on the motivic hypergeometric function, let us prove a motivic lift of the functional equation of the beta function.

LEMMA 10.4. *We have the identities*

$$\beta^{\mathfrak{m}}(s_0+1,s_1)=\frac{s_0}{s_0+s_1}\,\beta^{\mathfrak{m}}(s_0,s_1)\qquad\textit{and}\qquad\beta^{\mathfrak{m}}(s_0,s_1+1)=\frac{s_1}{s_0+s_1}\,\beta^{\mathfrak{m}}(s_0,s_1).$$

Proof. We describe an isomorphism

$$\alpha:M_{s_0+1,s_1}\xrightarrow{\sim}M_{s_0,s_1}.$$

At the level of de Rham components, it is induced by multiplication of (smooth) differential forms by x. The equality $\nabla_{s_0,s_1}(xf)=x\nabla_{s_0+1,s_1}(f)$ proves that this induces an isomorphism of de Rham complexes. At the level of Betti components, it is induced by multiplication by

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x. One easily checks that this gives rise to an isomorphism α in the category $\mathcal{T}$. One checks that it satisfies

$$\alpha_{\mathrm{B}}^{\vee}(\varphi) = \varphi \quad , \quad \alpha_{\mathrm{dR}}(\eta) = \frac{s_0}{s_0+s_1}\eta,$$

and the first identity follows. The second is proved in a similar way. □

PROPOSITION 10.5. *The (global) motivic coaction on the motivic hypergeometric function is*

$$\Delta F^{\mathfrak{m}}(a,b,c;y) = F^{\mathfrak{m}}(a,b,c;y) \otimes F^{\mathfrak{dr}}(a,b,c;y) - \frac{y}{1+c} F^{\mathfrak{m}}(1+a,1+b,2+c;y) \otimes G^{\mathfrak{dr}}(a,b,c;y) \ ,$$

where the terms $F^{\bullet}(a,b,c;y)$ and $G^{\mathfrak{dr}}(a,b,c;y)$ are as in Definition 10.3.

Proof. The coaction is multiplicative and thus satisfies

$$\Delta F^{\mathfrak{m}}(a,b,c;y) = (\Delta \beta^{\mathfrak{m}}(b,c-b))^{-1} \Delta \mathcal{F}^{\mathfrak{m}}(a,b,c;y).$$

The result follows from Proposition 10.1 and (108) and the equality

$$\beta^{\mathfrak{m}}(c-b,b)^{-1} = \frac{b(c-b)}{c(c+1)} \beta^{\mathfrak{m}}(c-b+1,b+1)^{-1},$$

which follows from Lemma 10.4. □

10.2 Motivic coaction for the hypergeometric function: the local point of view

We now study motivic lifts of the functions $\mathcal{F}(a,b,c;y)$ and $F(a,b,c;y)$ in the local setting, that is, by lifting the coefficients of their Laurent expansions around $a=b=c=0$ to motivic periods of mixed Tate motives over k. Our main theorem is that the global coaction formulae of Propositions 10.1 and 10.5 admit local counterparts. We use the shorthand notation

$$FL^{\bullet}_{i,j} = (FL^{\bullet}_{\{0,1,y^{-1}\}}(b,c-b,-a))_{i,j}$$

for $1 \leq i,j \leq 2$ and $\bullet \in \{\mathfrak{m}, \varpi\}$. Here, we assume that the class γ_1 corresponds to the line segment $(0,1)$ and γ_2 is the line segment $(0,y^{-1})$, assuming that $y \notin \mathbb{R}_{>0}$. We set

$$\mathcal{F}^{\bullet}_{\mathrm{loc}}(a,b,c;y) = \frac{c}{b(c-b)} FL^{\bullet}_{1,1} + \frac{1}{b} FL^{\bullet}_{1,2} \quad \in \quad \mathcal{P}^{\bullet}_{\mathcal{MT}(k)}[[a,b,c]][b^{-1}(c-b)^{-1}] \tag{109}$$

for $\bullet \in \{\mathfrak{m}, \varpi\}$. They are motivic lifts of the Laurent expansions of the functions $\mathcal{F}$ and $\mathcal{F}^{\mathbf{s}}$ around $a=b=c=0$, respectively; more precisely, we get from (79) and (101) and from Theorems 6.18 and 7.11 the equalities between Laurent series:

$$\mathrm{per}\, \mathcal{F}^{\mathfrak{m}}_{\mathrm{loc}}(a,b,c;y) = \mathcal{F}(a,b,c;y) \quad \text{and} \quad \mathbf{s}\, \mathcal{F}^{\varpi}_{\mathrm{loc}}(a,b,c;y) = \mathcal{F}^{\mathbf{s}}(a,b,c;y),$$

where the period map $\mathrm{per} : \mathcal{P}^{\mathfrak{m}}_{\mathcal{MT}(k)} \to \mathbb{C}$ and the single-valued period map $\mathbf{s} : \mathcal{P}^{\varpi}_{\mathcal{MT}(k)} \to \mathbb{C}$ are applied term by term to the Laurent series. One can also obtain $\mathcal{F}^{\varpi}_{\mathrm{loc}}(a,b,c;y)$ by applying the projection $\pi^{\mathfrak{m},+}_{\varpi}$ term by term to $\mathcal{F}^{\mathfrak{m}}_{\mathrm{loc}}(a,b,c;y)$:

$$\pi^{\mathfrak{m},+}_{\varpi}\, \mathcal{F}^{\mathfrak{m}}_{\mathrm{loc}}(a,b,c;y) = \mathcal{F}^{\varpi}_{\mathrm{loc}}(a,b,c;y).$$

We also introduce

$$\mathcal{F}^{\mathfrak{m}}_{\mathrm{loc}}(1+a,1+b,2+c;y) = -\frac{1}{ya} FL^{\mathfrak{m}}_{1,2}.$$

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REMARK 10.6. Strictly speaking, it does not make sense to multiply a motivic period by $\frac{1}{y}$ since the ring $\mathcal{P}^{\mathfrak{m}}_{\mathcal{MT}(k)}$ is only $\mathbb{Q}$-linear. This is because the Betti and (canonical) de Rham fiber functors take values in $\mathbb{Q}$-vector spaces. This prefactor comes about in the above definition because the form $\eta_2 = \frac{dx}{1-yx}$ is not in fact in the $\mathbb{Q}$-structure $H^1_{\varpi}(X_\Sigma)$, having residue $-y^{-1}$ at $x = y^{-1}$. However, the form $y\eta_2$ is $\mathbb{Q}$-rational, and therefore the term $y\mathcal{F}^{\mathfrak{m}}_{\mathrm{loc}}(1+a,1+b,2+c;y)$, which is the one which appears in the coaction formula (110), is a well-defined series of motivic periods in $\mathcal{P}^{\mathfrak{m}}_{\mathcal{MT}(k)}$, although $\mathcal{F}^{\mathfrak{m}}_{\mathrm{loc}}(1+a,1+b,2+c;y)$ itself is not. Alternatively, one could extend scalars and work with a (canonical) de Rham fiber functor valued in $\mathbb{Q}(y)$-vector spaces.

REMARK 10.7. A warning about the notation is in order. Note that $\mathcal{F}^{\mathfrak{m}}_{\mathrm{loc}}(1+a,1+b,2+c;y)$ is *not* obtained from $\mathcal{F}^{\mathfrak{m}}_{\mathrm{loc}}(a,b,c;y)$ by shifting the formal variables via $(a,b,c)\mapsto(1+a,1+b,2+c)$, which does not make sense in the setting of power series. Rather, the notation is justified by the fact that we have the equality between Laurent series:

$$\mathrm{per}\,\mathcal{F}^{\mathfrak{m}}_{\mathrm{loc}}(1+a,1+b,2+c;y) = \mathcal{F}(1+a,1+b,2+c;y),$$

which follows from Theorem 6.18 and the equality $\frac{dx}{1-yx} = -\frac{1}{ya}(a\omega_2)$. Note that the function $\mathcal{F}(1+a,1+b,2+c;y)$ can also be expressed as a combination of $\mathcal{F}(a,b,c;y)$ and its derivative with respect to y.

We will also need the following definition:

$$\mathcal{G}^{\varpi}_{\mathrm{loc}}(a,b,c;y) = \frac{-c}{b(c-a)}\left(FL^{\varpi}_{1,1} + \frac{c-b}{c}FL^{\varpi}_{1,2} - \frac{c}{c-b}FL^{\varpi}_{2,1} - FL^{\varpi}_{2,2}\right).$$

By (102) and Theorem 7.11, we get the equality between Laurent series:

$$\mathbf{s}\mathcal{G}^{\varpi}_{\mathrm{loc}}(a,b,c;y) = \mathcal{G}^{\mathbf{s}}(a,b,c;y).$$

We now turn to the local coaction formulae. As in §8.2, we consider the normalized motivic coaction

$$\Delta_{\mathrm{nor}} : \mathcal{P}^{\mathfrak{m}}_{\mathcal{MT}(k)}[[a,b,c]][(bc(c-b))^{-1}] \longrightarrow (\mathcal{P}^{\mathfrak{m}}_{\mathcal{MT}(k)} \otimes \mathcal{P}^{\varpi}_{\mathcal{MT}(k)})[[a,b,c]][(bc(c-b))^{-1}],$$

which consists in applying the coaction of $\mathcal{MT}(k)$ on each coefficient and coacting on the formal variables via

$$\Delta_{\mathrm{nor}}(a) = a\,(1\otimes(\mathbb{L}^{\varpi})^{-1}) \quad , \quad \Delta_{\mathrm{nor}}(b) = b\,(1\otimes(\mathbb{L}^{\varpi})^{-1}) \quad , \quad \Delta_{\mathrm{nor}}(c) = c\,(1\otimes(\mathbb{L}^{\varpi})^{-1}).$$

THEOREM 10.8. *We have the following (local) coaction formula:*

$$\begin{aligned}\Delta_{\mathrm{nor}}\mathcal{F}^{\mathfrak{m}}_{\mathrm{loc}}(a,b,c;y) \;=\; & \frac{b(c-b)}{c}\,\mathcal{F}^{\mathfrak{m}}_{\mathrm{loc}}(a,b,c;y) \otimes \mathcal{F}^{\varpi}_{\mathrm{loc}}((\mathbb{L}^{\varpi})^{-1}a,(\mathbb{L}^{\varpi})^{-1}b,(\mathbb{L}^{\varpi})^{-1}c;y) \\ & - y\,\frac{a(c-a)}{c}\,\mathcal{F}^{\mathfrak{m}}_{\mathrm{loc}}(1+a,1+b,2+c;y) \otimes \mathcal{G}^{\varpi}_{\mathrm{loc}}((\mathbb{L}^{\varpi})^{-1}a,(\mathbb{L}^{\varpi})^{-1}b,(\mathbb{L}^{\varpi})^{-1}c;y).\end{aligned} \tag{110}$$

Proof. We drop the Lefschetz de Rham periods $\mathbb{L}^{\varpi}$ from the notation; they can be taken care of by weight considerations as in the proof of Theorem 8.5. On the one hand, by (109) and Theorem 8.5, the left-hand side equals

$$\frac{c}{b(c-b)}\left(FL^{\mathfrak{m}}_{1,1}\otimes FL^{\varpi}_{1,1} + FL^{\mathfrak{m}}_{1,2}\otimes FL^{\varpi}_{2,1}\right) + \frac{1}{b}\left(FL^{\mathfrak{m}}_{1,2}\otimes FL^{\varpi}_{2,2} + FL^{\mathfrak{m}}_{1,1}\otimes FL^{\varpi}_{1,2}\right).$$

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On the other hand, the right-hand side is expressed as

$$\frac{b(c-b)}{c}\left(\frac{c}{b(c-b)}FL^{\mathfrak{m}}_{1,1}+\frac{1}{b}FL^{\mathfrak{m}}_{1,2}\right)\otimes\left(\frac{c}{b(c-b)}FL^{\varpi}_{1,1}+\frac{1}{b}FL^{\varpi}_{1,2}\right)$$
$$-y\,\frac{a(c-a)}{c}\left(-\frac{1}{ya}FL^{\mathfrak{m}}_{1,2}\right)\otimes\left(-\frac{c}{b(c-a)}\right)\left(FL^{\varpi}_{1,1}+\frac{c-b}{c}FL^{\varpi}_{1,2}-\frac{c}{c-b}FL^{\varpi}_{2,1}-FL^{\varpi}_{2,2}\right).$$

By expanding it, one sees that simplications occur and that both sides are equal. □

It is notable, and a priori not at all obvious, that this *local* motivic coaction formula is formally identical to the *global* motivic coaction formula obtained in a different context in Proposition 10.1.

We now turn to the hypergeometric function F and reinsert the beta factors. We recall the definition of the local motivic beta functions

$$\beta^{\bullet}_{\mathrm{loc}}(s_0,s_1)$$
$$=\frac{s_0+s_1}{s_0s_1}\exp\left(\sum_{n\geq 2}\frac{(-1)^{n-1}\zeta^{\bullet}(n)}{n}((s_0+s_1)^n-s_0^n-s_1^n)\right)\in\mathcal{P}^{\bullet}_{\mathcal{MT}(k)}[[s_0,s_1]][(s_0s_1)^{-1}],$$

for $\bullet\in\{\mathfrak{m},\varpi\}$. We have the local motivic coaction formula

$$\Delta_{\mathrm{nor}}\beta^{\mathfrak{m}}_{\mathrm{loc}}(s_0,s_1)=\frac{s_0s_1}{s_0+s_1}\beta^{\mathfrak{m}}_{\mathrm{loc}}(s_0,s_1)\otimes\beta^{\varpi}_{\mathrm{loc}}((\mathbb{L}^{\varpi})^{-1}s_0,(\mathbb{L}^{\varpi})^{-1}s_1). \tag{111}$$

In view of (78), we define

$$F^{\bullet}_{\mathrm{loc}}(a,b,c;y)=(\beta^{\bullet}_{\mathrm{loc}}(b,c-b))^{-1}\,\mathcal{F}^{\bullet}_{\mathrm{loc}}(a,b,c;y)\ \in\ \mathcal{P}^{\bullet}_{\mathcal{MT}(k)}[[a,b,c]][(bc(c-b))^{-1}],$$

for $\bullet\in\{\mathfrak{m},\varpi\}$. (The term $\beta^{\bullet}_{\mathrm{loc}}(b,c-b)$ is invertible since we have inverted c in the ring of power series.) They satisfy the equalities between Laurent series:

$$\mathrm{per}\,F^{\mathfrak{m}}_{\mathrm{loc}}(a,b,c;y)=F(a,b,c;y)\qquad\text{and}\qquad\mathbf{s}\,F^{\varpi}_{\mathrm{loc}}(a,b,c;y)=F^{\mathbf{s}}(a,b,c;y),$$

where the period map $\mathrm{per}:\mathcal{P}^{\mathfrak{m}}_{\mathcal{MT}(k)}\to\mathbb{C}$ and the single-valued period map $\mathbf{s}:\mathcal{P}^{\varpi}_{\mathcal{MT}(k)}\to\mathbb{C}$ are applied term by term to the Laurent series. One can also obtain $F^{\varpi}_{\mathrm{loc}}(a,b,c;y)$ by applying the projection $\pi^{\mathfrak{m},+}_{\varpi}$ term by term to $F^{\mathfrak{m}}_{\mathrm{loc}}(a,b,c;y)$:

$$\pi^{\mathfrak{m},+}_{\varpi}F^{\mathfrak{m}}_{\mathrm{loc}}(a,b,c;y)=F^{\varpi}_{\mathrm{loc}}(a,b,c;y).$$

We also introduce

$$F^{\mathfrak{m}}_{\mathrm{loc}}(1+a,1+b,2+c;y)=\frac{c(c+1)}{b(c-b)}\,(\beta^{\mathfrak{m}}_{\mathrm{loc}}(b,c-b))^{-1}\,\mathcal{F}^{\mathfrak{m}}_{\mathrm{loc}}(1+a,1+b,2+c;y).$$

The same warning as in Remark 10.7 is in order: since we are working with formal power series, $F^{\mathfrak{m}}_{\mathrm{loc}}(1+a,1+b,2+c;y)$ is *not* obtained from $F^{\mathfrak{m}}_{\mathrm{loc}}(a,b,c;y)$ by a shift of variables, which would not make sense. Rather, the notation is justified by the equality between formal Laurent series:

$$\mathrm{per}\,F^{\mathfrak{m}}_{\mathrm{loc}}(1+a,1+b,2+c;y)=F(1+a,1+b,2+c;y),$$

which follows from the functional equation $\beta(b+1,c-b+1)=\frac{b(c-b)}{c(c+1)}\beta(b,c-b)$ for the beta function.

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Lastly, we introduce

$$G^{\varpi}_{\mathrm{loc}}(a,b,c;y) = \frac{a(c-a)}{c}\,(\beta^{\varpi}_{\mathrm{loc}}(b,c-b))^{-1}\,\mathcal{G}^{\varpi}_{\mathrm{loc}}(a,b,c;y).$$

It satisfies

$$\mathbf{s}\,G^{\varpi}_{\mathrm{loc}}(a,b,c;y) = G^{\mathbf{s}}(a,b,c;y),$$

where $G^{\mathbf{s}}(a,b,c;y)$ was defined in Definition 9.11.

THEOREM 10.9. *We have the following (local) motivic coaction formula for the (local) motivic hypergeometric function:*

$$\begin{aligned}\Delta_{\mathrm{nor}}\,F^{\mathfrak{m}}_{\mathrm{loc}}(a,b,c;y) \;=\; & F^{\mathfrak{m}}_{\mathrm{loc}}(a,b,c;y)\otimes F^{\varpi}_{\mathrm{loc}}\left((\mathbb{L}^{\varpi})^{-1}a,(\mathbb{L}^{\varpi})^{-1}b,(\mathbb{L}^{\varpi})^{-1}c;y\right)\\ & -\frac{y}{1+c}\,F^{\mathfrak{m}}_{\mathrm{loc}}(1+a,1+b,2+c;y)\otimes G^{\varpi}_{\mathrm{loc}}\left((\mathbb{L}^{\varpi})^{-1}a,(\mathbb{L}^{\varpi})^{-1}b,(\mathbb{L}^{\varpi})^{-1}c;y\right).\end{aligned} \tag{112}$$

Proof. It follows from Theorem 10.8 by using the multiplicativity of the coaction and (111). □

Again, it is a priori not at all obvious, but certainly true, that this *local* motivic coaction formula is comparable with the *global* motivic coaction formula for the hypergeometric function obtained in Proposition 10.5.

REMARK 10.10. Up until this point, we have fixed a point $y \in k\backslash\{0,1\}$. It is also possible to view y as a variable by working with families; this will enable us to derive the hypergeometric differential equation from the coaction, which we presently explain. The local motivic and (canonical) de Rham hypergeometric functions can be viewed as power series with coefficients in a ring of families of motivic periods over the base scheme $S = \mathbb{P}^1\backslash\{0,1,\infty\}$ with coordinate y. For this, we need to work with a Tannakian category $\mathcal{M}(S)$ of *motives* over S, such as the category $\mathcal{MT}(S)$ of mixed Tate motives over S [EL] or, for the following application, the category $\mathcal{H}(S)$ considered in [B3, §7] is sufficient. We regard $F^{\bullet}_{\mathrm{loc}}$, and $G^{\varpi}_{\mathrm{loc}}$ as Laurent series with coefficients in $\mathcal{P}^{\bullet}_{\mathcal{M}(S)}$, where $\bullet = \{\mathfrak{m},\varpi\}$ relative to suitable fiber functors (whose definition is not important for the following discussion). To the lowest two orders, we find that

$$F^{\varpi}_{\mathrm{loc}}(a,b,c;y) = \beta^{\varpi}_{\mathrm{loc}}(b,c-b)^{-1}\mathcal{F}^{\varpi}_{\mathrm{loc}}(a,b,c;y) \;=\; 1-\frac{ab}{c}\log^{\varpi}(1-y)+\cdots,$$

$$\begin{aligned}G^{\varpi}_{\mathrm{loc}}(a,b,c;y) &= \frac{a(c-a)}{c}(\beta^{\varpi}_{\mathrm{loc}}(b,c-b))^{-1}\\ \mathcal{G}^{\varpi}_{\mathrm{loc}}(a,b,c;y) &= -\frac{ab(c-a)(c-b)}{c^2}\log^{\varpi}(1-y)+\cdots,\end{aligned}$$

where the $\cdots$ denote iterated integrals of length ≥ 2, and $\log^{\varpi}$ is the (canonical) de Rham logarithm (see [B3, §§5.3 and 7]). This is consistent with the Laurent expansions (82) and (85). The rings of families of motivic periods $\mathcal{P}^{\bullet}_{\mathcal{M}(S)}$ are equipped with a canonical differential operator $\nabla_{\partial/\partial y}$ which is compatible with the coaction $\Delta : \mathcal{P}^{\mathfrak{m}}_{\mathcal{M}(S)} \to \mathcal{P}^{\mathfrak{m}}_{\mathcal{M}(S)} \otimes \mathcal{P}^{\varpi}_{\mathcal{M}(S)}$ in the sense that the following equation holds:

$$\Delta\circ\nabla_{\partial/\partial y} = (\mathrm{id}\otimes\nabla_{\partial/\partial y})\circ\Delta.$$

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Furthermore, $\nabla_{\partial/\partial y}$ decreases the length of (canonical) de Rham iterated integrals by one, and one checks that

$$\nabla_{\partial/\partial y}\log^{\varpi}(1-y) = -\frac{1}{1-y} \qquad \text{and} \qquad \nabla_{\partial/\partial y}(1) = 0.$$

The ring of (canonical) de Rham periods $\mathcal{P}^{\varpi}_{\mathcal{M}(S)}$ is a Hopf algebra with counit ev, such that $(\mathrm{id}\otimes \mathrm{ev})\Delta = \mathrm{id}$. It satisfies $\mathrm{ev}(\mathbb{L}^{\mathfrak{d}\mathfrak{r}}) = 1$ and annihilates all (canonical) de Rham iterated integrals of length ≥ 1 which occur in the expansions above. Let us apply $\nabla_{\partial/\partial y}$ to the (unnormalized version of the) coaction formula (112). Therefore, by combining (112) with the formulae stated above, we find that

$$\begin{aligned}\nabla_{\partial/\partial y}F^{\mathfrak{m}}_{\mathrm{loc}}(a,b,c;y) &= ((\mathrm{id}\otimes \mathrm{ev})\Delta)\left(\nabla_{\partial/\partial y}F^{\mathfrak{m}}_{\mathrm{loc}}(a,b,c;y)\right) = (\mathrm{id}\otimes \mathrm{ev}\,\nabla_{\partial/\partial y})\Delta F^{\mathfrak{m}}_{\mathrm{loc}}(a,b,c;y)\\ &= \frac{ab}{c}F^{\mathfrak{m}}_{\mathrm{loc}}(a,b,c;y)\otimes\frac{1}{1-y} - \frac{y}{1+c}F^{\mathfrak{m}}_{\mathrm{loc}}(1+a,1+b,2+c;y)\otimes\frac{ab(c-a)(c-b)}{c^2}\frac{1}{1-y}.\end{aligned}$$

Since the connexion is compatible with the period homomorphism, we deduce that

$$\frac{\partial}{\partial y}F(a,b,c;y) = \frac{ab}{c}\frac{1}{1-y}F(a,b,c;y) - \frac{ab(c-a)(c-b)}{c^2(1+c)}\frac{y}{1-y}F(1+a,1+b,2+c;y),$$

as formal power series near $a=b=c=0$. One verifies that this is equivalent, by contiguity relations, to

$$\frac{\partial}{\partial y}F(a,b,c;y) = \frac{ab}{c}F(a+1,b+1,c+1;y),$$

or to the hypergeometric differential equation (76).

REMARK 10.11. It is also a general fact that the (local) motivic coaction is compatible with the monodromy of the (local) hypergeometric function. This, as well as the connection above, provides a priori constraints on the shape of the motivic coaction.

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