Proceedings of the Edinburgh Mathematical Society (2003) **46**, 279–292 ©
DOI:10.1017/S0013091502000391 Printed in the United Kingdom

EXISTENCE OF POSITIVE SOLUTIONS FOR SUPERLINEAR SEMIPOSITONE m-POINT BOUNDARY-VALUE PROBLEMS

RUYUN MA

Department of Mathematics, Northwest Normal University, Lanzhou 730070, Gansu, People's Republic of China (mary@nwnu.edu.cn)

(Received 18 April 2002)

Abstract In this paper we consider the existence of positive solutions to the boundary-value problems

$$(p(t)u')' - q(t)u + \lambda f(t,u) = 0, \quad r < t < R,$$
$$au(r) - bp(r)u'(r) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i),$$
$$cu(R) + dp(R)u'(R) = \sum_{i=1}^{m-2} \beta_i u(\xi_i),$$

where λ is a positive parameter, $a, b, c, d \in [0, \infty)$, $\xi_i \in (r, R)$, $\alpha_i, \beta_i \in [0, \infty)$ (for $i \in \{1, \dots m-2\}$) are given constants satisfying some suitable conditions. Our results extend some of the existing literature on superlinear semipositone problems. The proofs are based on the fixed-point theorem in cones.

Keywords: multipoint boundary-value problems; positive solutions; fixed-point theorem; cones

2000 *Mathematics subject classification:* Primary 34B10, 34B18, 34B15

1. Introduction

Multipoint boundary-value problems (BVPs) for ordinary differential equations arise in a variety of areas of applied mathematics and physics. For example, the vibrations of a guy wire of uniform cross-section and composed of N parts of different densities can be set up as a multipoint BVP [**7**]; also, many problems in the theory of elastic stability can be handled by multipoint problems [**9**].

In [**5**], Il'in and Moiseev studied the existence of solutions for a linear multipoint BVP. Motivated by that study, Gupta [**3**] studied certain three-point BVPs for nonlinear ordinary differential equations. Since then, more general nonlinear multipoint BVPs have been studied by several authors. We refer the reader to [**3**, **4**, **6**, **10**] for some references.

In this paper, we are interested in the existence of positive solutions for the second-order m-point BVP

$$\left.\begin{aligned} (p(t)u')' - q(t)u + \lambda f(t,u) = 0, \quad r < t < R, \\ au(r) - bp(r)u'(r) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \\ cu(R) + dp(R)u'(R) = \sum_{i=1}^{m-2} \beta_i u(\xi_i), \end{aligned}\right\} \tag{1.1}$$

where $p, q \in C([r,R],(0,\infty))$, $a,b,c,d \in [0,\infty)$, $\xi_i \in (Y,R)$, $\alpha_i, \beta_i \in (0,\infty)$ (for $i \in \{1,\dots m-2\}$) are given constants. If $q \equiv 0$ and $\alpha_i = \beta_i = 0$ for $i = 1,\dots,m-2$, then the m-point BVP (1.1) reduces to the two-point BVP

$$\left.\begin{aligned} (p(t)u')' + \lambda f(t,u) = 0, \quad r < t < R, \\ au(r) - bp(r)u'(r) = 0, \\ cu(R) + dp(R)u'(R) = 0. \end{aligned}\right\} \tag{1.2}$$

In 1996, Anuradha, Hai and Shivaji [**1**] studied the existence of positive solutions for (1.2) under the assumptions:

(A1) $p \in C([r,R],(0,\infty))$;

(A2) $a,b,c,d \in [0,\infty)$ with $ac + ad + bc > 0$;

(A3) $f : [r,R] \times [0,\infty) \to R$ is continuous and there exists an $M > 0$ such that $f(t,u) \geqslant -M$ for every $t \in [r,R]$, $u \geqslant 0$; and

(A4) $\lim_{u\to\infty}(f(t,u)/u) = \infty$ uniformly on a compact subinterval $[\alpha,\beta]$ of (r,R).

They established the following result for (1.2).

Theorem 1.1 (see Theorem 1 in [1]). *Suppose that (A1)–(A4) hold. Then (1.2) has a positive solution for $\lambda > 0$ sufficiently small.*

If $r = 0$, $R = 1$, $\lambda = 1$, $p(t) \equiv 1$, $q(t) \equiv 0$, $f(t,u) = h(t)\bar{f}(u)$, $a = c = 1$, $b = d = 0$, $\alpha_i = 0$ for $i = 1,\dots,m-2$, and $\beta_j = 0$ for $j = 2,\dots,m-2$, then (1.1) reduces to the three-point BVP

$$\left.\begin{aligned} u'' + h(t)\bar{f}(u) = 0, \quad 0 < t < 1, \\ u(0) = 0, \qquad u(1) = \beta_1 u(\xi_1). \end{aligned}\right\} \tag{1.3}$$

In 1999, Ma [**6**] obtained the following result for (1.3).

Theorem 1.2 (see Theorem 1 in [6]).

(H1) $0 < \beta_1 \xi_1 < 1$.

(H2) $\bar{f} \in C([0,\infty),[0,\infty))$.

(H3) *$h \in C([0,1],[0,\infty))$ and there exists $t_0 \in [\xi_1, 1]$ such that $h(t_0) > 0$.*

Then (1.3) has at least one positive solution in one of the two following cases:

(i) $\bar{f}_0 = 0$ *and* $\bar{f}_\infty = \infty$,

(ii) $\bar{f}_0 = \infty$ *and* $\bar{f}_\infty = 0$,

where

$$\bar{f}_0 := \lim_{u\to 0^+} \frac{\bar{f}(u)}{u}, \qquad \bar{f}_\infty := \lim_{u\to\infty} \frac{\bar{f}(u)}{u}.$$

Theorem 1.2 has been improved by Webb [**10**]. We remark that in the proof of Theorem 1.2 we rewrite (1.3) as the following equivalent integral equation:

$$\begin{aligned} u(t) = &-\int_0^t (t-s)h(s)\bar{f}(u(s))\,\mathrm{d}s - \frac{\beta_1 t}{1-\beta_1\xi_1}\int_0^{\xi_1}(\xi_1 - s)h(s)\bar{f}(u(s))\,\mathrm{d}s \\ &+ \frac{t}{1-\beta_1\xi_1}\int_0^1 (1-s)h(s)\bar{f}(u(s))\,\mathrm{d}s \\ := &\ (Au)(t). \end{aligned} \tag{1.4}$$

Clearly, $(Au)(t)$ contains one positive term and two negative terms. This form is not convenient for studying the existence of positive solutions. In fact, in order to apply the fixed-point theorem in cones, we need to show that

$$(Ay)(t) \geqslant 0, \quad \text{for all } y \in C([0,1],[0,\infty)) \text{ and } t \in [0,1]. \tag{1.5}$$

Since Ay contains two negative terms, it is not easy to show that (H1)–(H3) imply that (1.5) holds.

In this paper, we consider the more general m-point BVP (1.1). To deal with (1.1), we give a new integral equation which is equivalent to

$$\begin{gathered} (p(t)u')' - q(t)u + y(t) = 0, \quad r < t < R, \\ au(r) - bp(r)u'(r) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \\ cu(R) + dp(R)u'(R) = \sum_{i=1}^{m-2} \beta_i u(\xi_i), \end{gathered}$$

and contains two positive terms if $y \geqslant 0$. Our most important result (see Theorem 3.1 below) extends the main results of [**1**] in two directions:

(i) the m-point BVP (1.1) is considered; and

(ii) the case $q(t) > 0$ is studied.

By a positive solution of (1.1) we understand a function $u(t)$ which is positive on (r, R) and satisfies the differential equation and the boundary conditions in (1.1).

The main tool of this paper is the following well-known Guo–Krasnoselskii fixed-point theorem.

https://doi.org/10.1017/S0013091502000391 Published online by Cambridge University Press

Theorem 1.3 (see [2]). *Let E be a Banach space, and let $K \subset E$ be a cone. Assume Ω_1, Ω_2 are open bounded subsets of E with $0 \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$, and let*

$$A : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$$

be a completely continuous operator such that

(i) $\|Au\| \leqslant \|u\|$, $u \in K \cap \partial\Omega_1$, *and* $\|Au\| \geqslant \|u\|$, $u \in K \cap \partial\Omega_2$; *or*

(ii) $\|Au\| \geqslant \|u\|$, $u \in K \cap \partial\Omega_1$, *and* $\|Au\| \leqslant \|u\|$, $u \in K \cap \partial\Omega_2$.

Then A has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

2. Preliminary lemmas

In the rest of the paper, we make the following assumptions:

(C1) $p \in C^1([r, R], (0, \infty))$, $q \in C([r, R], (0, \infty))$; and

(C2) $a, b, c, d \in [0, \infty)$ with $ac + ad + bc > 0$, $\alpha_i, \beta_i \in [0, \infty)$ for $i \in \{1, \dots m - 2\}$.

To state and prove the main results of this paper, we need the following lemmas.

Lemma 2.1. *Let (C1) and (C2) hold. Let ψ and ϕ be the solutions of the linear problems*

$$\left.\begin{aligned} (p(t)\psi'(t))' - q(t)\psi(t) = 0,\\ \psi(r) = b, \qquad p(r)\psi'(r) = a \end{aligned}\right\} \tag{2.1}$$

and

$$\left.\begin{aligned} (p(t)\phi'(t))' - q(t)\phi(t) = 0,\\ \phi(R) = d, \qquad p(R)\phi'(R) = -c, \end{aligned}\right\} \tag{2.2}$$

respectively. Then

(i) *ψ is strictly increasing on $[r, R]$, and $\psi(t) > 0$ on $(r, R]$; and*

(ii) *ϕ is strictly decreasing on $[r, R]$, and $\phi(t) > 0$ on $[r, R)$.*

Proof. We shall give a proof for (i) only. The proof of (ii) follows in a similar manner. It is easy to see that (2.1) is equivalent to the problem

$$\left.\begin{aligned} \psi''(t) + \frac{p'(t)}{p(t)}\psi'(t) - \frac{q(t)}{p(t)}\psi(t) = 0,\\ \psi(r) = b, \qquad \psi'(r) = \frac{a}{p(r)}. \end{aligned}\right\} \tag{2.3}$$

Now we divide the proof into three steps.

Step 1. We show that there exists $\sigma \in (0, R-r)$ such that ψ is strictly increasing on $(r, r+\sigma)$.

If $a > 0$, then we are done. If $a = 0$, then we know from (C2) that $b > 0$. Therefore, we have from (2.3) that

$$\psi''(r) = \frac{q(r)}{p(r)}\psi(r) > 0,$$

which implies that there exists $\sigma > 0$ such that $\psi'(t) > 0$ on $(r, r+\sigma)$. Thus $\psi(t)$ is strictly increasing on $(r, r+\sigma)$.

Step 2. We show that ψ has no local maxima on all of (r, R).

In fact, by Step 1, ψ is positive and strictly increasing on $(r, r+\sigma)$. So we can apply the maximum principle (see [**8**, Theorem 1 of Chapter 1]) to show that there are no local maxima on (r, R). Moreover, ψ is non-decreasing on (r, R).

Step 3. We show that ψ is strictly increasing on $[r, R]$.

If there exists $t_2, t_3 \subset [r, R]$ with $t_2 < t_3$ such that $\psi(t_2) = \psi(t_3)$, then

$$\psi(t) \equiv \psi(t_3), \quad t \in [t_2, t_3].$$

This implies

$$\psi'(t) = \psi''(t) = 0, \quad t \in [t_2, t_3].$$

We note that by Steps 1 and 2, $\psi(t_3) > 0$. Thus from (2.3) we get

$$\psi''(t_3) = \frac{q(t_3)}{p(t_3)}\psi(t_3) > 0.$$

This contradicts the fact that $\psi''(t_3) = 0$. □

Notation. Set

$$\rho := p(r)\begin{vmatrix} \phi(r) & \psi(r) \\ \phi'(r) & \psi'(r) \end{vmatrix}, \qquad \Delta := \begin{vmatrix} -\sum_{i=1}^{m-2} \alpha_i \psi(\xi_i) & \rho - \sum_{i=1}^{m-2} \alpha_i \phi(\xi_i) \\ \rho - \sum_{i=1}^{m-2} \beta_i \psi(\xi_i) & -\sum_{i=1}^{m-2} \beta_i \phi(\xi_i) \end{vmatrix}.$$

Lemma 2.2. *Let (C1) and (C2) hold. Assume that*

(C3) $\Delta :\neq 0$.

Then for $y \in C[r, R]$, the problem

$$\left.\begin{aligned} (p(t)u'(t))' - q(t)u(t) + y(t) = 0, \quad r < t < R, \\ au(r) - bu'(r) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \\ cu(R) + du'(R) = \sum_{i=1}^{m-2} \beta_i u(\xi_i) \end{aligned}\right\} \tag{2.4}$$

https://doi.org/10.1017/S0013091502000391 Published online by Cambridge University Press

has a unique solution

$$u(t) = \int_r^R G(t,s)y(s)\,\mathrm{d}s + A(y)\psi(t) + B(y)\phi(t), \tag{2.5}$$

where

$$G(t,s) = \frac{1}{\rho}\begin{cases}\phi(t)\psi(s), & r \leqslant s \leqslant t \leqslant R,\\ \phi(s)\psi(t), & r \leqslant t \leqslant s \leqslant R,\end{cases} \tag{2.6}$$

$$A(y) := \frac{1}{\Delta}\begin{vmatrix} \sum_{i=1}^{m-2}\alpha_i\int_r^R G(\xi_i,s)y(s)\,\mathrm{d}s & \rho - \sum_{i=1}^{m-2}\alpha_i\phi(\xi_i) \\ \sum_{i=1}^{m-2}\beta_i\int_r^R G(\xi_i,s)y(s)\,\mathrm{d}s & -\sum_{i=1}^{m-2}\beta_i\phi(\xi_i)\end{vmatrix} \tag{2.7}$$

and

$$B(y) := \frac{1}{\Delta}\begin{vmatrix} -\sum_{i=1}^{m-2}\alpha_i\psi(\xi_i) & \sum_{i=1}^{m-2}\alpha_i\int_r^R G(\xi_i,s)y(s)\,\mathrm{d}s \\ \rho - \sum_{i=1}^{m-2}\beta_i\psi(\xi_i) & \sum_{i=1}^{m-2}\beta_i\int_r^R G(\xi_i,s)y(s)\,\mathrm{d}s\end{vmatrix}. \tag{2.8}$$

Proof. The proof follows by routine calculations. □

Lemma 2.3. *Let (C1) and (C2) hold. Assume*

(C4) $\Delta < 0$, $\rho - \sum_{i=1}^{m-2}\alpha_i\phi(\xi_i) > 0$, $\rho - \sum_{i=1}^{m-2}\beta_i\psi(\xi_i) > 0$.

Then for $y \in C[r,R]$ with $y \geqslant 0$, the unique solution u of the problem (2.4) satisfies

$$u(t) \geqslant 0, \quad \text{for } t \in [r,R]. \tag{2.9}$$

Proof. This is an immediate consequence of the facts that $G \geqslant 0$ on $[r,R]\times[r,R]$ and

$$A(y) \geqslant 0, \qquad B(y) \geqslant 0. \tag{2.10}$$

□

Lemma 2.4. *Let (C1), (C2) and (C4) hold. Let*

$$\tilde{q}(t) := \min\left\{\frac{\phi(t)}{\phi(r)}, \frac{\psi(t)}{\psi(R)}\right\}. \tag{2.11}$$

Then for $y \in C[r,R]$ with $y \geqslant 0$, the unique solution u of the problem (2.4) satisfies

$$u(t) \geqslant \tfrac{1}{2}\gamma(t)\|u\|,$$

https://doi.org/10.1017/S0013091502000391 Published online by Cambridge University Press

where $\|u\| = \max\{u(t) | t \in [r, R]\}$ *and*

$$\gamma(t) := \frac{1}{k_0}[\tilde{q}(t) + \tilde{A}\psi(t) + \tilde{B}\phi(t)] \tag{2.12}$$

with $k_0 \in N$ *a fixed integer such that*

$$\frac{1}{k_0}[\tilde{q}(t) + \tilde{A}\psi(t) + \tilde{B}\phi(t)] \leqslant 1, \quad \text{for all } t \in [r, R],$$

$$\tilde{A} := \frac{1}{\Delta}\begin{vmatrix} \sum_{i=1}^{m-2} \alpha_i \tilde{q}(\xi_i) & \rho - \sum_{i=1}^{m-2} \alpha_i \phi(\xi_i) \\ \sum_{i=1}^{m-2} \beta_i \tilde{q}(\xi_i) & -\sum_{i=1}^{m-2} \beta_i \phi(\xi_i) \end{vmatrix} \tag{2.13}$$

and

$$\tilde{B} := \frac{1}{\Delta}\begin{vmatrix} -\sum_{i=1}^{m-2} \alpha_i \psi(\xi_i) & \sum_{i=1}^{m-2} \alpha_i \tilde{q}(\xi_i) \\ \rho - \sum_{i=1}^{m-2} \beta_i \psi(\xi_i) & \sum_{i=1}^{m-2} \beta_i \tilde{q}(\xi_i) \end{vmatrix}. \tag{2.14}$$

Proof. We have from (2.6) that

$$0 \leqslant G(t, s) \leqslant G(s, s), \quad t \in [r, R],$$

which implies

$$u(t) \leqslant \int_r^R G(s, s)y(s)\,\mathrm{d}s + A(y)\psi(t) + B(y)\phi(t), \quad \text{for all } t \in [r, R]. \tag{2.15}$$

Applying (2.6), we have that for $t \in [r, R]$

$$\begin{aligned} \frac{G(t, s)}{G(s, s)} &= \begin{cases} \dfrac{\phi(t)}{\phi(s)}, & r \leqslant s \leqslant t \leqslant R, \\ \dfrac{\psi(t)}{\psi(s)}, & r \leqslant t \leqslant s \leqslant R, \end{cases} \\ &\geqslant \begin{cases} \dfrac{\phi(t)}{\phi(r)}, & r \leqslant s \leqslant t \leqslant R, \\ \dfrac{\psi(t)}{\psi(R)}, & r \leqslant t \leqslant s \leqslant R, \end{cases} \\ &\geqslant \tilde{q}(t), \end{aligned} \tag{2.16}$$

where $\tilde{q}(t)$ is as in (2.11). Combining (2.16) with (2.7) and (2.8), we can conclude that

$$A(y) \geqslant \tilde{A}\int_r^R G(s, s)y(s)\,\mathrm{d}s, \qquad B(y) \geqslant \tilde{B}\int_r^R G(s, s)y(s)\,\mathrm{d}s, \tag{2.17}$$

https://doi.org/10.1017/S0013091502000391 Published online by Cambridge University Press

where $\tilde{A}$ and $\tilde{B}$ are as in (2.13) and (2.14), respectively. Thus for $t \in [r, R]$,

$$
\begin{aligned}
u(t) &= \int_r^R G(t,s)y(s)\,\mathrm{d}s + A(y)\psi(t) + B(y)\phi(t) \\
&\geqslant \frac{1}{2}\biggl[\int_r^R G(t,s)y(s)\,\mathrm{d}s + A(y)\psi(t) + B(y)\phi(t)\biggr] + \tfrac{1}{2}[A(y)\psi(t) + B(y)\phi(t)] \\
&= \frac{1}{2}\biggl[\int_r^R \frac{G(t,s)}{G(s,s)} G(s,s)y(s)\,\mathrm{d}s + A(y)\psi(t) + B(y)\phi(t)\biggr] + \tfrac{1}{2}[A(y)\psi(t) + B(y)\phi(t)] \\
&\geqslant \frac{1}{2}\biggl[\tilde{q}(t)\int_r^R G(s,s)y(s)\,\mathrm{d}s + \tilde{A}\int_r^R G(s,s)y(s)\,\mathrm{d}s\psi(t) \\
&\qquad\qquad + \tilde{B}\int_r^R G(s,s)y(s)\,\mathrm{d}s\phi(t)\biggr] + \tfrac{1}{2}[A(y)\psi(t) + B(y)\phi(t)] \\
&= \tfrac{1}{2}k_0\gamma(t)\int_r^R G(s,s)y(s)\,\mathrm{d}s + \tfrac{1}{2}[A(y)\psi(t) + B(y)\phi(t)] \\
&\geqslant \tfrac{1}{2}\gamma(t)\biggl[\int_r^R G(s,s)y(s)\,\mathrm{d}s + A(y)\psi(t) + B(y)\phi(t)\biggr] \\
&\geqslant \tfrac{1}{2}\gamma(t)\|u\| \quad \text{(by (2.15))},
\end{aligned}
$$

where

$$
\gamma(t) := \frac{1}{k_0}[\tilde{q}(t) + \tilde{A}\psi(t) + \tilde{B}\phi(t)].
$$

□

Lemma 2.5. *Let (C1)–(C4) hold and Let $\bar{w}$ be the solution of*

$$
\left.
\begin{aligned}
(p(t)u'(t))' - q(t)u(t) + 1 = 0, \quad r < t < R, \\
au(r) - bu'(r) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \\
cu(R) + du'(R) = \sum_{i=1}^{m-2} \beta_i u(\xi_i).
\end{aligned}
\right\}
\tag{2.18}
$$

Then there exists a positive number C such that $\bar{w}(t) \leqslant C\gamma(t)$ for every $t \in [r, R]$.

Proof. By Lemma 2.2, we know that

$$
\begin{aligned}
\bar{w}(t) &= \int_r^R G(t,s)\,\mathrm{d}s + A(1)\psi(t) + B(1)\phi(t) \\
&= \frac{1}{\rho}\biggl[\int_r^t \phi(t)\psi(s)\,\mathrm{d}s + \int_r^t \psi(t)\phi(s)\,\mathrm{d}s\biggr] + A(1)\psi(t) + B(1)\phi(t)
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \frac{1}{\rho}\left[\int_r^t \phi(t)\psi(t)\,\mathrm{d}s + \int_t^R \psi(t)\phi(t)\,\mathrm{d}s\right] + A(1)\psi(t) + B(1)\phi(t) \\
&\leqslant \frac{1}{\rho}(R-r)\phi(t)\psi(t) + A(1)\psi(t) + B(1)\phi(t) \\
&\leqslant \frac{1}{\rho}(R-r)\phi(r)\psi(R)\tilde{q}(t) + A(1)\psi(t) + B(1)\phi(t) \\
&= \frac{1}{\rho}(R-r)\phi(r)\psi(R)\tilde{q}(t) + \frac{A(1)}{\tilde{A}}\tilde{A}\psi(t) + \frac{B(1)}{\tilde{B}}\tilde{B}\phi(t) \\
&\leqslant \mu[\tilde{q}(t) + \tilde{A}\psi(t) + \tilde{B}\phi(t)] \\
&= C\gamma(t),
\end{aligned}
$$

where $C := k_0\mu$ and

$$
\mu := \begin{cases}
\max\left\{\dfrac{1}{\rho}(R-r)\phi(r)\psi(R), \dfrac{A(1)}{\tilde{A}}, \dfrac{B(1)}{\tilde{B}}\right\}, & \text{if } \displaystyle\sum_{i=1}^{m-2}\alpha_i \neq 0, \quad \sum_{i=1}^{m-2}\beta_i \neq 0, \\
\max\left\{\dfrac{1}{\rho}(R-r)\phi(r)\psi(R), \dfrac{B(1)}{\tilde{B}}\right\}, & \text{if } \displaystyle\sum_{i=1}^{m-2}\alpha_i \neq 0, \quad \sum_{i=1}^{m-2}\beta_i = 0, \\
\max\left\{\dfrac{1}{\rho}(R-r)\phi(r)\psi(R), \dfrac{A(1)}{\tilde{A}}\right\}, & \text{if } \displaystyle\sum_{i=1}^{m-2}\alpha_i = 0, \quad \sum_{i=1}^{m-2}\beta_i \neq 0.
\end{cases} \tag{2.19}
$$

We note that

$$
\tilde{A} > 0 \quad \text{if } \sum_{i=1}^{m-2}\beta_i \neq 0
$$

and

$$
\tilde{B} > 0 \quad \text{if } \sum_{i=1}^{m-2}\alpha_i \neq 0.
$$

So the constant C in (2.19) is well defined. □

3. The main result

The main result of the paper is the following theorem.

Theorem 3.1. *Let (C1), (C2), (C4) and (A3) and (A4) hold. Then (1.1) has a positive solution for $\lambda > 0$ sufficiently small.*

Remark 3.2. Theorem 3.1 extends [**1**, Theorem 1] in two main directions:

(i) the m-point BVPs (1.1) are considered; and

(ii) the case $q(t) > 0$ is studied.

https://doi.org/10.1017/S0013091502000391 Published online by Cambridge University Press

Proof of Theorem 3.1. Let λ satisfy

$$0 < \lambda < \min\left\{\frac{1}{C_1\|\bar{w}\|}, \frac{1}{2CM}\right\}, \tag{3.1}$$

where $C_1 = \sup\{g(t,u) \mid r \leqslant t \leqslant R,\ 0 \leqslant u \leqslant 1\}$, $g(t,u) := f(t,u) + M$ and C is the constant defined in Lemma 2.5. Let $w = \lambda M \bar{w}$. Then u is a positive solution of (1.1) if and only if $\tilde{u} = u + w$ is a solution of

$$\left.\begin{aligned} (p(t)u')' - q(t)u + \lambda\tilde{g}(t, u - w) = 0, \quad r < t < R,\\ au(r) - bp(r)u'(r) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i),\\ cu(R) + dp(R)u'(R) = \sum_{i=1}^{m-2} \beta_i u(\xi_i), \end{aligned}\right\} \tag{3.2}$$

with $\tilde{u}(t) > w(t)$ on (r, R). Here

$$\tilde{g}(t,u) = \begin{cases} g(t,u), & \text{for } u \geqslant 0,\\ g(t,0), & \text{for } u < 0. \end{cases} \tag{3.3}$$

Let

$$K = \{u \in C[r,R] : u(t) \geqslant \tfrac{1}{2}\gamma(t)\|u\|,\ t \in [r,R]\}, \tag{3.4}$$

where γ is as in (2.12). For each $v \in K$, let $u = Tv$ be the solution of

$$\left.\begin{aligned} (p(t)u')' - q(t)u + \lambda\tilde{g}(t, v - w) = 0, \quad r < t < R,\\ au(r) - bp(r)u'(r) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i),\\ cu(R) + dp(R)u'(R) = \sum_{i=1}^{m-2} \beta_i u(\xi_i). \end{aligned}\right\} \tag{3.5}$$

By Lemma 2.2,

$$Tv = \lambda\left[\int_r^R G(t,s)\tilde{g}(s, v(s) - w(s))\,\mathrm{d}s + A(\tilde{g}(\cdot, v - w))\psi(t) + B(\tilde{g}(\cdot, v - w))\phi(t)\right]. \tag{3.6}$$

From Lemma 2.4, we know that $T : K \to K$. It is easy to check that T is completely continuous. We shall prove that T has a fixed point in K by using Theorem 1.3.

Define $\Omega_1 = \{u \in C[r,R] : \|u\| < 1\}$. For $u \in \partial\Omega_1 \cap K$,

$$\begin{aligned} (Tv)(t) &= \lambda\left[\int_r^R G(t,s)\tilde{g}(s, v(s) - w(s))\,\mathrm{d}s + A(\tilde{g}(\cdot, v - w))\psi(t) + B(\tilde{g}(\cdot, v - w))\phi(t)\right]\\ &\leqslant \lambda C_1\left[\int_r^R G(t,s)\,\mathrm{d}s + A(1)\psi(t) + B(1)\phi(t)\right]\\ &= \lambda C_1 \bar{w}(t)\\ &\leqslant 1, \end{aligned}$$

https://doi.org/10.1017/S0013091502000391 Published online by Cambridge University Press

since $0 \leqslant v - w \leqslant v \leqslant 1$. Thus

$$\|Tu\| \leqslant \|u\|, \quad \text{for } u \in \partial\Omega_1 \cap K.$$

Now choose a constant $\tilde{M} > 0$ such that

$$1 \leqslant \tfrac{1}{4}\lambda\tilde{M}\Gamma \inf_{r\leqslant t\leqslant R} \int_\alpha^\beta G(t,s)\,\mathrm{d}s, \tag{3.7}$$

where

$$\Gamma := \min_{\alpha\leqslant t\leqslant\beta} \gamma(t).$$

By (A4), we know that there is a constant $D > 0$ such that

$$\frac{\tilde{g}(t,s)}{s} \geqslant \tilde{M}, \quad \text{for } (t,s) \in [\alpha,\beta] \times [D,\infty). \tag{3.8}$$

Set

$$\rho_2 = \max\left\{4, 4\lambda CM, \frac{4D}{\Gamma}\right\}$$

and define

$$\Omega_2 = \{u \in C[r,R] : \|u\| < \rho_2\}.$$

For $u \in \partial\Omega_2 \cap K$, we have from Lemmas 2.5 and 2.4 that

$$\begin{aligned} u(s) - w(s) &= u(s) - \lambda M\bar{w}(s) \\ &\geqslant u(s) - \lambda MC\gamma(s) \\ &\geqslant u(s) - \frac{\lambda CM}{\rho_2} 2u(s) \\ &\geqslant \tfrac{1}{2}u(s) \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} \min_{\alpha\leqslant s\leqslant\beta}(u(s) - w(s)) &\geqslant \min_{\alpha\leqslant s\leqslant\beta} \tfrac{1}{2}u(s) \\ &\geqslant \min_{\alpha\leqslant s\leqslant\beta} \tfrac{1}{4}\|u\|\gamma(s) \\ &= \tfrac{1}{2}\rho_2\Gamma \geqslant D. \end{aligned} \tag{3.10}$$

Therefore, for $u \in \partial\Omega_2 \cap K$, we have

$$\begin{aligned} \min_{t\in[\alpha,\beta]}(Tu)(t) &= \lambda \min_{t\in[\alpha,\beta]} \int_r^R G(t,s)\tilde{g}(s,u-w)\,\mathrm{d}s \\ &\qquad + A(\tilde{g}(\cdot,u-w))\psi(t) + B(\tilde{g}(\cdot,u-w))\phi(t) \\ &\geqslant \lambda \min_{t\in[\alpha,\beta]} \int_r^R G(t,s)\tilde{g}(s,u-w)\,\mathrm{d}s \end{aligned}$$

https://doi.org/10.1017/S0013091502000391 Published online by Cambridge University Press

$$
\begin{aligned}
&\geqslant \lambda \min_{t\in[\alpha,\beta]} \int_r^R G(t,s)\tilde{M}(u(s)-w(s))\,\mathrm{d}s \\
&\geqslant \lambda \min_{t\in[\alpha,\beta]} \int_r^R G(t,s)\tilde{M}\tfrac{1}{2}u(s)\,\mathrm{d}s \\
&\geqslant \lambda \min_{t\in[\alpha,\beta]} \int_r^R G(t,s)\tilde{M}\tfrac{1}{4}\gamma(s)\,\mathrm{d}s\|u\| \\
&\geqslant \lambda \min_{t\in[\alpha,\beta]} \int_r^R G(t,s)\tilde{M}\tfrac{1}{4}\Gamma\,\mathrm{d}s\|u\| \\
&\geqslant \|u\|. \qquad (3.11)
\end{aligned}
$$

This implies

$$\|Au\| \geqslant \|u\| \quad \text{for } u \in \partial\Omega_2 \cap K.$$

By Theorem 1.3, T has a fixed point $\tilde{u}$ with $1 \leqslant \|\tilde{u}\| \leqslant \rho_2$. It follows that

$$\tilde{u}(t) \geqslant \tfrac{1}{2}\gamma(t) \geqslant \tfrac{1}{2}(2\lambda CM)\gamma(t) \geqslant \lambda M\bar{w}(t) = w(t),$$

and so $u = \tilde{u} - w$ is a positive solution of (1.1), completing the proof of Theorem 3.1. □

4. An example

Let us consider the three-point BVP

$$
\left.
\begin{aligned}
u'' - u + \lambda(u^5 - 2) = 0, \quad 0 < t < 1,\\
u(0) = \tfrac{1}{2}u(\tfrac{1}{2}), \qquad u(1) = \tfrac{1}{2}u(\tfrac{1}{2}).
\end{aligned}
\right\} \qquad (4.1)
$$

Clearly, (C1) and (C2) hold. It is easy to check that

$$\psi(t) = \tfrac{1}{2}(\mathrm{e}^{t} - \mathrm{e}^{-t}), \qquad \phi(t) = \tfrac{1}{2}(\mathrm{e}^{1-t} - \mathrm{e}^{t-1})$$

and

$$\rho = \begin{vmatrix} \phi(0) & \psi(0) \\ \phi'(0) & \psi'(0) \end{vmatrix} = \begin{vmatrix} \tfrac{1}{2}(\mathrm{e} - \mathrm{e}^{-1}) & 0 \\ \tfrac{1}{2}(-\mathrm{e} - \mathrm{e}^{-1}) & 1 \end{vmatrix} = \tfrac{1}{2}(\mathrm{e} - \mathrm{e}^{-1}).$$

Since

$$\Delta = \begin{vmatrix} -\tfrac{1}{2}\psi(\tfrac{1}{2}) & \rho - \tfrac{1}{2}\phi(\tfrac{1}{2}) \\ \rho - \tfrac{1}{2}\psi(\tfrac{1}{2}) & -\tfrac{1}{2}\phi(\tfrac{1}{2}) \end{vmatrix} = -\rho(\tfrac{1}{2}(\mathrm{e} - \mathrm{e}^{-1}) - \tfrac{1}{2}(\mathrm{e}^{1/2} - \mathrm{e}^{-1/2})) < 0,$$

$$\rho - \tfrac{1}{2}\phi(\tfrac{1}{2}) = \tfrac{1}{2}(\mathrm{e} - \mathrm{e}^{-1}) - \tfrac{1}{4}(\mathrm{e}^{1/2} - \mathrm{e}^{-1/2}) > 0$$

and

$$\rho - \tfrac{1}{2}\psi(\tfrac{1}{2}) = \tfrac{1}{2}(\mathrm{e} - \mathrm{e}^{-1}) - \tfrac{1}{4}(\mathrm{e}^{1/2} - \mathrm{e}^{-1/2}) > 0,$$

we know that (C4) is satisfied. Let $\bar{w}$ be the unique solution of

$$
\left.
\begin{aligned}
u'' - u + 1 = 0, \quad 0 < t < 1,\\
u(0) = \tfrac{1}{2}u(\tfrac{1}{2}), \qquad u(1) = \tfrac{1}{2}u(\tfrac{1}{2}),
\end{aligned}
\right\} \qquad (4.2)
$$

https://doi.org/10.1017/S0013091502000391 Published online by Cambridge University Press

then

$$\bar{w} = \frac{(1-\mathrm{e}^{-1})\mathrm{e}^{t} + (\mathrm{e}-1)\mathrm{e}^{-t}}{2[(\mathrm{e}^{1/2}-\mathrm{e}^{-1/2}) + (\mathrm{e}-\mathrm{e}^{-1})]} + 1.$$

Moreover,

$$\|w\| = w(\tfrac{1}{2}) \tag{4.3}$$

and

$$\|w\| \doteq 0.203\,347\,172\,171\,906\,298\,02. \tag{4.4}$$

From (2.11),

$$\tilde{q}(t) = \min\left\{\frac{\phi(t)}{\phi(0)}, \frac{\psi(t)}{\psi(1)}\right\} = \min\left\{\frac{\mathrm{e}^{1-t}-\mathrm{e}^{t-1}}{\mathrm{e}-\mathrm{e}^{-1}}, \frac{\mathrm{e}^{t}-\mathrm{e}^{-t}}{\mathrm{e}-\mathrm{e}^{-1}}\right\} \tag{4.5}$$

and

$$\tilde{q}(t) \leqslant \tilde{q}(\tfrac{1}{2}) = \frac{1}{\mathrm{e}^{1/2}+\mathrm{e}^{-1/2}}. \tag{4.6}$$

From (2.13) and (2.14), we know that

$$\tilde{A} := \frac{1}{\Delta}\begin{vmatrix} \frac{1}{2}\tilde{q}(\frac{1}{2}) & \rho - \frac{1}{2}\phi(\frac{1}{2}) \\ \frac{1}{2}\tilde{q}(\frac{1}{2}) & -\frac{1}{2}\phi(\frac{1}{2}) \end{vmatrix} = -\frac{1}{2\Delta}\rho\tilde{q}(\tfrac{1}{2}) \tag{4.7}$$

and

$$\tilde{B} := \frac{1}{\Delta}\begin{vmatrix} -\frac{1}{2}\psi(\frac{1}{2}) & \frac{1}{2}\tilde{q}(\frac{1}{2}) \\ \rho - \frac{1}{2}\psi(\frac{1}{2}) & \frac{1}{2}\tilde{q}(\frac{1}{2}) \end{vmatrix} = -\frac{1}{2\Delta}\rho\tilde{q}(\tfrac{1}{2}). \tag{4.8}$$

Clearly,

$$\tilde{A} = \tilde{B} \doteq 0.338\,943\,166\,556\,021\,992\,19.$$

Thus from (2.12)

$$\begin{aligned} \tilde{q}(t) + \tilde{A}\psi(t) + \tilde{B}\phi(t) &\leqslant \tilde{q}(\tfrac{1}{2}) + \tilde{A}\psi(1) + \tilde{B}\phi(0) \\ &= \tilde{q}(\tfrac{1}{2})\left\{1 + \frac{\rho}{-2\Delta}\left[\frac{\mathrm{e}^{1}-\mathrm{e}^{-1}}{2} + \frac{\mathrm{e}^{1}-\mathrm{e}^{-1}}{2}\right]\right\} \\ &= \frac{1}{\mathrm{e}^{1/2}+\mathrm{e}^{-1/2}}\left\{1 + \frac{\mathrm{e}^{1}-\mathrm{e}^{-1}}{\mathrm{e}-\mathrm{e}^{-1}-\mathrm{e}^{1/2}+\mathrm{e}^{-1/2}}\right\} \\ &\doteq 0.886\,818\,883\,970\,073\,908\,68. \end{aligned} \tag{4.9}$$

So we can take $k_0 = 1$ and

$$\gamma(t) = \tilde{q}(t) + \tilde{A}\psi(t) + \tilde{B}\phi(t) \tag{4.10}$$

in Lemma 2.4. By (4.10)

$$\begin{aligned} \gamma(t) &\geqslant \tilde{A}\psi(t) + \tilde{B}\phi(t) \\ &= \tilde{A}(\psi(t) + \phi(t)) \\ &\geqslant \tilde{A}\psi(\tfrac{1}{2}) \end{aligned} \tag{4.11}$$

https://doi.org/10.1017/S0013091502000391 Published online by Cambridge University Press

for all $t \in [0,1]$. This together with (4.3) imply that

$$\bar{w}(t) \leqslant C^* \gamma(t), \quad t \in [0,1], \tag{4.12}$$

where

$$C^* = \frac{\|w\|}{\tilde{A}\phi(\frac{1}{2})} \doteq 1.151\,314\,817\,609\,928\,832\,4.$$

Now, by the proof of Theorem 3.1, we know that (4.1) has at least one positive solution for each $\lambda \in (0, \Lambda)$ with

$$\Lambda = \min\left\{\frac{1}{C_1\|\bar{w}\|}, \frac{1}{2C^* M}\right\} \doteq 0.217\,143\,040\,440\,482\,925\,60,$$

where $C_1 = 1$, $M = 2$.

Acknowledgements. The author was supported by the NSFC, GG-110-10736-1003, NWNU-KJCXGC-212 and the Foundation of Major Projects of Science and Technology of the Chinese Education Ministry. The author is very grateful to the referee for his/her valuable suggestions.

References

1. V. Anuhadha, D. D. Hai and R. Shivaji, Existence results for superlinear semipositone boundary value problems, *Proc. Am. Math. Soc.* **124** (1996), 757–763.
2. D. Guo and V. Lakshmikantham, *Nonlinear problems in abstract cones* (Academic, 1988).
3. C. P. Gupta, Solvability of a three-point nonlinear boundary value problem for a second order ordinary differential equation, *J. Math. Analysis Applic.* **168** (1992), 540–551.
4. C. P. Gupta, A generalized multi-point boundary value problem for second order ordinary differential equations, *Appl. Math. Computat.* **89** (1998), 133–146.
5. V. A. Il'in and E. I. Moiseev, Nonlocal boundary value problem of the first kind for a Sturm–Liouville operator in its differential and finite difference aspects, *Diff. Eqns* **23** (1987), 803–810.
6. R. Ma, Positive solutions of a nonlinear three-point boundary-value problems, *Electron. J. Diff. Eqns* **34** (1999), 1–8.
7. M. Moshinsky, Sobre los problemas de condiciones a la frontiera en una dimension de caracteristicas discontinuas, *Bol. Soc. Mat. Mexicana* **7** (1950), 1–25.
8. M. H. Protter and H. F. Weinberger, *Maximum principles in differential equations* (Springer, 1984).
9. S. Timoshenko, *Theory of elastic stability* (McGraw-Hill, New York, 1961).
10. J. R. L. Webb, Positive solutions of some three-point boundary value problems via fixed point theory, *Nonlin. Analysis* **47** (2001), 4319–4332.