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FINITE DIMENSIONAL CHARACTERISTICS RELATED TO SUPERREFLEXIVITY OF BANACH SPACES

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One of the important problems of the local theory of Banach Spaces can be stated in the following way. We consider a condition on finite sets in normed spaces that makes sense for a finite set of any cardinality. Suppose that the condition is such that to each set satisfying it there corresponds a constant describing "how well" the set satisfies the condition.

The problem is: *Suppose that a normed space X has a set of large cardinality satisfying the condition with "poor" constant. Does there exist in X a set of smaller cardinality satisfying the condition with a better constant?*

In the paper this problem is studied for conditions associated with one of R.C. James's characterisations of superreflexivity.

1. INTRODUCTION

One of the most important and interesting classes of Banach spaces is the class of superreflexive Banach spaces. This class can be characterised in many different ways, see in particular [**2, 4, 7, 10, 11, 12, 22, 24, 25**]. See [**1, 5, 6**] for exposition of some of characterisations of superreflexivity.

In this paper we are interested in the following characterisation that is due to James (see [**1**, pp. 236, 237, 265, 270] and [**20**, pp. 200–201]). By B_X we denote the unit ball of a Banach space X.

THEOREM 1. *The following three conditions are equivalent.*

(i) *The space X is non-superreflexive.*

(ii) *For some $\delta > 0$ and each $n \in \mathbf{N}$ there is a sequence $\{x_i\}_{i=1}^n \subset B_X$ such that*

$$\left\|\sum_{i=1}^{n} a_i x_i\right\| \geqslant \delta \cdot \sum_{i=1}^{n} |a_i| \tag{1}$$

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for each sequence $\{a_i\}_{i=1}^n \in \mathbf{R}^n$ *with at most one change of signs.*

(iii) *For each* $0 < \delta < 1$ *and each* $n \in \mathbf{N}$ *there is a sequence* $\{x_i\}_{i=1}^n \subset B_X$ *such that* (1) *is satisfied for each sequence* $\{a_i\}_{i=1}^n \in \mathbf{R}^n$ *with at most one change of signs.*

To discuss this result it is convenient to introduce the following notation. We denote the unit sphere of l_1^n by S_1^n. We denote the subset of S_1^n consisting of all sequences with at most one change of signs by J_n. (When we speak about changes of signs we ignore zero terms.)

Definition 1: Let $A \subset S_1^n$. For a Banach space X we define the *index* $h(X,A)$ *of* A *in* X by

$$h(X,A) = \sup_{\{x_i\}_{i=1}^n \subset B_X} \inf_{\{a_i\} \in A} \left\| \sum_{i=1}^n a_i x_i \right\|.$$

With this definition Theorem 1 can be restated as

Theorem 2. *The following three conditions are equivalent.*

(i) *The space* X *is non-superreflexive.*

(ii) $\inf_n h(X, J_n) > 0$.

(iii) $h(X, J_n) = 1\ \forall n \in \mathbf{N}$.

Using the standard ultraproduct argument (it is described, for example, in [**1**, p. 228]) it can be shown that Theorem 2 implies the following result.

Theorem 3. *For each* $n \in \mathbb{N}$ *and every real numbers* α, β *satisfying* $1 > \alpha > \beta > 0$ *there is* $N = N(n,\alpha,\beta) \in \mathbb{N}$ *such that* $h(X, J_N) \geqslant \beta$ *implies* $h(X, J_n) \geqslant \alpha$.

It is clear that Theorem 3 implies the equivalence (ii)$\Leftrightarrow$(iii) from Theorem 2.

In the local theory of Banach spaces it is important to have (as precise as possible) estimates for finite-dimensional parameters of Banach spaces. The main purpose of this paper is to estimate the function $N(n,\alpha,\beta)$ from below. It is worth mentioning that using the argument of [**25**] it is possible to estimate $N(n,\alpha,\beta)$ from above. The estimate obtained on these lines is a very rapidly increasing function, it is obtained by repeated use of the Ramsey theorem.

The main result of this paper states that for any pair α, β satisfying $1 > \alpha > \beta > 0$ the value of $N(n,\alpha,\beta)$ considered as a function of n grows more rapidly than any polynomial. We state this result in the following way.

Theorem 4. *Let* $1 > \alpha > \beta > 0$. *Then for each* $m \in \mathbf{N}$ *there is* $n \in \mathbb{N}$ *such that* $h(X, J_{n^m}) \geqslant \beta$ *does not imply* $h(X, J_n) \geqslant \alpha$.

This result contrasts with the well-known lemma due to Giesy (see [**8**, Lemmas I.4 and I.6] or [**16**, p. 62]). Using the notation that we have introduced the lemma can be stated as

$$h(X, S_1^{n^2}) \geqslant \alpha \Rightarrow h(X, S_1^n) \geqslant \sqrt{\alpha}.$$

The estimate of $h(X, J_n)$ for a superreflexive space X different from Euclidean seems to be a very difficult problem. For this reason our proof is indirect, it is based on some of the known results on the Kadets path distance. Let us recall necessary definitions.

DEFINITION 2: Let X be a Banach space and let Y and Z be closed subspaces of X. The *opening* (sometimes called also *gap*) between Y and Z is defined to be the Hausdorff distance between their unit balls, that is,

$$\Lambda(Y, Z) = \max\bigl\{\sup_{y\in B_Y} \operatorname{dist}(y, B_Z), \sup_{z\in B_Z} \operatorname{dist}(z, B_Y)\bigr\}.$$

If X and Y are arbitrary Banach spaces we define the *Kadets distance*

$$d_K(X, Y) = \inf_{Z,U,V} \Lambda(UX, VY),$$

where the infimum is taken over all Banach spaces Z and all linear isometric embeddings $U : X \to Z$ and $V : Y \to Z$.

REMARKS. 1. The Kadets distance is a natural Banach-space analogue of the well-known Gromov-Hausdorff distance between metric spaces. This analogue was introduced and studied by Kadets [**13**] in 1975 (before the publication of the famous paper Gromov [**9**]). See [**15**] and references therein for information on the Kadets distance.

2. Using standard techniques it can be shown (see [**17**, p. 110 of the English translation]) that d_K satisfies the triangle inequality. It follows easily that the restriction of d_K to the set of finite-dimensional Banach spaces is a metric. In the infinite-dimensional case d_K does not separate points (there exist non-isomorphic X and Y such that $d_K(X, Y) = 0$, see [**17**, Theorem 1c] or [**15**, p. 37]).

DEFINITION 3: Let X and Y be Banach spaces. Suppose that there exists a continuous (with respect to the Kadets distance) mapping $Z : [0, 1] \to \{\text{Banach spaces}\}$ such that $Z(0) = X$ and $Z(1) = Y$. We call such mapping a *path joining X and Y*. The *Kadets path distance* $d_{KP}(X, Y)$ between X and Y is defined to be the infimum of the lengths with respect to the Kadets distance of all paths joining X and Y.

If there is no path joining X and Y, then $d_{KP}(X, Y)$ is defined to be ∞.

By the length of a path we mean the standard definition for metric geometry, see for example [**3**, p. 34]. The Kadets path distance was introduced and studied in [**21**].

PROOF OF THEOREM 4: The plan of the proof is the following.

1. We assume the contrary, that is, we assume that for some $0 < \beta < \alpha < 1$ and some $m \in \mathbf{N}$ the inequality $h(X, J_{n^m}) \geqslant \beta$ implies $h(X, J_n) \geqslant \alpha$ for every $n \in \mathbf{N}$.
2. We consider a sequence $\{X_n\}_{n=1}^{\infty}$ of spaces satisfying $h(X_n, J_n) = 1$ and $d(X_n, l_2^n) = O(\ln n)$ (by $d(X, Y)$ we denote the Banach-Mazur distance, the existence of spaces $\{X_n\}$ satisfying the conditions was shown by Kadets [**14**], another example was found by Pisier and Xu [**23**]).

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3. We estimate the Kadets path distances $d_{KP}(X_n, l_2^n)$ from below and from above. The obtained estimates contradict each other.

To obtain an estimate for $d_{KP}(X_n, l_2^n)$ from above we use the following result from [**21**].

THEOREM 5. *Let X and Y be isomorphic Banach spaces. If $\ln d(X,Y) \geqslant \pi$, then*

$$d_{KP}(X,Y) \leqslant \pi \ln\ln d(X,Y).$$

Hence

$$d_{KP}(X_n, l_2^n) = O(\ln\ln\ln n). \tag{2}$$

To obtain an estimate from below we need the following (almost immediate) consequence of the triangle inequality for the norm of a Banach space (see [**18**] and [**19**, p. 292]).

PROPOSITION 1. *Let X and Y be Banach spaces, $A \subset S_1^n$ for some $n \in \mathbf{N}$. Then*

$$d_K(X,Y) \geqslant \left|h(X,A) - h(Y,A)\right| \tag{3}$$

The inequality (3) can be used to find estimates for $d_{KP}(X,Y)$.

PROPOSITION 2. *Let X and Y be Banach spaces and let $A_1, \dots, A_{k+1}$ be subsets of S_1^n such that for some numbers $\alpha_1, \dots, \alpha_k$ and $\beta_1, \dots, \beta_k$ the following conditions are satisfied.*

(A) $h(X, A_j) \geqslant \beta_j$ *implies* $h(X, A_{j+1}) \geqslant \alpha_j$, $j = 1, \dots, k$.

(B) $\alpha_j > \beta_{j+1}$, $j = 1, \dots, k$

(C) $h(X, A_1) \geqslant \beta_1$.

(D) $h(Y, A_{k+1}) < \alpha_k$.

Then

$$d_{KP}(X,Y) \geqslant (h(X,A_1) - \beta_1) + \sum_{j=1}^{k}(\alpha_j - \beta_{j+1}) + \left(\alpha_k - h(Y, A_{k+1})\right).$$

OBSERVATION. The conditions (A), (B), and (D) imply

(E) $h(Y, A_j) < \beta_j \ \forall j = 1, \dots, k$.

In fact, otherwise we would have $h(Y, A_j) \geqslant \beta_j \Rightarrow h(Y, A_{j+1}) \geqslant \alpha_j > \beta_{j+1} \Rightarrow h(Y, A_{j+2}) \geqslant \alpha_{j+1} > \beta_{j+2} \Rightarrow \cdots \Rightarrow h(Y, A_{k+1}) \geqslant \alpha_k$, a contradiction.

PROOF OF PROPOSITION 2. Let $X : [0,1] \to \{\text{Banach spaces}\}$ be a path, continuous with respect to d_K, with $X(0) = X, X(1) = Y$.

We shall repeatedly use the following immediate consequence of (3):

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(F) For any fixed A the index $h(X,A)$ is a continuous function of X with respect to d_K.

By (F), (C), and (E) there exists $t_1 \in [0,1]$, such that $0 \leqslant t_1 < 1$ and $h(X(t_1),A_1) = \beta_1$. Hence $h(X(t_1),A_2) \geqslant \alpha_1$ (by (A)).

By (F), (B), and (E) there exists $t_2 \in [0,1]$ such that $t_1 < t_2 < 1$ and $h(X(t_2),A_2) = \beta_2$. Hence $h(X(t_2),A_3) \geqslant \alpha_2$ (by (A)).

We continue in an obvious way. At the last step we find t_k such that $t_{k-1} < t_k < 1$ and $h(X(t_k),A_k) = \beta_k$. Hence $h(X(t_k),A_{k+1}) \geqslant \alpha_k$ (by (A)).

By the definition of d_{KP} we get (by (3) used with different A's for different terms)

$$\begin{aligned} d_{KP}(X,Y) &\geqslant d_K\big(X,X(t_1)\big) + \sum_{j=1}^{k-1} d_K\big(X(t_j),X(t_{j+1})\big) + d_K\big(X(t_k),Y\big) \\ &\geqslant \big(h(X,A_1) - h\big(X(t_1),A_1\big)\big) + \sum_{j=1}^{k-1}\big(h(X(t_j),A_{j+1}) - h(X(t_{j+1}),A_{j+1})\big) \\ &\qquad + \Big(h\big(X(t_k),A_{k+1}\big) - h(Y,A_{k+1})\Big) \\ &\geqslant \big(h(X,A_1) - \beta_1\big) + \sum_{j=1}^{k-1}(\alpha_j - \beta_{j+1}) + \big(\alpha_k - h(Y,A_{k+1})\big), \end{aligned}$$

as was stated. ∎

REMARK. Proposition 2 is a generalisation of a result from [**21**, Section 4].

We return to our proof of Theorem 4. Let $p \in \mathbb{N}$ be such that $h(l_2,J_p) < \beta$. Such p exists because l_2 is superreflexive. To estimate $d_{KP}(X_n,l_2^n)$ from below we introduce the number

$$s = \max\{i \in \mathbf{N} :\ p^{m^i} \leqslant n\}.$$

It is easy to check that there exists an absolute constant $C > 0$ such that

$$s \geqslant C \cdot \ln\ln n \tag{4}$$

for n large enough. (We use the fact that both p and m are absolute constants in this argument.)

Let $r(i) = p^{m^i}$ $i = 0,1,2,\ldots,s$. We apply Proposition 2 with $k = s$,

$$A_1 = J_{r(s)}, A_2 = J_{r(s-1)}, \ldots, A_{s+1} = J_{r(0)},$$

$\alpha_1 = \cdots = \alpha_s = \alpha$, and $\beta_1 = \cdots = \beta_s = \beta$. The assumption made at the beginning of the proof of Theorem 4 implies that the conditions of Proposition 2 are satisfied.

OBSERVATIONS.

1. Since $r(s) \leqslant n$, then $h(X_n,A_1) = 1$.

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2. Since $r(0) = p$, then $h(l_2^n, A_{s+1}) < \beta$.

By Proposition 2 we get

$$d_{KP}(X_n, l_2^n) \geqslant (1-\beta) + (s-1)(\alpha-\beta) + (\alpha-\beta) > (s+1)(\alpha-\beta).$$

From (4) we get

$$d_{KP}(X_n, l_2^n) \geqslant c \cdot \ln\ln n \tag{5}$$

for some absolute constant $c > 0$ if n is large enough. It is clear that (5) contradicts (2). ∎

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