J. Austral. Math. Soc. Ser. B 34(1992), 18–25

CONTROLLABILITY OF NEUTRAL VOLTERRA INTEGRODIFFERENTIAL SYSTEMS

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(Received 29 May 1990; revised 2 January 1991)

Abstract

Sufficent conditions for controllability of nonlinear neutral Volterra integrodifferential systems are established. Controllability of an infinite-delay neutral Volterra system is also considered.

1. Introduction

Several authors [4, 6, 8, 12] have studied the theory of functional differential equations. In [2, 9] the problem of controllability of linear neutral systems has been investigated. Motivation for such control systems and its importance in other fields can be found in [8, 10]. Chukwu [3] and Angell [1] studied the functional controllability and Underwood and Chukwu [13], null controllability of nonlinear neutral systems. Onwuatu [11] discussed the problem for nonlinear systems of neutral functional differential equations with limited controls. Gahl [7] derived a set of sufficient conditions for controllability of nonlinear neutral systems through the fixed point method developed by Dauer [5]. In this paper, we shall study the controllability of neutral Volterra integrodifferential system and infinite delay neutral Volterra systems, by suitably adapting the technique of Dauer [5].

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2. Preliminaries

Let Q be the Banach space of all continuous functions

$$(x, u): [0, t_1] \times [0, t_1] \to R^n \times R^m$$

with the norm defined by

$$\|(x, u)\| = \|x\| + \|u\|$$

where $\|x\| = \sup|x(t)|$ for $t \in [0, t_1]$.

Consider the linear neutral Volterra integrodifferential system of the form

$$\begin{aligned}\frac{d}{dt}\left[x(t) - \int_0^t C(t-s)x(s)\,ds - g(t)\right] \\ = Ax(t) + \int_0^t G(t-s)x(s)\,ds + B(t)u(t)\end{aligned} \tag{1}$$

and the nonlinear system

$$\begin{aligned}\frac{d}{dt}\left[x(t) - \int_0^t C(t-s)x(s)\,ds - g(t)\right] \\ = Ax(t) + \int_0^t G(t-s)x(s)\,ds + B(t)u(t) + f(t, x(t), u(t))\end{aligned} \tag{2}$$

where $x \in R^n$, $u \in R^m$, $C(t)$ and $G(t)$ are $n \times n$ continuous matrix valued functions and $B(t)$ is a continuous $n \times m$ matrix valued function, A a constant $n \times n$ matrix, f and g are respectively continuous and absolutely continuous n-vector functions.

We consider the controllability on a bounded interval $J = [0, t_1]$ of the system (1) and (2). That is, system (1) or (2) is said to be controllable on J if for every $x(0)$, $x_1 \in R^n$ there exists a control function u, defined on J, such that the solution of (1) or (2) satisfies $x(t_1) = x_1$.

The solution of (1) can be written as [14]

$$\begin{aligned}x(t) = Z(t)[x(0) - g(0)] + g(t) + \int_0^t \dot{Z}(t-s)g(s)\,ds \\ + \int_0^t Z(t-s)B(s)u(s)\,ds\end{aligned} \tag{3}$$

where $Z(t)$ is an $n \times n$ continuously differentiable matrix satisfying

$$\frac{d}{dt}\left[Z(t) - \int_0^t C(t-s)Z(s)\,ds\right] = AZ(t) + \int_0^t G(t-s)Z(s)\,ds$$

with $Z(0) = I$ and the solution of the nonlinear system (2) is given by

$$\begin{aligned} x(t) = Z(t)[x(0) - g(0)] + g(t) + \int_0^t \dot{Z}(t-s)g(s)\,ds \\ + \int_0^t Z(t-s)[B(s)u(s) + f(s, x(s), u(s))]\,ds. \end{aligned} \tag{4}$$

Define the matrix W by

$$W(0, t) = \int_0^t Z(t-s)B(s)B^*(s)Z^*(t-s)\,ds \tag{5}$$

where the star denotes the matrix transpose.

3. Main results

THEOREM 1. *The system* (1) *is controllable on* J *iff* W *is nonsingular.*

PROOF. Assume W is nonsingular. Let the control function u be defined on J as

$$\begin{aligned} u(t) = B^*(t)Z^*(t_1 - t)W^{-1}(0, t_1)\Big[x_1 - Z(t_1)(x(0) - g(0)) \\ -g(t_1) - \int_0^{t_1} \dot{Z}(t_1 - s)g(s)\,ds\Big]. \end{aligned}$$

Then from (3), it follows that $x(t_1) = x_1$.

Conversely suppose that (1) is controllable. In order to show that W is nonsingular let us assume the contrary. Then, there exists a vector $v \neq 0$ such that $v^*Wv = 0$. It follows that

$$\int_0^{t_1} v^*Z(t_1 - s)B(s)(v^*Z(t_1 - s)B(s))^*\,ds = 0.$$

Therefore, $v^*Z(t_1 - s)B(s) = 0$ for $s \in J$. Consider the initial point, $x(0) = 0$, and the final point, $x_1 = v$. Take $g = 0$; since the system is controllable there exists a control $u(t)$ on J that steers the response to $x_1 = v$ at $t = t_1$, that is

$$x(t_1) = v = \int_0^{t_1} Z(t_1 - s)B(s)u(s)\,ds$$

and hence

$$v^*v = \int_0^{t_1} v^*Z(t_1 - s)B(s)u(s)\,ds = 0.$$

This is a contradiction for $v \neq 0$. Hence W is nonsingular.

Now we shall consider the nonlinear system (2). For this, take $p = (x, u) \in R^n \times R^m$ and set $|p| = |x| + |u|$.

THEOREM 2. *If the continuous function* f *satisfies the condition*

$$\lim_{|p|\to\infty} \frac{|f(t, p)|}{|p|} = 0$$

uniformly in $t \in J$ *and if the system* (1) *is controllable on* J, *then the system* (2) *is controllable on* J.

PROOF. Define $T: Q \to Q$ by

$$T(x, u) = (y, v),$$

where

$$\begin{aligned} v(t) = {} & B^*(t)Z^*(t_1 - t)W^{-1}(0, t_1) \\ & \times \Big[x_1 - Z(t_1)(x(0) - g(0)) - g(t_1) \\ & \quad - \int_0^{t_1} \dot{Z}(t_1 - s)g(s)\,ds \\ & \quad - \int_0^{t_1} Z(t_1 - s)f(s, x(s), u(s))\,ds\Big] \end{aligned} \tag{6}$$

and

$$\begin{aligned} y(t) = {} & Z(t)[x(0) - g(0)] + g(t) + \int_0^t \dot{Z}(t - s)g(s)\,ds \\ & + \int_0^t Z(t - s)B(s)v(s)\,ds + \int_0^t Z\ (t - s)f(s, x(s), u(s))\,ds]. \end{aligned} \tag{7}$$

Let

$$\begin{aligned} a_1 &= \sup|Z(t - s)B(s)|, \quad 0 \le s \le t \le t_1, \\ a_2 &= |W^{-1}(0, t_1)|, \\ a_3 &= \sup\left|Z(t)(x(0) - g(0)) + g(t) + \int_0^t \dot{Z}(t - s)g(s)\,ds\right| + |x_1|, \\ a_4 &= \sup|Z(t - s)|, \qquad (t, s) \in J \times J. \\ b &= \max\{t_1 a_1, 1\}, \\ c_1 &= 4ba_1a_2a_4t_1, \\ c_2 &= 4a_4t_1, \\ d_1 &= 4a_1a_2a_3b, \qquad d_2 = 4a_3, \\ c &= \max\{c_1, c_2\}, \qquad d = \max\{d_1, d_2\} \\ \sup|f| &= \sup[|f(s, x(s), u(s))| : s \in J]. \end{aligned}$$

Then,

$$\begin{aligned}|v(t)| &\le a_1 a_2 [a_3 + a_4 t_1 \sup|f|] \\ &= \frac{d_1}{4b} + \frac{c_1}{4b}\sup|f| \\ &\le \frac{1}{4b}[d + c\sup|f|]\end{aligned}$$

and

$$\begin{aligned}|y(t)| &\le a_3 + t_1 a_1 \|v\| + t_1 a_4 \sup|f| \\ &\le b\|v\| + \frac{d}{4} + \frac{c}{4}\sup|f|.\end{aligned}$$

By hypothesis, f satisfies the following condition (Proposition 1 in [5]): for each pair of positive constants c and d, there exists a positive constant r such that, if $|p| \le r$, then

$$c|f(t, p)| + d \le r \quad \text{for all } t \in J. \tag{8}$$

Also, for given c and d, if r is a constant such that (8) is satisfied, then any r_1 such that $r < r_1$ will also satisfy (8). Now take c and d as given above, and let r be chosen so that (8) is satisfied. Therefore, if $\|x\| \le r/2$ and $\|u\| \le r/2$ then $\|(x, u)\| \le r$. It follows that $d + c\sup|f| \le r$. Therefore,

$$|v(t)| \le \frac{r}{4b} \quad \text{for all } t \in J$$

and hence $\|v\| \le r/4b$. It follows that

$$|y(t)| \le r/4 + r/4 = r/2 \quad \text{for all } t \in J$$

and hence that $\|y\| \le r/2$. Thus we have proved that, if

$$H = \{(x, u) \in Q : \|x\| \le r/2 \text{ and } \|u\| \le r/2\},$$

then T maps H into itself. Since all the functions involved in the definition of the operator T are continuous, it follows that T is continuous. Using the Arzela-Ascoli theorem, it is easy to see that T is completely continuous. Since H is closed, bounded and convex, the Schauder fixed point theorem guarantees that T has a fixed point $(x, u) \in H$. It follows that

$$\begin{aligned}x(t) = Z(t)[x(0) - g(0)] + g(t) + \int_0^t \dot{Z}(t-s)g(s)\,ds \\ + \int_0^t Z(t-s)B(s)u(s)\,ds + \int_0^t Z(t-s)f(s, x(s), u(s))\,ds \\ \text{for } t \in J.\end{aligned}$$

Hence $x(t)$ is a solution of the system (2) and it is easy to verify that $x(t-1) = x_1$. Hence (2) is controllable on J.

4. Infinite neutral systems:

We shall consider the following neutral system represented by

$$\frac{d}{dt}\left[x(t)-\int_{-\infty}^{t} C(t-s)x(s)\,ds-g(t)\right] = Ax(t)+\int_{-\infty}^{t} G(t-s)x(s)\,ds+B(t)u(t), \tag{9}$$

$x(t) = \phi(t)$ on $(-\infty, 0]$ where the initial function ϕ is continuous and bounded on R^n. Equivalently, (9) takes the form

$$\frac{d}{dt}\left[x(t)-\int_{0}^{t} C(t-s)x(s)\,ds-g(t)-\int_{-\infty}^{0} C(t-s)\phi(s)\,ds\right] = Ax(t)+\int_{0}^{t} G(t-s)x(s)\,ds+\int_{-\infty}^{0} G(t-s)\phi(s)\,ds+B(t)u(t). \tag{10}$$

Using (3), the solution of (10) can be written as

$$\begin{aligned} x(t) = {} & Z(t)\left[x(0)-g(0)-\int_{-\infty}^{0} C(-s)\phi(s)\,ds\right]+g(t) \\ & +\int_{-\infty}^{0} C(t-s)\phi(s)\,ds+\int_{0}^{t} \dot{Z}(t-s)\left[g(s)+\int_{-\infty}^{0} C(s-\tau)\phi(\tau)\,d\tau\right] ds \\ & +\int_{0}^{t} Z(t-s)B(s)u(s)\,ds+\int_{0}^{t} Z(t-s)\int_{-\infty}^{0} G(s-\tau)\phi(\tau)\,d\tau\,ds. \end{aligned} \tag{11}$$

Here the system (9) is said to be controllable if for each initial function $\phi \in C_n(-\infty, 0]$ and for every $x_1 \in R^n$, there exists a control $u(t)$, defined on J, such that the solution $x(t)$ of (9) satisfies $x(t_1) = x_1$.

THEOREM 3. *System (9) is controllable on J iff W is nonsingular.*

PROOF. Assume W is nonsingular. Let the control function u be defined on J by

https://doi.org/10.1017/S0334270000007347 Published online by Cambridge University Press

$$
\begin{aligned}
u(t) = B^*(t)Z^*(t_1 - t)W^{-1}(0, t_1) \\
\times \Bigg[x_1 - Z(t_1)\left(x(0) - g(0) - \int_{-\infty}^{0} C(-s)\phi(s)\,ds \right) - g(t_1) \\
- \int_0^{t_1} C(t_1 - s)g(s)\,ds \\
- \int_0^{t_1} \dot{Z}(t_1 - s)\left[g(s) + \int_{-\infty}^{0} C(s-\tau)\phi(\tau)\,d\tau \right] ds \\
- \int_0^{t_1} Z(t_1 - s)\left[\int_{-\infty}^{0} G(s-\tau)\phi(\tau)\,d\tau \right] ds \Bigg].
\end{aligned}
$$

Then from (11), it follows that $x(t_1) = x_1$. The converse part follows as in Theorem 1.

REMARK. By similar argument, with the same condition on the nonlinear function f as in Theorem 2, one can establish the controllability relationship between the linear system (9) and its corresponding nonlinear system.

References

[1] T. S. Angell, "On controllability for nonlinear hereditary systems; a fixed-point approach," *Nonlinear Analysis, Theory, Methods and Applications* **4** (1980) 529–548.

[2] H. T. Banks and G. A. Kent, "Control of functional differential equations of retarded and neutral type to target sets in function space," *SIAM J. Control* **10** (1972) 567–593.

[3] E. N. Chukwu, "Functional inclusion and controllability of nonlinear neutral functional differential systems," *J. Optimiz. Theory Appl.* **29** (1979) 291–300.

[4] E. N. Chukwu and H. C. Simpson, "Perturbations of nonlinear systems of neutral type," *J. Differential Equations* **82** (1989) 28–59.

[5] J. P.Dauer, "Nonlinear perturbations of quasi-linear control systems," *J. Math. Anal. Appl.* **54** (1976) 717–725.

[6] E. Fuchs, "The degeneracy property in linear autonomous functional differential equations of neutral type," *J. Math. Anal. Appl.* **90** (1983) 527–549.

[7] R. D. Gahl, "Controllability of nonlinear systems of neutral type," *J. Math. Anal. Appl.* **63** (1978) 33–42.

[8] J. K. Hale, *Theory of functional differential equations* (Springer-Verlag, New York, 1977).

[9] M. Q. Jacobs and C. E. Langenhop, "Criteria for function space controllability of linear neutral systems," *SIAM J. Control Optim.* **14** (1976) 1009–1048.

[10] D. A. O'Connor, *State controllability and observability for linear neutral systems*, (D. Sc Thesis, Washington University, 1978).

[11] J. U. Onwuatu, "On the null-controllability in function space of nonlinear systems of neutral functional differential equations with limited controls," *J. Optimiz. Theory Appl.* **42** (1984) 397–420.

[12] G. Tadmore, "Functional differential equations of retarded and neutral type: analytic solutions and piecewise continuous controls," *J. Diff. Equations* **51** (1984) 151–181.
[13] R. G. Underwood and E. N. Chukwu, "Null controllability of nonlinear neutral differential equations," *J. Math. Anal. Appl.* **129** (1988) 326–345.
[14] J. Wu, "Globally stable periodic solutions of linear neutral Volterra integrodifferential equations," *J. Math. Anal. Appl.* **130** (1988) 474–483.