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P-ADIC INTERPOLATION OF DEDEKIND SUMS

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In this article we give an explicit representation of p-adic Dedekind sums and their reciprocity laws by using p-adic measure theory. We then study the consequences of the p-adic reciprocity law for particular positive integer values in which case we can recover a reciprocity law for Dedekind sums attached to particular Dirichlet characters. This gives a proof different from that of Nagasaka.

1. INTRODUCTION

In [**4**], the authors showed that by p-adically interpolating certain partial zeta functions, it is possible to interpolate the higher order Dedekind sums introduced by Apostol [**1**], thus obtaining p-adic Dedekind sums. The authors then showed that there is a reciprocity law for p-adic Dedekind sums, however they were not able to obtain an explicit representation of the reciprocity law for all p-adic integers. In this article, we obtain an explicit form for the reciprocity law for arbitrary p-adic integers. This is accomplished by the use of p-adic measure theory. We then study the consequences of this p-adic reciprocity law for particular integer values in which case we can recover a reciprocity law for Dedekind sums attached to particular types of Dirichlet characters. This gives a proof different from that of Nagasaka [**3**] for these special cases.

2. p-ADIC INTERPOLATION OF HIGHER ORDER DEDEKIND SUMS

The higher order Dedekind sums are defined as follows: let m, h and k be integers such that $m \geqslant 0$ and $k > 0$, then

$$s_m(h,k) = \sum_{\mu=0}^{k-1} \overline{B}_1\left(\frac{\mu}{k}\right)\overline{B}_m\left(\frac{h\mu}{k}\right)$$

where $\overline{B}_m(x)$ denotes the mth periodic Bernoulli function defined by

$$\sum_{m=0}^{\infty} B_m(x)\frac{z^m}{m!} = \frac{ze^{xz}}{e^z - 1} \qquad (|z| < 2\pi)$$

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for all real x and $\overline{B}_m(x) = B_m(x-[x])$. It is well-known that $B_m(x) = \sum_{j=0}^{m}\binom{m}{j}x^{m-j}B_j$ or, symbolically, $B_m(x) = (x+B)^m$ and $B_m(0) = B_m$, the mth Bernoulli number.

Throughout this section, let p denote a fixed prime which, for convenience, we assume to be odd. Let $\mathbb{Z}_p$ and $\mathbb{Q}_p$ denote the set of p-adic integers and p-adic rationals, respectively. Let $||_p$ denote the p-adic norm, normalised so that $|p|_p = \frac{1}{p}$. Recall that the group of p-adic units $\mathbb{Z}_p^* \simeq V \times (1+p\mathbb{Z}_p)$ where V is the group of $(p-1)$st roots of unity in $\mathbb{Z}_p$ and $1+p\mathbb{Z}_p$ is the so called group of principal units. If $x \in \mathbb{Z}_p^*$, then we denote by $\omega(x)$ and $< x >$ the projections of x onto V and $1+p\mathbb{Z}_p$, respectively. Furthermore let A_p denote the set

$$\{f(z) = \sum_{m=0}^{\infty} a_m z^m \in K[[z]]:\ \lim_{m\to\infty} a_m = 0\},$$

where K is a finite extension of $\mathbb{Q}_p$. We define a linear functional $d\beta$ from A_p to K by

$$\int f(z)d\beta(z) = \int \sum_{m=0}^{\infty} a_m z^m d\beta(z) = \sum_{m=0}^{\infty} a_m B_m.$$

Notice that this series converges since $|B_m|_p \leqslant p$ by the von Staudt-Clausen theorem. We then have the following proposition.

Proposition 1. *For all integers m, a and k such that $m \geqslant 0$, $k \neq 0$*

$$k^m B_m\left(\frac{a}{k}\right) = \int (a+kz)^m d\beta(z).$$

If in addition $a \neq 0$, then

$$k^m B_m\left(\frac{a}{k}\right) = a^m \int \left(1+\frac{kz}{a}\right)^m d\beta(z).$$

Proof:

$$\int (a+kz)^m d\beta(z) = \int \sum_{j=0}^{m}\binom{m}{j} a^{m-j}k^j z^j d\beta(z) = \sum_{j=0}^{m}\binom{m}{j} a^{m-j}k^j B_j.$$

$$= k^m \sum_{j=0}^{m}\binom{m}{j}\left(\frac{a}{k}\right)^{m-j} B_j = k^m B_m\left(\frac{a}{k}\right)$$

The second part is equally obvious. ∎

Now, if $p \mid k$ but $p \nmid a$, then it is easy to see how we can extend $k^m B_m(\frac{a}{k})$ to a continuous function for "p-adic m", namely, by $< a >^s \int \left(1+\frac{kz}{z}\right)^s d\beta(z)$ or more

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generally by $\omega^{-n}(a) < a >^s \int (1+\frac{kz}{a})^s d\beta(z)$ for some fixed integer n. Here $s \in \mathbb{Z}_p$ and $(1+\frac{kz}{a})^s = \sum_{m=0}^{\infty} \binom{s}{m}(\frac{k}{a})^m z^m \in A_p$ since $\binom{s}{m} \in \mathbb{Z}_p$ and $k/a \in p\mathbb{Z}_p$. From this we see easily how to interpolate $k^m s_m(h,k)$ when $p \mid k$, $p \nmid a$:

DEFINITION: Let h, k be integers such that $k > 0$, $p \mid k$ but $p \nmid h$. Then $S_p(s;h,k) = \sum_{\substack{\mu=0 \\ p\nmid\mu}}^{k-1} B_1(\frac{\mu}{k})\omega^{-1}(h\mu) < (h\mu)_k >^s \int \left(1+\frac{kz}{(h\mu)_k}\right)^s d\beta(z)$ for all $s \in \mathbb{Z}_p$. $(a)_k$ denotes the integer $x \in [0,k)$ such that $a \equiv x (\operatorname{mod} k)$.

We introduced the factor $\omega^{-1}(h\mu)$ in the above definition in order to recover the classical reciprocity law for higher order Dedekind sums, as we shall see later.

PROPOSITION 2. *For any integers m, h and k such that $m \geqslant 0$, $k > 0$ and $p \mid k$ but $p \nmid h$,*

$$S_p(m;h,k) = \sum_{\substack{\mu=0 \\ p\nmid\mu}}^{k-1} \overline{B}_1\left(\frac{\mu}{k}\right)\omega^{-m-1}(h\mu)k^m \overline{B}_m\left(\frac{h\mu}{k}\right).$$

Moreover, if $m+1 \equiv 0 \pmod{p-1}$,

$$S_p(m;h,k) = k^m s_m(h,k) - p^m (k/p)^m s_m(h,k/p).$$

PROOF:

$$S_p(m;h,k) = \sum_{\substack{\mu=0 \\ p\nmid\mu}}^{k-1} \overline{B}_1\left(\frac{u}{k}\right)\omega^{-1}(h\mu)\left(\omega^{-1}((h\mu)_k)(h\mu)_k\right)^m \int \left(1+\frac{kz}{(h\mu)_k}\right)^m d\beta(z)$$

$$= \sum_{\substack{\mu=0 \\ p\nmid\mu}}^{k-1} \overline{B}_1\left(\frac{\mu}{k}\right)\omega^{-m-1}(h\mu)k^m \overline{B}_m\left(\frac{h\mu}{k}\right)$$

by Proposition 1 and the observation that $\omega^{-m}((h\mu)_k) = \omega^{-m}(h\mu)$ since $p \mid k$ and ω^{-m} has period dividing p. If $m+1 \equiv 0 \pmod{p-1}$, then $\omega^{-m-1}(h\mu) = 1$ for all μ.

Thus

$$S_p(m;h,k) = \sum_{\mu=0}^{k-1} \overline{B}_1\left(\frac{\mu}{k}\right)k^m \overline{B}_m\left(\frac{h\mu}{k}\right) - \sum_{\substack{\mu=0 \\ p\mid\mu}}^{k-1} \overline{B}_1\left(\frac{\mu}{k}\right)k^m \overline{B}_m\left(\frac{h\mu}{k}\right).$$

$$= k^m s_m(h,k) - \sum_{\mu=0}^{\frac{k}{p}-1} \overline{B}_1\left(\frac{\mu p}{k}\right)k^m \overline{B}_m\left(\frac{h\mu p}{k}\right)$$

$$= k^m s_m(h,k) - k^m s_m(h,k/p)$$

∎

We shall now define $S_p(s;h,k)$ when $p\nmid hk$. We proceed as above by replacing k by pk and appealing to Raabe's theorem:

$$k^m\overline{B}_m\left(\frac{h\mu}{k}\right)=p^{m-1}k^m\sum_{j=0}^{p-1}\overline{B}_m\left(\frac{h\mu+kj}{pk}\right).$$

Each term on the right-hand side may be interpolated p-adically provided $h\mu+kj\not\equiv 0$ $(\bmod p)$.

Definition: Let h, k be integers such that $k>0$ and $p\nmid hk$. Then

$$S_p(s;h,k)=\sum_{\mu=0}^{k-1}\overline{B}_1\left(\frac{\mu}{k}\right)\frac{1}{p}\sum_{\substack{j=0\\ p\nmid h\mu+kj}}^{p-1}\omega^{-1}(h\mu+kj)$$
$$<(h\mu+kj)_{pk}>^s\int\left(1+\frac{pkz}{(h\mu+kj)_{pk}}\right)^s d\beta(z)$$

for all $s\in\mathbb{Z}_p$.

Proposition 3. *For any integers m, h and k such that $m\geqslant 0$, $k>0$ and $p\nmid hk$.*

$$S_p(m;h,k)=\sum_{\mu=0}^{p-1}\overline{B}_1\left(\frac{\mu}{k}\right)\frac{1}{p}\sum_{\substack{j=0\\ p\nmid h\mu+kj}}^{p-1}\omega^{-m-1}(h\mu+kj)(pk)^m\overline{B}_m\left(\frac{h\mu+kj}{pk}\right).$$

Moreover, if $m+1\equiv 0\pmod{p-1}$,

$$S_p(m;h,k)=k^m s_m(h,k)-p^{m-1}k^m s_m\left(\left(p^{-1}h\right)_k,k\right)$$

where $\left(p^{-1}h\right)_k$ denotes the integer $x\in[0,k)$ such that $px\equiv h\pmod k$.

Proof: The first formula follows just as in the proof of Proposition 2. If $m+1\equiv 0$ $(\bmod p-1)$, then

$$S_p(m;h,k)=\sum_{\mu=0}^{k-1}B_1\left(\frac{\mu}{k}\right)p^{m-1}k^m\sum_{j=0}^{p-1}\overline{B}_m\left(\frac{h\mu+kj}{pk}\right)$$
$$-\sum_{\mu=0}^{k-1}\overline{B}_1\left(\frac{\mu}{k}\right)\frac{1}{p}\sum_{\substack{j=0\\ p\mid h\mu+kj}}^{p-1}(kp)^m\overline{B}_m\left(\frac{k\mu+kj}{pk}\right)$$
$$=\sum_{\mu=0}^{k-1}\overline{B}_1\left(\frac{\mu}{k}\right)k^m\overline{B}_m\left(\frac{h\mu}{k}\right)-\sum_{\mu=0}^{k-1}\overline{B}_1\left(\frac{\mu}{k}\right)p^{m-1}k^m\overline{B}_m\left(\frac{\left(p^{-1}h\right)_k\mu}{k}\right)$$

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since $h\mu + kj \equiv 0 \pmod p$ and $h\mu + kj \equiv h\mu \pmod k$ implies that $h\mu + kj \equiv p(p^{-1}h)_k\mu \pmod{pk}$. Thus the Proposition. ∎

We now review the reciprocity law for higher order Dedekind sums and then see how to interpolate it. Recall that for all integers m, h and k such that $m \geqslant 0$, $h > 0$, $k > 0$ and $(h,k) = 1$

$$hk^m s_m(h,k) + kh^m s_m(k,h) = \frac{m}{m+1}B_{m+1} + \frac{1}{m+1}(hB+kB)^{m+1}$$

where $(hB+kB)^{m+1}$ is written symbolically.

We would like to determine explicitly $hS_p(s;h,k) + kS_p(s;k,h)$ when $p \nmid hk$. To this end we have the following Proposition.

PROPOSITION 4. *Let m, h and k be positive integers such that $(h,k)=1$, $p \nmid hk$ and $m+1 \equiv 0 \pmod{p-1}$. Then*

$$\begin{aligned} hS_p(m;h,k) + kS_p(m;k,h) = {}& \frac{m}{m+1}(1-p^m)B_m + \frac{1}{m+1}(kB-hB)^{m+1} \\ & - \frac{1}{m+1}p^{m-1}(ks-hs)^{m+1}\binom{hk}{p} \end{aligned}$$

where $(ks-hs)^{m+1}\binom{hk}{p} = \sum_{j=0}^{m+1}\binom{m+1}{j}(-h)^{m+1-j}k^j s_{j,m+1-j}\binom{hk}{p}$ *with* $s_{m,n}\binom{hk}{p} = \sum_{\lambda=0}^{p-1}\overline{B}_m\left(\frac{h\lambda}{p}\right)\overline{B}_n\left(\frac{k\lambda}{p}\right)$, *see* [2].

PROOF: By Proposition 3, we have for $m+1 \equiv 0 \pmod{p-1}$

$$\begin{aligned} hS_p(mh,k) + kS_p(m;k,h) = {}& hk^m s_m(h,k) + kh^m s_m(k,h) \\ & - p^{m-1}\left(hk^m s_m\left((p^{-1}h)_k, k\right) + kh^m s_m\left((p^{-1}k)_h, h\right)\right). \end{aligned}$$

The sum of the first two terms on the right-hand side is given by the reciprocity law above.

We now consider the remaining terms. Notice that

$$\begin{aligned} s_m\left((p^{-1}h)_k, k\right) &= \sum_{\mu=0}^{k-1}\overline{B}_1\left(\frac{\mu}{k}\right)\overline{B}_m\left(\frac{(p^{-1}h)_k\mu}{k}\right) \\ &= \sum_{\mu=0}^{k-1}\overline{B}_1\left(\frac{p\mu}{k}\right)\overline{B}_m\left(\frac{h\mu}{k}\right) = s_{1,m}\binom{ph}{k}. \end{aligned}$$

Similarly, $s_m\left((p^{-1}k)_h, h\right) = s_{1,m}\binom{pk}{h}$. But by [2] (5.6) we have the following reciprocity law:

$$hk^m s_{1,m}\binom{ph}{k} + kh^m s_{1,m}\binom{pk}{h} = \frac{m}{m+1}pB_{m+1} - hk^m B_m + \frac{1}{m+1}(ks-hs)^{m+1}\binom{hk}{p}.$$

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Putting the two terms together we obtain

$$hS_p(m;h,k)+kS_p(m;k,h)=\frac{m}{m+1}(1-p^m)B_{m+1}$$
$$+\frac{1}{m+1}(kB+hB)^{m+1}-\frac{p^{m-1}}{m+1}(ks-hs)^{m+1}\binom{hk}{p}+p^{m-1}hk^mB_m.$$

This yields the proposition since $(kB+hB)^{m+1}=(kB-hB)^{m+1}-hk^mB_m$ and m is odd since $m+1\equiv 0 \pmod{p-1}$. ∎

We are now in a position to state and prove our main theorem.

THEOREM. *Let h, k be positive integers such that $(h,k)=1$ and $p\nmid hk$. For any $s\in\mathbb{Z}_p$, let*

$$I_p(s)=\frac{1}{p}\sum_{\mu=1}^{p-1}<\mu>^{s+1}\int\left(1+\frac{pz}{\mu}\right)^{s+1}d\beta(z),$$
$$K_p(s)=\frac{1}{p^2}\sum_{\substack{i,j=0\\ i\neq j}}^{p-1}<k(hj)_p-h(ki)_p>^{s+1}$$
$$\times\iint\left(1+\frac{p(kz-hw)}{k(hj)_p-h(ki)_p}\right)^{s+1}d\beta(z)d\beta(w).$$

Then

$$hS_p(s;h,k)+kS_p(s;k,h)=\frac{s}{s+1}I_p(s)+\frac{1}{s+1}K_p(s).$$

PROOF: We show that for $m+1\equiv 0 \pmod{p-1}$, the theorem reduces to Proposition 4. Thus by continuity and the fact that $\{m\in\mathbb{N}\mid m+1\equiv 0 \pmod{p-1}\}$ is dense in $\mathbb{Z}_p$, the theorem will follow.

Thus assume m is a positive integer such that $m+1\equiv 0 \pmod{p-1}$. Then

$$I_p(m)=\frac{1}{p}\sum_{\mu=1}^{p-1}\mu^{m+1}\int\left(1+\frac{pz}{\mu}\right)^{m+1}d\beta(z)=\frac{1}{p}\sum_{\mu=1}^{p-1}p^{m+1}\overline{B}_{m+1}\left(\frac{\mu}{p}\right)$$
$$=p^m\sum_{\mu=0}^{p-1}\overline{B}_{m+1}\left(\frac{\mu}{p}\right)-p^mB_{m+1}=(1-p^m)B_{m+1}.$$

On the other hand,

$$K_p(m) = \frac{1}{p^2}\sum_{\substack{i,j=0\\ i\neq j}}^{p-1}\left(k(hj)_p - h(ki)_p\right)^{m+1}\iint\left(1+\frac{p(kz-hw)}{k(hj)_p - h(ki)_p}\right)^{m+1} d\beta(z)d\beta(w)$$

$$= \frac{1}{p^2}\sum_{i\neq j}\iint\left(k(hj)_p - h(ki)_p + p(kz-hw)\right)^{m+1} d\beta(z)d\beta(w)$$

$$= \frac{1}{p^2}\sum_{i\neq j}\sum_{l=0}^{m+1}\binom{m+1}{l}$$

$$\times \iint\left(k[(hj)_p + pz]\right)^{m+1-l}\left(-h[(ki)_p + pw]\right)^{l} d\beta(z)d\beta(w)$$

$$= \sum_{l=0}^{m+1}\binom{m+1}{l}k^{m+1-l}(-h)^l p^{m-1}\sum_{i\neq j}\overline{B}_{m+1-l}\left(\frac{hj}{p}\right)\overline{B}_l\left(\frac{ki}{p}\right)$$

$$= \sum_{l=0}^{m+1}\binom{m+1}{l}k^{m+1-l}(-h)^l$$

$$\times\left(\sum_{i,j=0}^{p-1}p^{m-l}\overline{B}_{m+1-l}\left(\frac{hi}{p}\right)p^{l-1}\overline{B}_l\left(\frac{kj}{p}\right) - p^{m-1}\sum_{i=0}^{p-1}\overline{B}_{m+1-l}\left(\frac{hi}{p}\right)\overline{B}_l\left(\frac{ki}{p}\right)\right)$$

$$= \sum_{l=0}^{m+1}\binom{m+1}{l}k^{m+1-l}(-h)^l B_{m+1-l}B_l - p^{m-1}(ks-hs)^{m+1}\left(\frac{hk}{p}\right).$$

Therefore $hS_p(m;h,k) + kS_p(m;k,h) = \frac{m}{m+1}I_p(m) + \frac{1}{m+1}K_p(m)$. This in turn establishes the theorem. ∎

In particular the theorem is true for any integer m. We obtain the following Corollary to the Theorem.

COROLLARY. *Let m be any nonnegative integer such that $m+1 \not\equiv 0 \pmod{p-1}$ and let h, k be positive integers such that $(h,k)=1$ and $p \nmid hk$, then*

$$hk^m\sum_{\mu=0}^{k-1}\overline{B}_1\left(\frac{\mu}{k}\right)p^{m-1}\sum_{j=0}^{p-1}\omega^{-m-1}(h\mu+kj)\overline{B}_m\left(\frac{h\mu+kj}{pk}\right)$$

$$+\, kh^m\sum_{\nu=0}^{h-1}\overline{B}_1\left(\frac{\nu}{h}\right)p^{m-1}\sum_{i=0}^{p-1}\omega^{-m-1}(k\nu+hi)\overline{B}_m\left(\frac{k\nu+hi}{ph}\right)$$

$$= \frac{m}{m+1}B_{m+1,\omega^{-m-1}}$$

$$+\, \frac{p^{m-1}}{m+1}\omega^{-m-1}(hk)\sum_{i,j=0}^{p-1}\omega^{-m-1}(j-i)\left(k\overline{B}\left(\frac{hj}{p}\right) - h\overline{B}\left(\frac{ki}{p}\right)\right)^{m+1}$$

where $B_{m+1,\omega^{-m-1}}$ is the $(m+1)$st generalised Bernoulli number associated with the character ω^{-m-1}, that is, $B_{m,\chi}$ is defined by

$$\sum_{m=0}^{\infty} B_{m,\chi}\frac{z^m}{m!} = \sum_{a=0}^{f-1}\frac{\chi(a)ze^{az}}{e^{fz}-1}$$

where f is a modulus of χ. The expression

$$\left(k\overline{B}\left(\frac{hj}{p}\right) - h\overline{B}\left(\frac{ki}{p}\right)\right)^{m+1} = \sum_{l=0}^{m+1}\binom{m+1}{l}k^{m+1-l}(-h)^l\overline{B}_{m+1-l}\left(\frac{hj}{p}\right)\overline{B}_l\left(\frac{ki}{p}\right).$$

PROOF: Let $s=m$ be as in the statement of the Corollary. Then by Proposition 3,

$$\begin{aligned} &hS_p(m;h,k) + kS_p(m;k,h) \\ &= hk^m\sum_{\mu=0}^{k-1}\overline{B}_1\left(\frac{\mu}{k}\right)p^{m-1}\sum_{j=0}^{p-1}\omega^{-m-1}(h\mu+kj)\overline{B}_m\left(\frac{h\mu+kj}{pk}\right) \\ &+ kh^m\sum_{\nu=0}^{h-1}\overline{B}_1\left(\frac{\nu}{h}\right)p^{m-1}\sum_{i=0}^{p-1}\omega^{-m-1}(k\nu+hi)\overline{B}_m\left(\frac{k\nu+hi}{ph}\right). \end{aligned}$$

Moreover,

$$\begin{aligned} I_p(m) &= \frac{1}{p}\sum_{\mu=0}^{p-1} < \mu >^{m+1}\int\left(1+\frac{pz}{\mu}\right)^{m+1}d\beta(z) \\ &= \frac{1}{p}\sum_{\mu=1}^{p-1}\omega^{-m-1}(\mu)\mu^{m+1}\int\left(1+\frac{pz}{\mu}\right)^{m+1}d\beta(z) \\ &= \frac{1}{p}\sum_{\mu=1}^{p-1}\omega^{-m-1}(\mu)p^{m+1}\overline{B}_{m+1}\left(\frac{\mu}{p}\right) \\ &= p^m\sum_{\mu=1}^{p-1}\omega^{-m-1}(\mu)\overline{B}_{m+1}\left(\frac{\mu}{p}\right) = B_{m+1,\omega^{-m-1}}. \end{aligned}$$

The last equality follows from the definitions of $B_{m+1,\omega^{-m-1}}$ and $B_{m+1}(x)$ in terms of their generating functions.

By an argument similar to the one in the proof of the theorem we obtain

$$\begin{aligned} K_p(m) &= \sum_{i,j=0}^{p-1}\omega^{-m-1}\left(k(hj)_p - h(ki)_p\right) \\ &\times\sum_{l=0}^{m+1}\binom{m+1}{l}k^{m+1-l}(-h)^l p^{m-1}\overline{B}_{m+1-l}\left(\frac{hj}{p}\right)\overline{B}_l\left(\frac{ki}{p}\right) \\ &= p^{m-1}\omega^{-m-1}(hk)\sum_{i,j=0}^{p-1}\omega^{-m-1}(j-i)\left(k\overline{B}\left(\frac{hj}{p}\right) - h\overline{B}\left(\frac{ki}{p}\right)\right)^{m+1}. \end{aligned}$$

The corollary now follows easily. ∎

The corollary to the theorem suggests a definition of Dedekind sums attached to characters somewhat different (although equivalent) to that of Nagasaka [**3**].

DEFINITION: Let χ be a numerical character on $\mathbb{Z}$ of modulus dividing f. For any integer $m \geqslant 0$, define

$$\overline{B}_{m,\chi}(x) = f^{m-1} \sum_{a=0}^{f-1} \chi(a+x) \overline{B}_m \left(\frac{a+x}{f} \right)$$

for any rational number x with denominator relatively prime to f.
(Notice χ extends without ambiguity to such x by multiplicativity).

DEFINITION: Let χ be a numerical character of modulus f, let h, k be integers such that $k > 0$ and $(k, f) = 1$. Then for any integer $m \geqslant 0$, define

$$s^{\chi}_{n,m}(h,k) = \sum_{\mu=0}^{k-1} \overline{B}_n \left(\frac{\mu}{k} \right) \overline{B}_{m,\chi} \left(\frac{h\mu}{k} \right).$$

It is easy to see that $s^{\chi}_{n,m}(h,k)$ is independent of the choice of representatives of $\mu (\mathrm{mod}\ k)$. Then the corollary is equivalent to the following result for $\chi = \omega^{-m-1}$.

Let χ be a primitive character of conductor f. Let m be any integer with $m \geqslant 0$ and h and k positive integers such that $(h,k) = 1$ and $(hk, f) = 1$. Then

$$\begin{aligned} & hk^m \chi(k) s^{\chi}_{1,m}(h,k) + kh^m \chi(h) s^{\chi}_{1,m}(k,h) \\ &= \frac{m}{m+1} B_{m+1,\chi} + \frac{f^{m-1}}{m+1} \sum_{i,j=0}^{f-1} \chi(hi+kj) \left(h\overline{B}\left(\frac{i}{f}\right) + k\overline{B}\left(\frac{j}{f}\right) \right)^{m+1}. \end{aligned}$$

We shall not prove this statement since an equivalent form may be found in [**3**].

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