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LOCAL HOMOMORPHISMS OF TOPOLOGICAL GROUPS

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Abstract

A mapping $f : G \to S$ from a left topological group G into a semigroup S is a *local homomorphism* if for every $x \in G \setminus \{e\}$, there is a neighborhood U_x of e such that $f(xy) = f(x)f(y)$ for all $y \in U_x \setminus \{e\}$. A local homomorphism $f : G \to S$ is *onto* if for every neighborhood U of e, $f(U \setminus \{e\}) = S$. We show that

(1) every countable regular left topological group containing a discrete subset with exactly one accumulation point admits a local homomorphism onto $\mathbb{N}$,
(2) it is consistent that every countable topological group containing a discrete subset with exactly one accumulation point admits a local homomorphism onto any countable semigroup,
(3) it is consistent that every countable nondiscrete maximally almost periodic topological group admits a local homomorphism onto the countably infinite right zero semigroup.

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1. Introduction

A group G endowed with a topology is *left topological* if for each $a \in G$, the left shift $G \ni a \mapsto ax \in G$ is continuous or equivalently, a homeomorphism.

Let G be a left topological group with identity e. A mapping $f : G \to S$ from G into a semigroup S is a *local homomorphism* if for every $x \in G \setminus \{e\}$ there is a neighborhood U_x of e such that $f(xy) = f(x)f(y)$ for all $y \in U_x \setminus \{e\}$ [9]. A local homomorphism $f : G \to S$ is *onto* if $f(U \setminus \{e\}) = S$ for every neighborhood U of e. All topologies are assumed to be Hausdorff.

Local homomorphisms of G are important because they induce continuous homomorphisms of the *ultrafilter semigroup* Ult(G) of G. It consists of all nonprincipal ultrafilters on G converging to e and is a closed subsemigroup in the *Stone-Čech compactification* βG_d of G as a discrete semigroup. (Given any discrete semigroup S, the operation can be extended to βS by

$$pq = \lim_{x \to p} \lim_{y \to q} xy$$

where $x, y \in S$, which makes βS a compact right topological semigroup with S contained in its topological center. A semigroup T endowed with a topology is *right topological* if for each $p \in T$, the right shift $T \ni x \mapsto xp \in T$ is continuous. The *topological center* $\Lambda(T)$ of a right topological semigroup T consists of all $a \in T$ such that the left shift $T \ni x \mapsto ax \in T$ is continuous. An elementary introduction to the semigroup βS can be found in [4]. For confirmation that $\mathrm{Ult}(G)$ is a closed subsemigroup in βG_d, see [4, Exercise 9.2.3].)

LEMMA 1.1. *Let $f : G \to T$ be a local homomorphism from G into a compact right topological semigroup T such that $f(G) \subseteq \Lambda(T)$, let $\overline{f} : \beta G_d \to T$ be the continuous extension of f, and let $f^* = \overline{f}|_{\mathrm{Ult}(G)}$. Then $f^* : \mathrm{Ult}(G) \to T$ is a continuous homomorphism. If, in addition, for every neighborhood U of e, $f(U \setminus \{e\})$ is dense in T, then f^* is onto.*

PROOF. It suffices to check the first statement. We show more: for every $p \in \beta G_d$ and $q \in \mathrm{Ult}(G)$, one has $\overline{f}(pq) = \overline{f}(p)\overline{f}(q)$. Indeed,

$$\begin{aligned}
\overline{f}(pq) &= \overline{f}(\lim_{x \to p} \lim_{y \to q} xy) = \lim_{x \to p} \lim_{y \to q} f(xy) \\
&= \lim_{x \to p} \lim_{y \to q} f(x) f(y) \quad \text{because } f \text{ is a local homomorphism} \\
&= \lim_{x \to p} f(x)\overline{f}(q) \qquad \text{because } f(x) \in \Lambda(T) \\
&= \overline{f}(p)\overline{f}(q).
\end{aligned}$$

□

Thus if, for example, G admits a local homomorphism onto the countably infinite right zero semigroup S, then $\mathrm{Ult}(G)$ admits a continuous homomorphism onto βS which is also a right zero semigroup, and consequently $\mathrm{Ult}(G)$ is a disjoint union of $2^{\mathfrak{c}}$ closed right ideals. Recall that *right (left) zero* semigroups are defined by the identity $xy = y$ $(xy = x)$.

Local homomorphisms are also interesting for their own sake.

LEMMA 1.2. *Let $f : G \to S$ be a mapping such that $f(U \setminus \{e\}) = S$ for every neighborhood U of e, and for every $s \in S$, let $A_s = f^{-1}(s) \setminus \{e\}$. (Equivalently, let $\{A_s : s \in S\}$ be a partition of $G \setminus \{e\}$ such that $e \in \mathrm{cl}\, A_s$ for each $s \in S$, and let $f : G \to S$ be such that $f(x) = s$ if $x \in A_s$.)*

Then

(1) *f is a local homomorphism if and only if for every $s \in S$ and $x \in G \setminus \{e\}$ there is a neighborhood U_x of e such that*

$$x(U_x \cap A_s) \subseteq A_{f(x)s}.$$

In particular,

(2) *if S is a left zero semigroup then f is a local homomorphism if and only if A_s is open for every $s \in S$.*

(3) *if S is a right zero semigroup then f is a local homomorphism if and only if A_s is is dense in G for every $s \in S$, and moreover, for every $x \in G \setminus \{e\}$ there is a neighborhood U_x of $e \in G$ such that*

$$x(U_x \cap A_s) = (x(U_x \setminus \{e\})) \cap A_s.$$

PROOF. (1) To see the necessity, let U_x be a neighborhood of e from the definition of a local homomorphism and such that $x^{-1} \notin U_x$, and let $y \in U_x \cap A_s$. Then $xy \neq e$ and $f(xy) = f(x)f(y) = f(x)s$, so $xy \in A_{f(x)s}$.

To see the sufficiency, let $y \in U_x \setminus \{e\}$. Then $y \in U_x \cap A_s$ for some $s \in S$, so $xy \in x(U_x \cap A_s) \subseteq A_{f(x)s}$ and hence $f(xy) = f(x)s = f(x)f(y)$.

(2) By (1), f is a local homomorphism if and only if for every $x \in G \setminus \{e\}$ there is a neighborhood U_x of e such that $x(U_x \cap A_s) \subseteq A_{f(x)}$ for all $s \in S$, so if and only if $xU_x \subseteq A_{f(x)}$.

(3) By (1), f is a local homomorphism if and only if for every $s \in S$ and $x \in G \setminus \{e\}$, there is a neighborhood U_x of e such that $x(U_x \cap A_s) \subseteq (x(U_x \setminus \{e\})) \cap A_s$. It is clear that if $x(U_x \cap A_s) = (x(U_x \setminus \{e\})) \cap A_s$ then f is a local homomorphism. Conversely, we suppose that f is a local homomorphism and show that the equality holds. Let U_x be a neighborhood of e from the definition of a local homomorphism, $y \in U_x \setminus \{e\}$ and $xy \in A_s$. Then $f(y) = f(x)f(y) = f(xy) = s$, so $y \in A_s$. □

It follows from Lemma 1.2 that if G admits a local homomorphism onto the 2-element left zero semigroup then G is not *extremally disconnected* (that is, the closures of disjoint open subsets are disjoint), and if G admits a local homomorphism onto the 2-element left zero semigroup, then G is *resolvable* (that is, can be partitioned into two dense subsets). It is a difficult open problem whether there is in ZFC a countable nondiscrete extremally disconnected topological group (see [1]). If we proved that it is consistent that every countable nondiscrete topological group admits a local homomorphism onto the 2-element left zero semigroup, then the answer would be negative. The corresponding question about irresolvable topological groups (see [2]) has already been solved: it is consistent that every countable nondiscrete topological group is *ω-resolvable* (that is, can be partitioned into ω dense subsets) (see [6, 10]).

(In [6] this result has been obtained for Abelian groups, not necessarily countable, and in [10] it has been shown that every countable nondiscrete ω-irresolvable topological group contains an open Boolean subgroup.) But the proof does not give any hint of how to construct local homomorphisms onto right zero semigroups.

In this paper we show the following:

(1) Every countable regular left topological group containing a discrete subset with exactly one accumulation point admits a local homomorphism onto $\mathbb{N}$ (Corollary 2.3).
(2) It is consistent that every countable topological group containing a discrete subset with exactly one accumulation point admits a local homomorphism onto any countable semigroup (Corollary 2.8).
(3) It is consistent that every countable nondiscrete maximally almost periodic topological group admits a local homomorphism onto the countably infinite right zero semigroup (Corollary 3.3).

Recall that a topological group is *maximally almost periodic* if it can be continuously embedded into a compact topological group.

2. Producing local homomorphisms by discrete subsets

Consider the countably infinite Boolean group $\mathbb{B} = \bigoplus_\omega \mathbb{Z}_2$ endowed with the topology induced by the product topology on $\prod_\omega \mathbb{Z}_2$. Observe that each nonzero element $x \in \mathbb{B}$ has a unique *canonical* representation in the form $x = x_0 + \cdots + x_k$, where

(a) for every $l \leq k$, $\operatorname{supp}(x_l)$ is a nonempty interval in ω,
(b) for every $l \leq k-1$, $\max\operatorname{supp}(x_l) + 2 \leq \min\operatorname{supp}(x_{l+1})$.

As usual, for any $x = (a_n)_{n<\omega} \in \mathbb{B}$, $\operatorname{supp}(x) = \{n < \omega : a_n \neq 0\}$. If $\operatorname{supp}(x)$ is a nonempty interval in ω, we say that x is *basic*. It follows that any mapping from the set of basic elements of $\mathbb{B}$ into a semigroup S can be extended to the mapping $g : \mathbb{B} \to S$ by $g(x) = g(x_0)\cdots g(x_k)$, where $x = x_0 + \cdots + x_k$ is the canonical representation. We have that $g(x+y) = g(x)g(y)$ whenever $\max\operatorname{supp}(x) + 2 \leq \min\operatorname{supp}(y)$, so g is a local homomorphism.

The following result is a strengthened version of [9, Theorem 2].

THEOREM 2.1. *Let G be a countable nondiscrete regular left topological group and let D be a discrete subset of G such that $e \notin D$ and* $\operatorname{cl} D \setminus D \subseteq \{e\}$. *Then there is a continuous bijection $h : G \to \mathbb{B}$ with $h(e) = 0$ such that*

(1) *$h(xy) = h(x)h(y)$ whenever* $\max\operatorname{supp}(h(x)) + 2 \leq \min\operatorname{supp}(h(y))$,
(2) *for every $x \in D$, $h(x)$ is basic.*

If G is first countable then h can be chosen to be a homeomorphism.

PROOF. Let W be the set of words on the letters 0 and 1 with empty word $\emptyset$. For any $w = a_0 \cdots a_n$ and $v = b_0 \cdots b_m$, define $w + v = c_0 \cdots c_k$ by $k = \max\{n, m\}$ and

$$c_l = \begin{cases} a_l & \text{if } l \leq n \\ b_l & \text{otherwise} \end{cases} .$$

Notice that each nonempty $w \in W$ has a unique *canonical* representation in the form $w = w_0 + \cdots + w_k$, where

(a) for every $l \leq k$, $w_l = 0^{i_l} 1^{j_l}$ with $i_l, j_l < \omega$ and $i_l + j_l > 0$,
(b) for every $l \leq k - 1$, $j_l > 0$,
(c) for every $l \leq k - 1$, $i_l + j_l < i_{l+1}$.

The words of the form $0^i 1^j$, where $i, j < \omega$ and $j > 0$, are called *basic*. The words of the form 0^i, where $i < \omega$, are called *zero*. From now on, when we write $w = w_0 + \cdots + w_k$ we mean that this is the canonical representation. For any $w \in W$, $|w|$ denotes the length of w.

Enumerate $G \setminus \{e\}$ as $\{x_n : n < \omega\}$. We shall assign to each $w \in W$ a point $x(w) \in G$ and a clopen neighborhood $X(w)$ of $x(w)$ so that

(i) $x(0^n) = e$ and $X(\emptyset) = G$,
(ii) $X(w^\frown 0) \cap X(w^\frown 1) = \emptyset$ and $X(w^\frown 0) \cup X(w^\frown 1) = X(w)$,
(iii) $x(w) = x(w_0) \cdots x(w_k)$ and $X(w) = x(w_0) \cdots x(w_{k-1}) X(w_k)$, where $w = w_0 + \cdots + w_k$,
(iv) $x_n \in \{x(w) : |w| = n + 1\}$,
(v) $X(w^\frown 0) \cap D \subseteq \{x(w)\}$ for all nonzero w.

Pick as $X(0)$ a clopen neighborhood of e such that $x_0 \notin X(0)$. Put $X(1) = X \setminus X(0)$ and $x(0) = e$ and $x(1) = x_0$.

Suppose that $X(w)$ and $x(w)$ have been constructed for all w with $|w| \leq n$ such that conditions (i)-(v) hold.

Notice that the subsets $X(w)$ with $|w| = n$ form a partition of X. So, one of them, say $X(u)$, contains x_n. Let $u = u_0 + \cdots + u_r$. Then $X(u) = x(u_0) \cdots x(u_{r-1})(X(u_r))$ and $x_n = x(u_0) \cdots x(u_{r-1}) y_n$ for some $y_n \in X(u_r)$. If $y_n = x(u_r)$, we choose as $X(0^n)$ a clopen neighborhood of e such that for each basic w with $|w| = n$, $X(w) \setminus x(w) X(0^{n+1}) \neq \emptyset$, and following condition (v), for each nonzero w with $|w| = n$, $x(w) \cdot X(0^{n+1}) \cap D \subseteq \{x(w)\}$. For each basic w, put $x(w^\frown 0) = x(w)$, $X(w^\frown 0) = x(w) X(0^n)$ and $X(w^\frown 1) = X(w) \setminus X(w^\frown 0)$, and pick as $x(w^\frown 1)$ any element of $X(w^\frown 1)$. If $y_n \neq x(u_r)$, choose $X(0^n)$ in addition such that $y_n \notin x(u_r) X(0^n)$ and put $x(u_r^\frown 1) = y_n$. For all nonbasic w with $|w| = n + 1$, we define $X(w)$ and $x(w)$ by condition (iii).

Then

$$x(w) = x(w_0)\cdots x(w_k) \in x(w_0)\cdots x(w_{k-1})X(w_k) = X(w),$$

and if $x_n \notin \{x(w) : |w| = n\}$ then

$$x_n = x(u_0)\cdots x(u_{r-1})x(u_r^\frown 1) = x(u^\frown 1) \in \{x(w) : |w| = n+1\}.$$

To check (ii), let $|w| = n$. Then

$$\begin{aligned} x(w^\frown 0) &= x(w_0 + \cdots + w_k + 0^n) = x(w_0)\cdots x(w_k)x(0^n) = x(w_0)\cdots x(w_k) \\ &= x(w), \\ X(w^\frown 0) &= x(w)X(0^n) = x(w_0)\cdots x(w_k)X(0^n) = x(w_0)\cdots x(w_{k-1})X(w_k^\frown 0), \\ X(w^\frown 1) &= x(w_0)\cdots x(w_{k-1})X(w_k^\frown 1), \end{aligned}$$

so

$$\begin{aligned} X(w^\frown 0) \cup X(w^\frown 1) &= x(w_0)\cdots x(w_{k-1})\left[X(w_k^\frown 0) \cup Y(w_k^\frown 1)\right] \\ &= x(w_0)\cdots x(w_{k-1})X(w_k) = X(w), \\ X(w^\frown 0) \cap X(w^\frown 1) &= \emptyset. \end{aligned}$$

Now, for every $x \in G$, there is $w \in W$ with nonzero last letter such that $x = x(w)$, so $\{v \in W : x = x(v)\} = \{w^\frown 0^n : n < \omega\}$. It follows that we can define $h : X \to \mathbb{B}$ by putting

$$h(x(w)) = \overline{w} = (a_0, \ldots, a_n, 0, 0, \ldots)$$

for every $w = a_0 \cdots a_n \in W$.

It is clear that h is a bijection with $h(e) = 0$. Since, for every $w = a_0 \cdots a_n$, $X(w)$ consists of all elements $x \in G$ such that

$$h(x) = (a_0, \ldots, a_n, \ldots),$$

is continuous. To check (1), let $x = x(w)$ and let $y = x(v) \in X\left(0^{|w|+1}\right)$. Then

$$h(x(w)x(v)) = h(x(w+v)) = \overline{w+v} = \overline{w} + \overline{v} = h(x(w)) + h(x(v)).$$

To check (2), let $x = x(w) \in D$ with $w = w_0 + \cdots + w_k$. Then

$$x \in x(w_0)X\left(0^{|w_0|+1}\right) \quad \text{and} \quad x(w_0)X\left(0^{|w_0|+1}\right) \cap D \subseteq \{x(w_0)\},$$

so $x = x(w_0)$. Hence $h(x) = \overline{w_0}$ is basic.

If G is first countable, we can choose $\{X(0^n) : n < \omega\}$ to be a neighborhood base of e, and then h will be a homeomorphism. □

COROLLARY 2.2. *Let G be a countable regular left topological group and let D be a discrete subset of G such that $e \notin D$ and* $\operatorname{cl} D \setminus D \subseteq \{e\}$. *Then every mapping from D into a semigroup S can be extended to a local homomorphism $f : G \to S$.*

PROOF. Let $h : G \to \mathbb{B}$ be a local homomorphism guaranteed by Theorem 2.1 and let f_0 be a mapping from D into S. Define a local homomorphism $g : \mathbb{B} \to S$ as follows. Let $x \in \mathbb{B}$. If $h^{-1}(x) \in D$, put $g(x) = f_0(x)$. For any other basic $x \in \mathbb{B}$, pick $g(x)$ arbitrarily. If x is nonbasic, put $g(x) = g(x_0) \cdots g(x_k)$, where $x = x_0 + \cdots + x_k$ is the canonical representation of x. We then define $f = g \circ h$.

Obviously, $f|_D = f_0$. To see that f is a local homomorphism, let $x \in G \setminus \{e\}$. Put

$$U_x = \{y \in G : \min \operatorname{supp}(h(y)) \geq \max \operatorname{supp}(h(x)) + 2\} \cup \{e\}.$$

Then U_x is a neighborhood of e because h is continuous, and for every $y \in U_x \setminus \{e\}$,

$$f(xy) = g(h(x) + h(y)) = g(h(x)) \cdot g(h(y)) = f(x)f(y). \qquad \square$$

COROLLARY 2.3. *Every countable regular left topological group containing a discrete subset with exactly one accumulation point admits a local homomorphism onto* $\mathbb{N}$.

PROOF. Let G be a countable regular left topological group and let D be a discrete subset of G with $\operatorname{cl} D \setminus D = \{e\}$. For every $x \in D$, put $f_0(x) = 1 \in \mathbb{N}$. By Corollary 2.2, the mapping $f_0 : D \to \mathbb{N}$ can be extended to a local homomorphism $f : G \to \mathbb{N}$. Since $e \in \operatorname{cl} D$ and $f(D) = \{1\}$, f is onto. $\square$

Corollary 2.3 can be specified by using the following notion.

Given a space X with a distinguished element $e \in X$, let $\delta(X)$ denote the least cardinal such that for every discrete subset D of X with $D' = \{e\}$ where D' is the set of all accumulation points of $D \subseteq X$ and for every partition $\{D_i : i \in I\}$ of D with $D_i' = \{e\}$ for each $i \in I$, one has $|I| \leq \delta(X)$.

Notice that

(1) $\delta(X) = 0$ if and only if X has no discrete subset D with $D' = \{e\}$, and

(2) $0 < \delta(X) = n < \omega$ if and only if there is a discrete $D \subset X$ with a partition $\{D_i : i < n\}$ of D such that $D' = D_i' = \{e\}$ for each $i \in I$, and for every discrete $C \subset X$ with $C' = \{e\}$ there exist a neighborhood U of e and a $J \subseteq n$ such that $C \cap U = \left(\bigcup_{i \in J} D_i\right) \cap U$.

LEMMA 2.4. *Let X be a countable space with a distinguished element $e \in X$. Then $\delta(X) = \omega$ if and only if there is a discrete $D \subset X$ with a partition $\{D_i : i < \omega\}$ of D such that $D' = D_i' = \{e\}$ for each $i \in I$.*

PROOF. The sufficiency is obvious. To see the necessity, let $(n_i)_{i<\omega}$ be an increasing sequence of natural numbers and let $(C_i)_{i<\omega}$ be a sequence of discrete subsets of G such that for every $i < \omega$ there is a partition $\{C_{ij} : j < n_i\}$ of C_i with $C_{ij}' = C_i' = \{e\}$. Since X is countable and Hausdorff, there exists a decreasing sequence $(U_i)_{i<\omega}$ of closed neighborhoods of e with $\bigcap_{i<\omega} U_i = \{e\}$. Put $D = \bigcup_{i<\omega}(D_i \cap U_i)$. Clearly, $D' = \{e\}$ and there is a partition $\{D_i : i < \omega\}$ of D with $D_i' = \{e\}$. $\square$

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COROLLARY 2.5. *Every countable regular left topological group G with $\delta(G) > 0$ admits a local homomorphism onto any semigroup generated by a subset whose cardinality is less than or equal to $\delta(G)$.*

PROOF. Let S be a semigroup with a generating set I such that $|I| \leq \delta(G)$. Then there is a discrete subset D of G with a partition $\{D_i : i \in I\}$ of D such that $D' = D_i' = \{e\}$ (Lemma 2.4). Define the mapping $f_0 : D \to I$ by $f_0(x) = i$ if $x \in D_i$. By Corollary 2.2, the mapping f_0 can be extended to a local homomorphism $f : G \to S$. Clearly, f is onto. □

We now are going to show that it is consistent that countable topological groups with $0 < \delta(G) < \omega$ do not exist.

A point of a topological space is called a *P-point* if the intersection of any countable family of its neighborhoods is again its neighborhood. A nonprincipal ultrafilter p on ω is a P-point in $\omega^* = \beta\omega \setminus \omega$ if and only if for every partition $\{A_n : n < \omega\}$ of ω with $A_n \notin p$ there exists $A \in p$ such that $|A \cap A_n| < \omega$ for all n. It is consistent with ZFC that there are no P-points in ω^* (see [7]).

THEOREM 2.6. *Let G be a countable topological group and let D be a discrete subset of G with $D' = \{e\}$. Assume that*

(1) *there is only one ultrafilter on G containing D and converging to e,*

(2) *there is an open $U \supset D$ such that for every discrete $C \subset U$ with $C' = \{e\}$ we have $C \cap D \neq \emptyset$.*

Then there is a P-point in ω^*.

PROOF. Enumerate D as $\{x_n : n < \omega\}$. Choose a decreasing sequence $(U_n)_{n<\omega}$ of clopen neighborhoods of e such that $x_n \notin U_n$ and $x_n U_n \subseteq U$ and all $x_n U_n$ are pairwise disjoint and $\operatorname{cl}\left(\bigcup_{n<\omega} x_n U_n\right) = \left(\bigcup_{n<\omega} x_n U_n\right) \cup \{e\}$. Define the mapping $f : G \to \omega$ by

$$f(x) = \begin{cases} n & \text{if } x \in U_n \setminus U_{n+1} \\ 0 & \text{if } x \notin U_0. \end{cases}$$

Let p be the ultrafilter on G containing D and converging to e and let $\overline{f} : \beta G \to \beta\omega$ be the continuous extension of f and let $q = \overline{f}(p)$, so that q is an ultrafilter on ω with a base of subsets $f(B)$, where $B \in p$. Clearly q is nonprincipal. We shall prove that q is a P-point in ω^*.

Let $\{A_m : m < \omega\}$ be a partition of ω with $A_m \notin q$ and let $D_m = f^{-1}(A_m)$. Then $\{D_m : m < \omega\}$ is a partition of D with $D_m \notin p$, and consequently $e \notin \operatorname{cl} D_m$. Every element $x_n \in D$ belongs to some subset D_{m_n} of the partition. Put $C_n = x_n(U_n \cap D_{m_n})$

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and $C = \bigcup_{n<\omega} C_n$. Then $C_n \subset x_n U_n$ and $C_n' = \emptyset$ and $x_n \notin C_n$. It follows that $C' \subseteq \{e\}$ and $C \cap D = \emptyset$. Hence $e \notin \operatorname{cl} C$. Choose neighborhoods V and W of e such that $V \cap C = \emptyset$ and $W^2 \subseteq V$. Put $E = D \cap W$ and $A = f(E)$. We claim that all $A \cap A_m$ are finite. Indeed, otherwise there exist $x_n \in E \cap D_m$ and $x_k \in U_n \cap E \cap D_m$. Then, on the one hand, $x_n x_k \in E^2 \subseteq W^2 \subseteq V$ and, on the other hand, $x_n x_k \in x_n(U_n \cap D_m) \subseteq C \subseteq G \setminus V$, which is a contradiction. □

COROLLARY 2.7. *If there are no P-points then there are no countable topological groups G with $0 < \delta(G) < \omega$.*

PROOF. Let $\delta(G) = n$ and let D be a discrete subset of G with a partition $\{D_i : i < n\}$ of D such that $D' = D_i' = \{e\}$. It is clear that for each $i < n$, there is only one ultrafilter on G containing D_i and converging to e. Fix $i < n$ and for every $x \in D_i$, choose an open neighborhood U_x of x such that $U_x \cap (D \setminus D_i) = \emptyset$, and put $U = \bigcup_{x \in D_i} U_x$. Then for every discrete $C \subset U$ with $C' = \{e\}$, there exists a neighborhood V of e such that $C \cap V = D_i \cap V$. Hence, one can apply Theorem 2.6. □

Combining Corollary 2.5 and Corollary 2.7 gives the following result.

COROLLARY 2.8. *If there are no P-points then every countable topological group containing a discrete subset with exactly one accumulation point admits a local homomorphism onto any countable semigroup.*

REMARK. Given any nonprincipal ultrafilter p on ω, one can define the group topology $\mathscr{T}_p$ on $\mathbb{B} = \bigoplus_\omega \mathbb{Z}_2$ by taking as a neighborhood base of zero the subgroups $\{x \in \mathbb{B} : \operatorname{supp} x \subset A\}$, where $A \in p$. If p is a selective ultrafilter then $G = (\mathbb{B}, \mathscr{T}_p)$ is an extremally disconnected topological group (see [8]). It can be shown that $\delta(G) = 1$ and G admits local homomorphisms only onto cyclic semigroups.

Recall that an ultrafilter p on ω is called *selective* if for every partition $\{A_n : n < \omega\}$ of ω with $A_n \notin p$, there exists $A \in p$ such that $|A \cap A_n| \leq 1$ for all n. Obviously, every nonprincipal selective ultrafilter is a P-point. A nonprincipal selective ultrafilter can be constructed using Martin's Axiom (see, for example, [3]).

3. Local homomorphisms onto right zero semigroups

For any sequence $(m_n)_{n<\omega}$ of integers ≥ 2, let the group $\bigoplus_{n<\omega} \mathbb{Z}_{m_n}$ be endowed with the topology induced by the product topology on $\prod_{n<\omega} \mathbb{Z}_{m_n}$.

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THEOREM 3.1. *Let G be a countable nondiscrete maximally almost periodic topological group. Then there exist a sequence $(m_n)_{n<\omega}$ of integers ≥ 2 and a continuous bijection $h : G \to \bigoplus_{n<\omega} \mathbb{Z}_{m_n}$ with $h(e) = 0$ such that*

(1) *$h(xy) = h(x) + h(y)$ whenever $r(x) + 2 \leq l(y)$, where $r(x) = \max \operatorname{supp}(h(x))$ and $l(y) = \min \operatorname{supp}(h(y))$,*

(2) *$r(x^i y^j) \leq M + 1$ for all $i, j \in \{1, -1\}$, where $M = \max\{r(x), r(y)\}$, and consequently, if $|r(x) - r(y)| \geq 2$, then $r(x^i y^j) \geq M - 1$,*

(3) *if $m \geq 1$ and $\operatorname{pr}_n(x) = \operatorname{pr}_n(y)$ for all $n \leq m$, then $l(x^{-1}y), l(xy^{-1}) \geq m - 1$, where $\operatorname{pr}_n(x) = (h(x))_n$.*

If G is first countable, then h can be chosen to be a homeomorphism.

PROOF. The proof is similar to that of Theorem 2.1.

Given a sequence $(m_n)_{n<\omega}$ of integers ≥ 2, let $W = W((m_n)_{n\in\mathbb{N}})$ be the set of words of the form $w = a_0 \cdots a_n$, where $a_i \in \mathbb{Z}_{m_i}$, including the empty word $\emptyset$. A word $w = a_0 \cdots a_n$ is *basic* if $\{i \leq n : a_i \neq 0\}$ is a nonempty final interval in $\{0, \ldots, n\}$. For any $w = a_0 \cdots a_n$ and $v = b_0 \cdots b_m$, define $w + v = c_0 \cdots c_k$ by $k = \max\{n, m\}$ and

$$c_i = \begin{cases} a_i & \text{if } i \leq n \\ b_i & \text{otherwise}. \end{cases}$$

Each nonempty $w \in W$ has a unique *canonical* representation in the form $w = w_0 + \cdots + w_k$ where

(a) for every $l \leq k - 1$, w_l is basic,
(b) w_k is either basic or zero,
(c) for every $l \leq k - 1$, $|w_l| < |w_{l+1}|$ and the first $|w_l| + 1$ letters of w_{l+1} are zero.

From now on, when we write $w = w_0 + \cdots + w_k$ we mean that this is the canonical representation.

Enumerate $G \setminus \{e\}$ as $\{x_n : n < \omega\}$. We shall construct a sequence $(m_n)_{n<\omega}$ of integers ≥ 2 and assign to each $w \in W = W((m_n)_{n<\omega})$ a point $x(w) \in X$ and a clopen neighborhood $X(w)$ of $x(w)$ so that

(i) $x(0^n) = e$ and $X(\emptyset) = G$,
(ii) $\{X(wa) : a \in \mathbb{Z}_{m_n}\}$ is a partition of $X(w)$, where $n = |w|$,
(iii) $x(w) = x(w_0) \cdots x(w_k)$ and $X(w) = x(w_0) \cdots x(w_{k-1}) X(w_k)$, where $w = w_0 + \cdots + w_k$,
(iv) $\{x_n\} \cup X_{n-1}^{\pm 1} X_{n-1}^{\pm 1} \subseteq X_n$, where $X_n = \{x(w) : |w| = n + 1\}$,
(v) $X(w) \subseteq x(w) X\left(0^{|w|}\right)$,
(vi) $X(0^n) = \left(X(0^n)\right)^{-1}$ and $x\left(X(0^{n+1})\right)^2 x^{-1} \subseteq X(0^n)$ for all $x \in G$.

Without loss of generality we may assume that G is totally bounded.

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Suppose that $X(w)$ and $x(w)$ have been constructed for all w with $|w| \leq n$ such that conditions (i) to (vi) are satisfied.

Denote $Y_n = X_{n-1}^{\pm 1} X_{n-1}^{\pm 1} \cup \{x_n\}$ and let $y \in Y_n$. Notice that the subsets $X(w)$ with $|w| = n$ form a partition of G. So exactly one of them, say $X(u)$, contains y. Let $u = u_0 + \cdots + u_r$. Then $X(u) = x(u_0) \cdots x(u_{r-1}) X(u_r)$ and $y = x(u_0) \cdots x(u_{r-1}) y'$ for some unique $y' \in X(u_r)$. Put $Z_n = \{y' : y \in Y_n\} \setminus X_{n-1}$. Choose $X(0^{n+1})$ by (vi) and such that for every basic u with $|u| = n$,

$$X(u) \setminus x(u) X(0^{n+1}) \neq \emptyset \text{ and } x(u) X(0^{n+1}) \cap Z_n = \emptyset.$$

Choose $m_n \geq 2$ large enough to ensure that that for every basic u, there exists a finite $(m_n - 1)$-element subset $F(u) \subset X(u) \setminus x(u) X(0^{n+1})$ with

$$Z_n \cap X(u) \subseteq F(u) \quad \text{and} \quad X(u) \setminus x(u) X\left(0^{n+1}\right) \subseteq F(u) \cdot X\left(0^{n+1}\right).$$

Enumerate $F(u)$ as $\left\{x(ua) : a \in \mathbb{Z}_{m_n} \setminus \{0\}\right\}$ and choose as $\left\{X(ua) : a \in \mathbb{Z}_{m_n} \setminus \{0\}\right\}$ a clopen partition of $X(u) \setminus x(u) X(0^{n+1})$ such that $x(ua) \in X(ua) \subseteq x(ua) X(0^{n+1})$. For nonzero nonbasic w with $|w| = n + 1$, define $X(w)$ and $x(w)$ by condition (3). Then

$$x(w) = x(w_0) \cdots x(w_k) \in x(w_0) \cdots x(w_{k-1}) X(w_k) = X(w),$$

and if $y \in Y_n \setminus X_{n-1}$, then

$$y = x(u_0) \cdots x(u_{r-1}) x(u_r a) = x(ua) \in X_n, \quad \text{and}$$

$$X(w) = x(w_0) \cdots x(w_{k-1}) X(w_k) \subseteq x(w_0) \cdots x(w_{k-1}) x(w_k) X(0^{|w|}) = x(w) X(0^{|w|}).$$

Similarly, to check (ii), let $|w| = n$. Then

$$x(w0) = x(w_0 + \cdots + w_k + 0^n) = x(w_0) \cdots x(w_k) x(0^n) = x(w_0) \cdots x(w_k) = x(w),$$
$$X(w0) = x(w) X(0^n) = x(w_0) \cdots x(w_k) X(0^n) = x(w_0) \cdots x(w_{k-1}) X(w_k 0),$$
$$X(wa) = x(w_0) \cdots x(w_{k-1}) X(w_k a).$$

Now, for every $x \in G$ there is $w \in W$ with nonzero last letter such that $x = x(w)$, so $\{v \in W : x = x(v)\} = \{w^\frown 0^n : n < \omega\}$. It follows that we can define $h : G \to \bigoplus_{n<\omega} \mathbb{Z}_{m_n}$ by putting, for every $w = a_0 \cdots a_n \in W$,

$$h(x(w)) = \overline{w} = (a_0, \ldots, a_n, 0, 0, \ldots).$$

As in the proof of Theorem 2.1, it can be easily checked that h is a continuous bijection satisfying (1). The condition (2) is the inclusion $X_M^{\pm 1} X_M^{\pm 1} \subseteq X_{M+1}$. To see (3), let $x, y \in X(w)$ for some w with $|w| = m + 1$. Then by (v), $x, y \in x(w) X(0^{m+1})$, and then by (vi),

$$x^{-1} y \in \left(X\left(0^{m+1}\right)\right)^{-1} (x(w))^{-1} x(w) X\left(0^{m+1}\right) = \left(X\left(0^{m+1}\right)\right)^{-1} X\left(0^{m+1}\right) \subseteq X(0^m),$$
$$x y^{-1} \in x(w) X\left(0^{m+1}\right) \left(X\left(0^{m+1}\right)\right)^{-1} (x(w))^{-1} = x(w) \left(X\left(0^{m+1}\right)\right)^2 (x(w))^{-1} \subseteq X(0^m).$$

If G is first countable, we can choose $\{X(0^n) : n < \omega\}$ to be a neighborhood base of e, and then h will be a homeomorphism. □

THEOREM 3.2. *Let G be a countable nondiscrete maximally almost periodic topological group and let $h : G \to \bigoplus_{n<\omega} \mathbb{Z}_{m_n}$ be a bijection guaranteed by Theorem 3.1. Define the subset $F \subseteq \omega^*$ by $F = \overline{r}(\mathrm{Ult}(G))$, where $\overline{r} : \beta G \setminus \{e\} \to \beta\omega$ is the continuous extension of $r : G \setminus \{e\} \to \omega$. Assume that G has no subsets with exactly one accumulation point and that F is finite. Then every point from F is a P-point.*

PROOF. Let $\mathscr{F}$ be the filter on ω with a base consisting of subsets of the form $r(U) = \{r(x) : x \in U \setminus \{e\}\}$, where U is a neighborhood of $e \in G$. Observe that F consists of all ultrafilters on ω containing $\mathscr{F}$. For every $s \in F$, choose $A(s) \in s$ so that the subsets $A(s)$, where $s \in F$, are pairwise disjoint, and put

$$A = \bigcup_{s \in F} A(s).$$

It is clear that $A \in \mathscr{F}$. Let $\{A_n(s) : n < \omega\}$ be any partition of $A(s)$ with $A_n(s) \notin s$. Without loss of generality we may assume that if $s = t + 1$ for $s, t \in F$, then $A_n(s) = A_n(t) + 1$. For every $n < \omega$, put

$$A_n = \bigcup_{s \in F} A_n(s).$$

Then $\{A_n : n < \omega\}$ is a partition of A and for every $m < \omega$,

$$\bigcup_{m \le n < \omega} A_n \in \mathscr{F}.$$

Define $g : A \to \omega$ by

$$g(x) = n \quad \text{if } n \in A_n \setminus A_{n+1}.$$

Notice that if both x and $x + 1 \in A$ then $g(x) = g(x + 1)$. To prove the theorem, it is enough to find $C \in \mathscr{F}$ such that $g|_C$ is finite-to-one.

Define the subset $U \subseteq G$ by

$$U = \{x \in G : g(r(x)) \ge l(x)\} \cup \{e\}.$$

We claim that U is a neighborhood of e. Indeed, assume the contrary and let $P = G \setminus U$. Then $P \in p$ for some $p \in \mathrm{Ult}(G)$ and $g(r(x)) < l(x)$ for all $x \in P$. Since e is an accumulation point of P, there exists one more accumulation point $a \ne e$ of P. Consequently, there exists $q \in \mathrm{Ult}(G)$ such that $P \in aq$. Choose $Q \in q$ such that $r(a) + 2 \le l(x)$ for all $x \in Q$ and $aQ \subseteq P$. Then for every $x \in Q$ we have $r(x) = r(ax)$ and $g(r(ax)) < l(ax) = l(a)$, hence $g(r(x)) < l(a)$ which contradicts

$$\bigcup_{l(a) \le n < \omega} A_n \in \mathscr{F}.$$

Now choose a neighborhood V of e such that $VV^{-1} \subseteq U$ and put $C = r(V)$. Assume on the contrary that $g|_C$ is not finite-to-one. Then there exist $m < \omega$ and a sequence $(a_n)_{n<\omega}$ of elements in V such that $r(a_n) + 2 \leq r(a_{n+1})$ and $g(r(a_n)) = m$. Since $(a_n)_{n<\omega}$ is infinite and $\bigoplus_{n \leq m+1} \mathbb{Z}_{m_n}$ is finite, there exist $n < k < \omega$ such that $\mathrm{pr}_i(g(a_n)) = \mathrm{pr}_i(g(a_k))$ for all $i \leq m + 1$. Then we obtain that $l(a_k a_n^{-1}) \geq m$,

$$r\left(a_k a_n^{-1}\right) \in r(a_k) + \{-1, 0, 1\}$$

and $a_k a_n^{-1} \in U$, hence

$$g(r(a_k)) = g\left(r\left(a_k a_n^{-1}\right)\right) > l\left(a_k a_n^{-1}\right) \geq m,$$

which is a contradiction. □

COROLLARY 3.3. *If there are no P-points then every countable nondiscrete maximally almost periodic topological group admits a local homomorphism onto the countably infinite right zero semigroup.*

PROOF. Let G be any countable nondiscrete maximally almost periodic topological group. If G has a discrete subset with exactly one accumulation point then, by Corollary 2.8, G admits a local homomorphism onto any countable semigroup. Assume that G has no discrete subset with exactly one accumulation point. Let $h : G \to \bigoplus_{n<\omega} \mathbb{Z}_{m_n}$ be a bijection guaranteed by Theorem 3.1. Then, by Theorem 3.2, the subset $F = \overline{r}(\mathrm{Ult}(G)) \subseteq \omega^*$ is infinite. Consequently, one can choose a partition $\{A_n : n < \omega\}$ of ω such that for every neighborhood U of e we have $r(U) \cap A_n \neq \emptyset$. Let $S = \{a_n : n < \omega\}$ be the countably infinite right zero semigroup. Define $f : G \to S$ by

$$f(x) = a_n \text{ if } r(x) \in A_n.$$

To see that f is a local homomorphism, let

$$U_x = \{y \in G : r(x) + 2 \leq l(y)\} \cup \{e\}.$$

Then U_x is a neighborhood of e and for every $y \in U_x \setminus \{e\}$, $r(xy) = r(y)$, so $f(xy) = f(y) = f(x)f(y)$.

To see that f is onto, let U be any neighborhood of e and let $a_n \in S$. Pick $x \in U \setminus \{e\}$ with $r(x) \in A_n$. Then $f(x) = n$. □

REMARK. Under Martin's Axiom, there is a group topology $\mathscr{T}$ on $\mathbb{B} = \bigoplus_\omega \mathbb{Z}_2$ with exactly one nonprincipal ultrafilter converging to the zero (see [5]). It is clear that $(\mathbb{B}, \mathscr{T})$ admits a local homomorphism only onto the 1-element semigroup.

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References

[1] A. Arhangel'skiĭ, 'Every extremally disconnected bicompactum of weight $\mathfrak{c}$ is inhomogeneous', *Soviet Math. Dokl.* **8** (1967), 897–900.
[2] W. Comfort and J. van Mill, 'Groups with only resolvable group topologies', *Proc. Amer. Math. Soc.* **120** (1994), 687–696.
[3] D. Fremlin, *Consequences of Martin's axiom* (Cambridge University Press, 1984).
[4] N. Hindman and D. Strauss, *Algebra in the Stone-Čech compactification* (De Gruyter, Berlin, 1998).
[5] V. Malykhin, 'Extremally disconnected and similar groups', *Soviet Math. Dokl.* **16** (1975), 21–25.
[6] I. Protasov, 'Indecomposable topologies on groups', *Ukranian Math. J.* **50** (1998), 1879–1887.
[7] S. Shelah, *Proper forcing* (Springer-Verlag, Berlin, 1982).
[8] S. Sirota, 'The product of topological groups and extremal disconnectedness', *Math. USSR Sbornik* **8** (1969), 169–180.
[9] Y. Zelenyuk, 'Finite groups in $\beta\mathbb{N}$ are trivial', *Semigroup Forum* **55** (1997), 131–132.
[10] ———, 'On partitions of groups into dense subsets', *Topology Appl.* **126** (2002), 327–339.

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