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Radial solvability for Pucci-Lane-Emden systems in annuli

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We establish a priori bounds, existence and qualitative behaviour of positive radial solutions in annuli for a class of nonlinear systems driven by Pucci extremal operators and Lane-Emden coupling in the superlinear regime. Our approach is purely nonvariational. It is based on the shooting method, energy functionals, spectral properties, and on a suitable criteria for locating critical points in annular domains through the moving planes method that we also prove.

Keywords: Fully nonlinear systems; radial positive solutions; annulus; shooting method

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1. Introduction

In this note we study a priori bounds, existence and qualitative behaviour of positive radial solutions for fully nonlinear elliptic partial differential systems such as

$$\begin{cases} \mathcal{M}^{\pm}(D^2u) + v^p = 0 & \text{in } A_{\mathfrak{a},\mathfrak{b}} \\ \mathcal{M}^{\pm}(D^2v) + u^q = 0 & \text{in } A_{\mathfrak{a},\mathfrak{b}} \\ u, v > 0 & \text{in } A_{\mathfrak{a},\mathfrak{b}} \\ u, v = 0 & \text{on } \partial A_{\mathfrak{a},\mathfrak{b}} \end{cases} \tag{1.1}$$

in the superlinear regime $pq > 1$ for p, $q > 0$ where for some $\mathfrak{a}$, $\mathfrak{b} > 0$,

$$A_{\mathfrak{a},\mathfrak{b}} = \{x \in \mathbb{R}^N : \mathfrak{a} < |x| < \mathfrak{b}\}, \text{ with } 0 < \mathfrak{a} < \mathfrak{b} < +\infty,$$

is an annulus in $\mathbb{R}^N$ for $N \geqslant 2$. Here $\mathcal{M}^{\pm}$ are the Pucci's extremal operators, which play an essential role in stochastic control theory and mean field games. We deal with classical solutions of (1.1) that are C^2 in $A_{\mathfrak{a},\mathfrak{b}}$.

The analysis of the associated ODE problem for proving existence of annular or exterior domain solutions has been performed in many papers in the semilinear case [**1**, **11**, **14**]. Up to our knowledge, for the Lane-Emden system, only the case when $p,\, q > 1$ is available, see [**7**] whose proof is based on degree theoretic methods. It is worth mentioning that the change of variables employed in [**11**], which eliminates u' e v' from the ODE problem, does not work for Pucci's extremal operators. Therefore, a completely different approach in this case is required.

In the case of the Lane-Emden system involving the Laplacian operator, since

$$W^{1,s}_{rad}(A_{\mathfrak{a},\mathfrak{b}}) \hookrightarrow C(\overline{A_{\mathfrak{a},\mathfrak{b}}}) \text{ for any } s > 1,$$

its possible to use the standard Mountain Pass theorem to prove existence of a positive radial solution. Nevertheless, regarding a priori bounds, only partial results are known in the Lipschitz superlinear case $p,\, q \geqslant 1$, see [**4**]. The proof there explores a differentiability notion of the nonlinearities besides relying on the variational formulation of the problem. Here we obtain new results in order to give a full picture for the standard Lane-Emden system. Furthermore, since our approach is nonvariational, we are able to develop an existence theory for operators with fully nonlinear structure. We mention that, under no radial symmetry assumptions on the domain, the best known existence result for systems involving Pucci operators requires bounds from above on the exponents p and q, see [**15**].

In the sequel we state our main result.

THEOREM 1.1. *For any $p,\, q > 0$ with $pq > 1$, and $0 < \mathfrak{a} < \mathfrak{b} < +\infty$, problem* (1.1) *has a radial solution pair in the annulus $A_{\mathfrak{a},\mathfrak{b}}$. Moreover, there exists a constant $C = C(\mathfrak{a},\, \mathfrak{b},\, N,\, \lambda,\, \Lambda,\, p,\, q) > 0$ that bounds the L^{∞}-norm of all solutions. In addition, for a fixed $\mathfrak{a} > 0$, the two components of any solution blow up as $\mathfrak{b} \to \mathfrak{a}$.*

The proof of theorem 1.1 consists on a careful study of the ODE problem through the shooting method, asymptotics, energy and topological arguments, spectral properties, and on a suitable criteria for critical points produced via the moving planes method that we also prove. We highlight that degree theory in cones and fully nonlinear operators in the scalar setting were also combined in [**8**, **16**].

More than that, we prove uniform bounds for the maximum positive inclination of the solutions

$$C_2(\mathfrak{a},\mathfrak{b},N,\lambda,\Lambda,p,q) \geqslant u'(\mathfrak{a}),\ v'(\mathfrak{a}) \geqslant C_1(\mathfrak{a},\mathfrak{b},N,\lambda,\Lambda,p,q) > 0, \tag{1.2}$$

which ends up characterizing the admissible shooting parameters in the respective ODE problem in order to produce solutions in the annulus. This is particularly interesting feature since uniqueness of solutions is a delicate issue when it refers to systems.

We highlight that all arguments in this paper could be performed in order to address Hardy-Henón type weights $|x|^a,\ |x|^b$ with $a,\, b \in \mathbb{R}$. Indeed, the energy estimates are a little bit more involved, see [**12**] for a single equation. However, the

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difficulty in obtaining the a priori bounds remains the same since one just plug $\mathfrak{a} \leqslant |x| \leqslant \mathfrak{b}$ on the estimates. In addition, we could also treat more general radial fully nonlinear operators, in light of [9]. We prefer to skip overload notation to keep the presentation simpler and to concentrate in the difficulties produced by the nature of the system above all.

The paper is organized as follows. In § 2 we introduce some basic properties of the second order ODE problem associated to (1.1). In § 3 we obtain the crucial a priori bounds for the solutions in terms of estimates for the corresponding shooting parameters. Finally, § 4 is devoted to the existence statement in theorem 1.1.

2. Auxiliary tools

We start by recalling that the Pucci's extremal operators $\mathcal{M}^{\pm}_{\lambda,\Lambda}$, for $0 < \lambda \leqslant \Lambda$,

$$\mathcal{M}^{+}_{\lambda,\Lambda}(X) := \sup_{\lambda I \leqslant A \leqslant \Lambda I} \operatorname{tr}(AX), \quad \mathcal{M}^{-}_{\lambda,\Lambda}(X) := \inf_{\lambda I \leqslant A \leqslant \Lambda I} \operatorname{tr}(AX),$$

where A, X are $N \times N$ symmetric matrices, and I is the identity matrix. Equivalently, if we denote by $\{e_i\}_{1 \leqslant i \leqslant N}$ the eigenvalues of X, we can define the Pucci's operators as

$$\mathcal{M}^{+}_{\lambda,\Lambda}(X) = \Lambda \sum_{e_i > 0} e_i + \lambda \sum_{e_i \leqslant 0} e_i, \quad \mathcal{M}^{-}_{\lambda,\Lambda}(X) = \lambda \sum_{e_i > 0} e_i + \Lambda \sum_{e_i \leqslant 0} e_i. \tag{2.1}$$

From now on we will drop writing the parameters λ, Λ in the notations for the Pucci's operators.

When u is a radial function, for ease of notation we set $u(|x|) = u(r)$ for $r = |x|$. If in addition u is C^2, the eigenvalues of the Hessian matrix D^2u are given by $\{u'', \frac{u'(r)}{r}, \dots, \frac{u'(r)}{r}\}$ where $\frac{u'(r)}{r}$ is repeated $N-1$ times.

The system (1.1) for $\mathcal{M}^{+}$ and positive solutions is written in radial coordinates as

$$\begin{cases} u'' = M_{+}(-r^{-1}(N-1)\, m_{+}(u') - v^{p}), \\ v'' = M_{+}(-r^{-1}(N-1)\, m_{+}(v') - u^{q}), \quad u,\, v > 0, \end{cases} \tag{P_+}$$

while for $\mathcal{M}^{-}$ one has

$$\begin{cases} u'' = M_{-}(-r^{-1}(N-1)\, m_{-}(u') - v^{p}), \\ v'' = M_{-}(-r^{-1}(N-1)\, m_{-}(v') - u^{q}), \quad u,\, v > 0, \end{cases} \tag{P_-}$$

which are understood in the maximal interval where u, v are both positive.

Let us have in mind the following initial value problem with positive shooting parameters δ, μ, which produces the radial solutions of (1.1),

$$\begin{cases} u'' = M_{\pm}\left(-r^{-1}(N-1)\, m_{\pm}(u') - |v|^{p-1}v\right), \quad u(\mathfrak{a}) = 0,\ u'(\mathfrak{a}) = \delta, \quad \delta > 0, \\ v'' = M_{\pm}\left(-r^{-1}(N-1)\, m_{\pm}(v') - |u|^{q-1}u\right), \quad v(\mathfrak{a}) = 0,\ v'(\mathfrak{a}) = \mu, \quad \mu > 0, \end{cases} \tag{2.2}$$

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where $M_\pm$ and $m_\pm$ are the Lipschitz functions

$$m_+(s) = \begin{cases} \lambda s \text{ if } s \leqslant 0 \\ \Lambda s \text{ if } s > 0 \end{cases} \quad \text{and} \quad M_+(s) = \begin{cases} s/\lambda \text{ if } s \leqslant 0 \\ s/\Lambda \text{ if } s > 0, \end{cases} \tag{2.3}$$

$$m_-(s) = \begin{cases} \Lambda s \text{ if } s \leqslant 0 \\ \lambda s \text{ if } s > 0 \end{cases} \quad \text{and} \quad M_-(s) = \begin{cases} s/\Lambda \text{ if } s \leqslant 0 \\ s/\lambda \text{ if } s > 0. \end{cases} \tag{2.4}$$

Here we denote such a solution by $(u_{\delta,\mu}, v_{\delta,\mu})$. That is, a radial solution of (1.1) in the annulus $A_{\mathfrak{a},\mathfrak{b}}$ satisfies (2.2) for some δ, $\mu > 0$ with $u(\mathfrak{b}) = v(\mathfrak{b}) = 0$. We shall omit the dependence on the parameters δ, μ whenever it is clear from the context.

Next we look at monotonicity properties for solutions $(u_{\delta,\mu}, v_{\delta,\mu})$ of (2.2) as follows.

LEMMA 2.1. *For any δ, $\mu > 0$ such that $(u_{\delta,\mu}, v_{\delta,\mu})$ is a positive solution of* (1.1) *in the annulus $A_{\mathfrak{a},\mathfrak{b}}$, there exist numbers $\tau_u = \tau_u(\delta, \mu)$, and $\tau_v = \tau_v(\delta, \mu)$, with τ_u, $\tau_v \in (\mathfrak{a}, \mathfrak{b})$, such that the solution pair (u, v) of* (2.2) *satisfies*

$$u'(r) > 0 \text{ for } r < \tau_u, \quad u'(\tau_u) = 0, \quad u'(r) < 0 \text{ for } \tau_u < r < b,$$
$$v'(r) > 0 \text{ for } r < \tau_v, \quad v'(\tau_v) = 0, \quad v'(r) < 0 \text{ for } \tau_v < r < b.$$

Proof. Since we have a positive solution pair (u, v) in the annulus, and both functions start positive and increasing, a critical point must exist for both u and v by Rolle's theorem.

The uniqueness of τ_u follows from the fact that, since v is positive, any critical point of u is a strict local maximum; likewise for τ_v and v. □

Notation. Here and onward in the text we write

$$\tau_* = \min\{\tau_u, \tau_v\}, \quad \tau^* = \max\{\tau_u, \tau_v\}.$$

As a consequence of this monotonicity, the problems (P_+) and (P_-) can be better specified. In the interval where $u' \geqslant 0$ and $v' \geqslant 0$ we write

$$\text{for } \mathcal{M}^+ \text{ in } [\mathfrak{a}, \tau_*]: \quad \begin{cases} \lambda u'' = -\Lambda r^{-1}(N-1)u' - v^p, \\ \lambda v'' = -\Lambda r^{-1}(N-1)v' - u^q, \quad u,\, v > 0; \end{cases} \tag{2.5}$$

$$\text{for } \mathcal{M}^- \text{ in } [\mathfrak{a}, \tau_*]: \quad \begin{cases} \Lambda u'' = -\lambda r^{-1}(N-1)u' - v^p, \\ \Lambda v'' = -\lambda r^{-1}(N-1)v' - u^q, \quad u,\, v > 0; \end{cases} \tag{2.6}$$

while in the interval where $u' \leqslant 0$ and $v' \leqslant 0$ it yields

$$\text{for } \mathcal{M}^+ \text{ in } [\tau^*, \mathfrak{b}]: \quad \begin{cases} u'' = M_+(-\lambda r^{-1}(N-1)\, u' - r^a v^p), \\ v'' = M_+(-\lambda r^{-1}(N-1)\, v' - r^b u^q), \quad u,\, v > 0; \end{cases} \tag{2.7}$$

$$\text{for } \mathcal{M}^- \text{ in } [\tau^*, \mathfrak{b}]: \quad \begin{cases} u'' = M_-(-\Lambda r^{-1}(N-1)\, u' - r^a v^p), \\ v'' = M_-(-\Lambda r^{-1}(N-1)\, v' - r^b u^q), \quad u,\, v > 0. \end{cases} \tag{2.8}$$

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Moreover, in between, one of the following situations takes a place for the operators $\mathcal{M}^{\pm}$:

$$\text{for } \mathcal{M}^{+} \text{ in } [\tau_u, \tau_v]: \quad \begin{cases} u'' \;=\; M_{+}(-\lambda r^{-1}(N-1)\, u' - v^p), \\ \lambda v'' \;=\; -\Lambda r^{-1}(N-1) v' - u^q, \qquad u,\, v > 0; \end{cases} \tag{2.9}$$

$$\text{for } \mathcal{M}^{-} \text{ in } [\tau_u, \tau_v]: \quad \begin{cases} u'' \;=\; M_{-}(-\Lambda r^{-1}(N-1)\, u' - v^p), \\ \Lambda v'' \;=\; -\lambda r^{-1}(N-1) v' - u^q, \quad u,\, v > 0; \end{cases} \tag{2.10}$$

if $\tau_* = \tau_u$ and $\tau^* = \tau_v$; while

$$\text{for } \mathcal{M}^{+} \text{ in } [\tau_v, \tau_u]: \quad \begin{cases} \lambda u'' \;=\; -\Lambda r^{-1}(N-1) u' - v^p, \\ v'' \;=\; M_{+}(-\lambda r^{-1}(N-1)\, v' - u^q), \quad u,\, v > 0; \end{cases} \tag{2.11}$$

$$\text{for } \mathcal{M}^{-} \text{ in } [\tau_v, \tau_u]: \quad \begin{cases} \Lambda u'' \;=\; -\lambda r^{-1}(N-1) u' - v^p, \\ v'' \;=\; M_{-}(-\Lambda r^{-1}(N-1)\, v' - u^q), \quad u,\, v > 0. \end{cases} \tag{2.12}$$

if $\tau_* = \tau_v$ and $\tau^* = \tau_u$.

The next theorem gives us a better precision on the location of the critical points. It says no critical points exist in the closure of the half annulus $A_{\frac{1}{2}(\mathfrak{a}+\mathfrak{b}),\,\mathfrak{b}}$.

THEOREM 2.2. *Let* $(u,\, v)$ *be a positive* C^2 *solution pair of* (1.1) *in the annulus* $A_{\mathfrak{a},\mathfrak{b}}$*, with* $u = v = 0$ *on* $\partial A_{\mathfrak{a},\mathfrak{b}}$*. Then* $\partial_r u < 0$ *and* $\partial_r v < 0$ *for all* $r \in \left[\frac{1}{2}(\mathfrak{a}+\mathfrak{b}),\, \mathfrak{b}\right]$*, where* $r = |x|$*.*

The proof is accomplished through the moving planes method as in [**10**, theorem 2], properly adapted to the Lane-Emden system in light of [**6**, **17**]. It is worth observing that a classical Gidas-Ni-Nirenberg type symmetry result does not hold for annular domains in order to conclude that solutions of (1.1) are radial. However, the moving planes method can still be applied to obtain strict monotonicity in a half portion of the annulus.

Proof. We revisit the moving planes method as performed in [**6**] in order to treat the general range $pq > 1$. The lack of C^1 or even Lipschitz continuity on the nonlinearities is allowed there, namely when either $p < 1$ or $q < 1$. Accordingly to the notation in [**6**], for the annulus we have

$$\Lambda_1 = \Lambda_2 = \frac{1}{2}(\mathfrak{a}+\mathfrak{b}),$$

see § 2 in [**6**] for the corresponding definitions.

For any direction $\gamma > 0$ (as positive axis $\{x_1 = 0\}$) it follows as in [**6**, Step 1 of the proof of proposition 3.1, p.4183] that $\gamma \cdot Du < 0$ in the maximal cap Σ_{Λ_1}. The union of these maximal caps, originated by all directions $\gamma = \frac{x}{|x|}$, $x \neq 0$, produces the half annulus $A_{\Lambda_1,\mathfrak{b}}$. In particular, no critical points exist in the open annulus $A_{\Lambda_1,\mathfrak{b}}$.

If we had $\partial_r u(x_0) = 0$ or $\partial_r v(x_0) = 0$ for some x_0 with $|x_0| = \Lambda_1$, then the Hopf lemma in [**6**, lemma 3.3] would give us $U^{\Lambda_1} \equiv 0$ or $V^{\Lambda_1} \equiv 0$ in Σ_{Λ_1}. Since the system is strongly coupled, this means $U^{\Lambda_1} = V^{\Lambda_1} \equiv 0$ in Σ_{Λ_1}. So $u(x^{\Lambda_1}) = v(x^{\Lambda_1}) = 0$ for all $x \in \Sigma_{\Lambda_1} \cap \partial A_{\mathfrak{a},\mathfrak{b}}$ due to the boundary condition on $|x| = \mathfrak{b}$, but this is impossible since the solution is positive. □

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Onward in the text and proofs, to fix the ideas we are going to consider problem (P_+) driven by the operator $\mathcal{M}^+$. However, everything can be repeated for the respective (P_-), or even for a problem involving both $\mathcal{M}^+$ and $\mathcal{M}^-$, with slight modifications.

The next result concerns the monotonicity of some associated energy functions. We point out that related monotonicity properties of energy-like functions for fully nonlinear operators have been already observed for scalar equations in [**2**, **9**].

We recall the dimension-like numbers $\tilde{N}_- = (N-1)\frac{\Lambda}{\lambda}+1$ and $\tilde{N}_+ = (N-1)\frac{\lambda}{\Lambda}+1$.

Proposition 2.3. *Let δ, $\mu > 0$ be such that $(u_{\delta,\mu}, v_{\delta,\mu})$ is a positive solution of (1.1) in the annulus $A_{\mathfrak{a},\mathfrak{b}}$. We set*

$$\mathcal{E}_{\mathfrak{s}}(r) = u'v' + \frac{1}{\mathfrak{s}(p+1)}v^{p+1} + \frac{1}{\mathfrak{s}(q+1)}u^{q+1}, \quad \mathfrak{s} > 0.$$

Then $\mathcal{E}_\lambda(r)$ is monotone decreasing in $[\mathfrak{a}, \tau_*] \cup [\tau^*, \mathfrak{b})$, and it is increasing in $[\tau_*, \tau^*]$. Further,

$$E_1^\lambda(r) = r^{2(\tilde{N}_- -1)}\, \mathcal{E}_\lambda(r) \ \textit{in}\ [\mathfrak{a}, \tau_*]$$
$$E_1^\Lambda(r) = r^{2(\tilde{N}_- -1)}\, \mathcal{E}_\Lambda(r) \quad \textit{in}\ [\tau^*, \mathfrak{b})$$

are monotone increasing functions.

Proof. We recall that in $[\mathfrak{a}, \tau_*]$ we have u', $v' \geqslant 0$, $u'' \leqslant 0$, and

$$u''v' + \tfrac{v^p v'}{\lambda} = -\tfrac{(\tilde{N}_- -1)u'v'}{r}, \quad v''u' + \tfrac{u^q u'}{\lambda} = -\tfrac{(\tilde{N}_- -1)u'v'}{r}.$$

In $[\tau^*, \mathfrak{b})$ we have u', $v' < 0$ and

$$u''v' + \frac{v^p v'}{\Lambda} \geqslant u''v' + \frac{v^p v'}{\sigma} = -\frac{(\hat{N}-1)u'v'}{r} \geqslant -\frac{(\tilde{N}_- -1)u'v'}{r},$$

where $(\sigma, \hat{N})$ is either (λ, N) or $(\Lambda, \tilde{N}_+)$, and analogously $v''u' + \frac{u^q u'}{\Lambda} \geqslant -\frac{(\tilde{N}_- -1)u'v'}{r}$. Thus, for $\kappa = 2(\tilde{N}_- - 1)$, we obtain in $[\tau^*, \mathfrak{b})$,

$$(E_1^\Lambda)'(r) = \kappa r^{\kappa-1}\left\{u'v' + \frac{v^{p+1}}{\Lambda(p+1)} + \frac{u^{q+1}}{\Lambda(q+1)}\right\} + r^\kappa\left\{u''v' + u'v'' + \frac{v^p v'}{\Lambda} + \frac{u^q u'}{\Lambda}\right\} \geqslant 0,$$

while in $(\mathfrak{a}, \tau_*]$ it yields

$$(E_1^\lambda)'(r) = \kappa r^{\kappa-1}\left\{u'v' + \frac{v^{p+1}}{\lambda(p+1)} + \frac{u^{q+1}}{\lambda(q+1)}\right\} + r^\kappa\left\{u''v' + u'v'' + \frac{v^p v'}{\lambda} + \frac{u^q u'}{\lambda}\right\} \geqslant 0.$$

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On the other hand, in $[\mathfrak{a}, \tau_*]$ one writes

$$u''v' + \frac{v^p v'}{\lambda} \leqslant -\frac{(\tilde{N}_+ - 1)u'v'}{r}, \quad v''u' + \frac{u^q u'}{\lambda} \leqslant -\frac{(\tilde{N}_+ - 1)u'v'}{r},$$

while in $[\tau^*, \mathfrak{b})$,

$$u''v' + \frac{v^p v'}{\lambda} \leqslant u''v' + \frac{v^p v'}{\sigma} \leqslant -\frac{(\tilde{N}_+ - 1)u'v'}{r}, \quad v''u' + \frac{u^q u'}{\lambda} \leqslant -\frac{(\tilde{N}_+ - 1)u'v'}{r},$$

so anyways $\mathcal{E}_\lambda'(r) \leqslant -\frac{2(\tilde{N}_+ - 1)}{r} u'v' \leqslant 0$.

Let us now analyse the interval $[\tau_*, \tau^*]$; to fix the ideas say $\tau_* = \tau_v$ and $\tau^* = \tau_u$. Then, in $[\tau_v, \tau_u]$ we have $u' > 0$ and $v' < 0$ (recall that $\tau^* \leqslant \mathfrak{b}$). Hence

$$u''v' + \frac{v^p v'}{\lambda} \geqslant -\frac{(\tilde{N}_+ - 1)u'v'}{r},$$
$$v''u' + \frac{u^q u'}{\lambda} \geqslant v''u' + \frac{u^q u'}{\sigma} = -\frac{(\hat{N} - 1)u'v'}{r} \geqslant -\frac{(\tilde{N}_+ - 1)u'v'}{r}.$$

Thus, for $\kappa = 2(\tilde{N}_+ - 1)$ we get $\mathcal{E}_\lambda'(r) \geqslant -\frac{\kappa}{r} u'v' \geqslant 0$. The reasoning is analogous when instead $\tau_* = \tau_u$ and $\tau^* = \tau_v$. □

As a consequence of the energy, we derive some useful shooting estimates.

LEMMA 2.4. *Let δ, $\mu > 0$ be such that $(u_{\delta,\mu}, v_{\delta,\mu})$ is a positive solution of (1.1) in the annulus $A_{\mathfrak{a},\mathfrak{b}}$. Then, for some $C_0 = C_0(\mathfrak{a}, \mathfrak{b}, N, \lambda, \Lambda, p, q)$, the following estimates hold*:

$$\tau_v^{\tilde{N}_-} \geqslant C_0 \, \frac{\mu^{\frac{1}{q+1}}}{\delta^{\frac{q}{q+1}}}, \tag{2.13}$$

$$\tau_u^{\tilde{N}_-} \geqslant C_0 \, \frac{\delta^{\frac{1}{p+1}}}{\mu^{\frac{p}{p+1}}}. \tag{2.14}$$

Proof. By proposition 2.3 we have $\mathcal{E}_\lambda(r) \leqslant \mathcal{E}_\lambda(\mathfrak{a})$ for all $r \leqslant \tau_*$, that is,

$$\frac{1}{p+1} v^{p+1}(r) + \frac{1}{q+1} u^{q+1}(r) \leqslant \lambda\,\delta\mu \quad \text{for all } [\mathfrak{a}, \tau_*], \tag{2.15}$$

since $u'v' \geqslant 0$ in $[\mathfrak{a}, \tau_*]$. Observe that (2.15) implies

$$u^q(r) \leqslant C(\delta\mu)^{\frac{q}{q+1}} \quad \text{for all}\, r \in [\mathfrak{a}, \tau_v], \quad v^p(r) \leqslant C(\delta\mu)^{\frac{p}{p+1}} \quad \text{for all}\, r \in [\mathfrak{a}, \tau_u], \tag{2.16}$$

since τ_u (resp. τ_v) is the maximum point for u (resp. v) in $[\mathfrak{a}, \rho_u]$ (resp. in $[\mathfrak{a}, \rho_v]$).

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Next we write the equation for v in $[\mathfrak{a},\, \tau_v]$ as $(v' r^{\tilde{N}_- -1})' = -\frac{u^q}{\lambda} r^{\tilde{N}_- -1}$, and so

$$0 = v'(\tau_v)\, \tau_v^{\tilde{N}_- -1} = \mu\, \mathfrak{a}^{\tilde{N}_- -1} - \frac{1}{\lambda} \int_{\mathfrak{a}}^{\tau_v} r^{\tilde{N}_- -1} u^q(r)\, \mathrm{d}r. \tag{2.17}$$

By combining the estimate for u in (2.16) and equality (2.17) we obtain

$$\mu = \frac{1}{\lambda \mathfrak{a}^{\tilde{N}_- -1}} \int_{\mathfrak{a}}^{\tau_v} r^{\tilde{N}_- -1} u^q(r)\, \mathrm{d}r \leqslant \frac{C}{\tilde{N}_-} (\delta\mu)^{\frac{q}{q+1}}\, \tau_v^{\tilde{N}_-},$$

from which we derive (2.13).

Analogously, in $[\mathfrak{a},\, \tau_u]$ one writes $(u' r^{\tilde{N}_- -1})' = -\frac{v^p}{\lambda} r^{\tilde{N}_- -1}$ and so

$$0 = u'(\tau_u)\, \tau_u^{\tilde{N}_- -1} = \delta\, \mathfrak{a}^{\tilde{N}_- -1} - \frac{1}{\lambda} \int_{\mathfrak{a}}^{\tau_u} r^{\tilde{N}_- -1} v^p(r)\, \mathrm{d}r. \tag{2.18}$$

Thus, using the estimate for v in (2.16) into (2.18) one reaches (2.14) out. □

Note that if the product $\delta\mu \to 0$, then $u(\tau_*)$, $v(\tau_*) \to 0$. Indeed, by (2.15),

$$\frac{1}{p+1} v^{p+1}(\tau_*) + \frac{1}{q+1} u^{q+1}(\tau_*) \leqslant \lambda\, \delta\mu \to 0 \text{ whenever } \delta\mu \to 0. \tag{2.19}$$

We show in the next corollary that such a property is never true for solutions of (1.1), by verifying the lower estimate in (1.2).

Corollary 2.5. *Let $\delta,\, \mu > 0$ be such that $(u_{\delta,\mu},\, v_{\delta,\mu})$ is a positive solution of* (1.1) *in the annulus $A_{\mathfrak{a},\mathfrak{b}}$. Then*

$$\delta, \mu \geqslant C(\mathfrak{a}, \mathfrak{b}, N, \lambda, \Lambda, p, q) > 0. \tag{2.20}$$

Proof. Set $C_1 = (\mathfrak{b}^{\tilde{N}_-}/C_0)^{q+1}$ and $C_2 = (\mathfrak{b}^{\tilde{N}_-}/C_0)^{p+1}$. By (2.13) and (2.14) we derive

$$\mu \leqslant C_1\, \delta^q \quad \textit{and} \quad \delta \leqslant C_2\, \mu^p.$$

The combination of these two estimates then implies

$$\delta^{pq-1} \geqslant \frac{1}{C_1^p C_2}, \quad \mu^{pq-1} \geqslant \frac{1}{C_1 C_2^q},$$

which gives us the lower bound (2.20). □

3. A priori bounds and blow-up

In the first part of this section we show that there exists $C > 0$ such that

$$\|u\|_{L^\infty(A_{\mathfrak{a},\mathfrak{b}})}, \quad \|v\|_{L^\infty(A_{\mathfrak{a},\mathfrak{b}})} \leqslant C$$

for all positive solution pairs $(u,\, v)$ of problem (1.1) in the annulus $A_{\mathfrak{a},\mathfrak{b}}$.

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Our strategy is to combine concavity properties with a uniform bound on the shooting parameters. On the one hand, from the concavity of u and v in $[\mathfrak{a},\, \tau_u]$ and $[\mathfrak{a},\, \tau_v]$ respectively, for any solution pair of (1.1) in the annulus $A_{\mathfrak{a},\mathfrak{b}}$ we have

$$\|u\|_\infty = u(\tau_u) \leqslant \delta(\mathfrak{b}-\mathfrak{a}), \quad \|v\|_\infty = v(\tau_u) \leqslant \mu(\mathfrak{b}-\mathfrak{a}). \tag{3.1}$$

Then it only remains to prove the following estimate by above for δ and μ. Combined with (2.20), this establishes the estimates in (1.2).

Lemma 3.1. *Given* $0<\mathfrak{a}<\mathfrak{b}<+\infty$*, let* $\delta,\, \mu>0$ *be such that* $(u_{\delta,\mu},\, v_{\delta,\mu})$ *is a positive solution of* (1.1) *in the annulus* $A_{\mathfrak{a},\mathfrak{b}}$*. Then* $\delta \leqslant C$ *and* $\mu \leqslant C$ *for some universal* $C=C(\mathfrak{a},\, \mathfrak{b},\, \lambda,\, \Lambda,\, p,\, q,\, N)$.

Proof. We fix the annulus $A_{\mathfrak{a},\mathfrak{b}}$ with $0<\mathfrak{a}<\mathfrak{b}<+\infty$. Assume by contradiction that there exists a sequence of shooting parameters $(\delta_k,\, \mu_k)$ with respective solutions $(u_k,\, v_k)$ of (2.2) in $A_{\mathfrak{a},\mathfrak{b}}$ such that at least one of them converges to infinity, that is $\delta_k \to +\infty$ or $\mu_k \to +\infty$. The first step is to show that both of them approach infinity in this case. Step (1) $\delta_k \to +\infty$ and $\mu_k \to +\infty$.

Assume on the contrary that either $\delta_k \to \infty$ or $\mu_k \to \infty$, and the other one is bounded. To fix the ideas we suppose $\delta_k \to +\infty$ and $\mu_k \leqslant C$ for all k. Then, by (2.14) we obtain $\mathfrak{b} \geqslant \tau_u \to +\infty$, which is impossible since the annulus $A_{\mathfrak{a},\mathfrak{b}}$ is fixed. Analogously, if $\mu_k \to +\infty$ and $\delta_k \leqslant C$ for all k, one finds the absurdity $\mathfrak{b} \geqslant \tau_v \to +\infty$ by (2.13).

We set

$$w_k(r) := \frac{1}{p+1}\, v_k^{p+1}(r) + \frac{1}{q+1}\, u_k^{q+1}(r).$$

Step (2) $w_k(\tau_*^k) \to +\infty$ and $w_k(\tau_k^*) \to +\infty$. We already know that the energy E_1^λ is increasing in $[\mathfrak{a},\, \tau_*^k]$ by proposition 2.3, and the annulus is fixed so that $\mathfrak{a} \leqslant \tau_*^k \leqslant \mathfrak{b}$ for all k. Thus,

$$w_k(\tau_*^k) \geqslant C_0\, \delta_k \mu_k \text{ for all } k, \text{ where } C_0 = C_0(\mathfrak{a}, \mathfrak{b}, N, \lambda, \Lambda, p, q). \tag{3.2}$$

On the other hand, $w_k(\tau_k^*) \geqslant w_k(\tau_*^k)$ by proposition 2.3, since the energy $\mathcal{E}_\lambda$ is increasing in $[\tau_*^k,\, \tau_k^*]$. Now, by Step 1 we have $\delta_k\mu_k \to +\infty$. This proves Step 2. Step (3) $\|u_k\|_\infty \to +\infty$ and $\|v_k\|_\infty \to +\infty$. By Step 2 we know that at least one of the norms sequences satisfies $\|u_k\|_\infty \to +\infty$ or $\|v_k\|_\infty \to +\infty$. Without loss we assume $\|u_k\|_\infty \to +\infty$.

Suppose by contradiction that $\|v_k\|_{L^\infty(A)} \leqslant C$ is bounded in the annulus $A = A_{\mathfrak{a},\mathfrak{b}}$. Recall that u_k solves $-\mathcal{M}^{\pm}(D^2 u_k) = v_k^p$ in A, with $u_k = 0$ on ∂A. Now we are going to use the Alexandrov-Bakelman-Pucci estimate (ABP), which can be found for instance in [**3**]. By ABP we then get $u_k \leqslant C$ in $A_{\mathfrak{a},\mathfrak{b}}$, which is impossible. Thus, $\|v_k\|_\infty \to +\infty$.

Step (4) $\lim_{k\to\infty} \tau_k^* = \mathfrak{b}$. Otherwise we may write $\mathfrak{b} > (1+\epsilon)\tau_k^*$ for all k, up to a subsequence, for some $\epsilon>0$. In particular, $u_k,\, v_k$ are both positive and decreasing in the interval $[\tau_k^*,\, (1+\epsilon)\tau_k^*]$.

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We consider the annulus $A_k = A_{\tau_k^*, r}$. Then $U_k = t_k u_k$ and v_k solve

$$-\mathcal{M}^{\pm}(D^2 U_k) \geqslant t_k v_k^p,\ -\mathcal{M}^{\pm}(D^2 v_k) \geqslant u_k^q \geqslant t_k\, U_k^{1/p} \text{ in } A_k,\ U_k, v_k > 0 \text{ in } A_k;$$

while u_k and $V_k = s_k v_k$ satisfy

$$-\mathcal{M}^{\pm}(D^2 u_k) \geqslant v_k^p \geqslant s_k\, V_k^{1/q}, \quad -\mathcal{M}^{\pm}(D^2 V_k) \geqslant s_k u_k^q \quad \text{in } A_k, \quad u_k, V_k > 0 in A_k,$$

where

$$t_k = \min_{A_k} u_k^{\frac{pq-1}{p+1}} = u_k^{\frac{pq-1}{p+1}}(r), \quad s_k = \min_{A_k} v_k^{\frac{pq-1}{q+1}} = v_k^{\frac{pq-1}{q+1}}(r).$$

Hence, by the definition of first eigenvalue $\lambda_1^+(\mathcal{D}) = \lambda_1^+(\mathcal{M}^{\pm},\, \mathcal{M}^{\pm},\, \mathcal{D})$ for the fully nonlinear Lane-Emden systems in [**13**], we have

$$u_k^{\frac{pq-1}{p+1}}(r),\, v_k^{\frac{pq-1}{q+1}}(r) \leqslant \lambda_1^+(A_k) \leqslant \lambda_1^+(A_{\tau_k^*, i_k}) \leqslant \frac{1}{\mathfrak{a}^2}\lambda_1^+\left(A_{1,1+\frac{\epsilon}{2}}\right) =: C_1, \tag{3.3}$$

for all $r \in I_k = [i_k,\, j_k]$, where $i_k := \left(1 + \frac{\epsilon}{2}\right)\tau_k^*$ and $j_k := (1+\epsilon)\tau_k^*$, since $\epsilon > 0$ is fixed. Recall the energy E_1^{Λ} is increasing in $[\tau_k^*,\, \mathfrak{b}]$ by proposition 2.3. Thus, $E_1^{\Lambda}(\tau_k^*) \leqslant E_1^{\Lambda}(r)$ for all $r \in I_k$. Hence, this, (3.2), and (3.3) give us for $r \in I_k$

$$(u_k' v_k')(r) \geqslant \tfrac{1}{\Lambda}\left(\tfrac{\mathfrak{a}}{\mathfrak{b}}\right)^{2(\tilde{N}_- - 1)} w_k(\tau_k^*) - \tfrac{1}{\Lambda} w_k(r) \geqslant c_0\, \delta_k \mu_k \tag{3.4}$$

for large k. Recall that $u_k' < 0$ and $v_k' < 0$ in I_k. In particular, $u_k'(r)$ or $v_k'(r)$ goes to $-\infty$ as $k \to \infty$, for all $r \in I_k$.

Observe that $(u_k')^2(r) + (v_k')^2(r) \geqslant 2(u_k' v_k')(r) \geqslant 2c_0\, \delta_k \mu_k$ for all $r \in I_k$. Then for each k we have either

$$|\mathcal{C}_k| = |\,\{r \in I_k : (u_k')^2(r) \geqslant c_0\, \delta_k \mu_k\}\,| \geqslant \frac{1}{2}|I_k|$$

or

$$|\mathcal{D}_k| = |\,\{r \in I_k : (v_k')^2(r) \geqslant c_0\, \delta_k \mu_k\}\,| \geqslant \frac{1}{2}|I_k|.$$

So we can extract a subsequence for which one of the two situations occurs, namely the first one $|\mathcal{C}_k| \geqslant \frac{1}{2}|I_k|$.

Next, the Fundamental Theorem of Calculus and the Lebesgue integration for this subsequence imply

$$\begin{aligned} u_k(i_k) \geqslant u_k(i_k) - u_k(\mathfrak{b}) &= \int_{i_k}^{\mathfrak{b}} (-u_k') \geqslant \int_{\mathcal{C}_k} (-u_k') \\ &= |\mathcal{C}_k|\,(c_0\, \delta_k \mu_k)^{1/2} \geqslant \frac{1}{2}|I_k|\,(c_0\, \delta_k \mu_k)^{1/2} \geqslant \frac{\epsilon}{4}\mathfrak{a}\,(c_0\, \delta_k \mu_k)^{1/2} \to +\infty \end{aligned}$$

as $k \to +\infty$ by using the fact that $\epsilon > 0$ is fixed fulfilling $|I_k| = \frac{\varepsilon}{2}\tau_k^* \geqslant \frac{\varepsilon}{2}\mathfrak{a}$. Hence we reach a contradiction with the estimate (3.3). The case when $|\mathcal{D}_k| \geqslant \frac{1}{2}|I_k|$ is analogous.

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Step (5) Conclusion

We reach a contradiction by putting together theorem 2.2 with Step 4, since $\mathfrak{b} > \mathfrak{a}$. □

We point out that, in order to obtaining a priori bounds, it is essential to have a fixed minimum distance between the radii of the annulus, that is $\mathfrak{b} - \mathfrak{a} \geqslant c_0$, as shows the next proposition.

PROPOSITION 3.2. *If* $\mathfrak{b} \to \mathfrak{a}$ *then* $u(\tau_u)$, $v(\tau_v) \to +\infty$.

Proof. Let (u, v) be a solution pair of (2.2) and denote $A = A_{\mathfrak{a},\mathfrak{b}}$. We set $U = tu$, with $t > 0$, and write

$$\begin{cases} -\mathcal{M}^{\pm}(D^2 U) \leqslant t\, v^p \\ -\mathcal{M}^{\pm}(D^2 v) \leqslant u^{q-\frac{1}{p}}\, u^{\frac{1}{p}} \leqslant t^{-\frac{1}{p}}\, \|u\|_{\infty}^{\frac{pq-1}{p}}\, U^{\frac{1}{p}} = t\, U^{\frac{1}{p}} \end{cases}$$

since $pq > 1$, as long as we choose $t = \|u\|_{L^{\infty}(A)}^{\frac{pq-1}{p+1}}$. Hence, by applying the ABP estimate in the domain A for each of the scalar PDE inequalities above we obtain

$$\sup_A U \leqslant C\, t\, \sup_A v^p\, |A|^{1/N}, \quad \sup_A v \leqslant C\, t\, \sup_A U^{\frac{1}{p}}\, |A|^{1/N}.$$

Then by taking the $1/p$ power of the inequality above for U, and replacing it into the inequality satisfied by v, one finds

$$\sup_A v \leqslant C^{\frac{p+1}{p}}\, t^{\frac{p+1}{p}} \sup_A v^+\, |A|^{\frac{p+1}{Np}} \quad \Rightarrow \quad t \geqslant \frac{1}{C|A|^{1/p}} \to \infty \quad \text{as } |A| \to 0.$$

On the other hand, by writing $v^p = v^{p-\frac{1}{q}}\, v^{\frac{1}{q}} \leqslant \|v\|_{\infty}^{\frac{pq-1}{q}}\, v^{\frac{1}{q}}$ and arguing similarly with the pair (u, sV), where $V = sv$ for $s = \|v^+\|_{L^{\infty}(A)}^{\frac{pq-1}{q+1}}$ we get $v(\tau_v) = \|v^+\|_{L^{\infty}(A)} \to \infty$ as $|A| \to 0$ as well. □

4. Existence result

We are going to prove the existence of a classical solution in the annulus $A_{\mathfrak{a},\mathfrak{b}}$ so that $u = v = 0$ on $\partial A_{\mathfrak{a},\mathfrak{b}}$. The tactics is to use a suitable Krasnosel'skii degree theoretical argument, similar to those employed in [**4**, **5**].

PROPOSITION 4.1. *Let* K *be a cone in a Banach space* X *and* $\Phi : K \to K$ *a completely continuous operator such that* $\Phi(0) = 0$. *For* $\mathcal{B}_{\mathfrak{s}} = \{w \in K : \|w\| < \mathfrak{s}\}$, *assume that there exist* $0 < r < R$ *so that*

(i) $w \neq \theta\Phi(w)$ *for all* $\theta \in [0, 1]$ *and* $w \in K$ *such that* $\|w\| = r$;

(ii) *there exists a compact map* $F : \overline{\mathcal{B}}_R \times [0, \infty) \to K$ *with* $F(w, 0) = \Phi(w)$, $F(w, t) \neq w$ *for* $\|w\| = R$ *and* $0 \leqslant t < \infty$, *while* $F(w, t) = w$ *has no solution* $w \in \overline{\mathcal{B}}_R$ *for* $t \geqslant t_0$.

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Then if $\mathcal{U} = \{w \in K : r < \|w\| < R\}$, *one has*

$$i_K(\Phi, \mathcal{B}_R) = 0, \quad i_K(\Phi, \mathcal{B}_r) = 1, \quad i_K(\Phi, \mathcal{U}) = -1,$$

where $i_K(\Phi, \mathcal{W})$ *is the index of* Φ *on* $\mathcal{W}$. *In particular,* Φ *has a fixed point in* $\mathcal{U}$.

Proof of the existence in the annulus via degree theory. We consider $X = C(\bar{A}_{\mathfrak{a},\mathfrak{b}}) \times C(\bar{A}_{\mathfrak{a},\mathfrak{b}})$, with the norm $\|(u, v)\| := \max\{\|u\|_{L^\infty(A_{\mathfrak{a},\mathfrak{b}})}, \|v\|_{L^\infty(A_{\mathfrak{a},\mathfrak{b}})}\}$.

Let $K = \{(u, v) \in X : u, v \geqslant 0\}$, and denote $\mathcal{B}_{\mathfrak{s}} = \{(u, v) \in K : \|(u, v)\| < \mathfrak{s}\}$.

For any $(u, v) \in K$ and $t \geqslant 0$ we define the operator $F(t, u, v) = (U, V)$, with $U = U_t$ and $V = V_t$, as the unique solution of the problem

$$-\mathcal{M}^\pm(D^2U) = (v + t)^p, \quad -\mathcal{M}^\pm(D^2V) = (u + t)^q \text{ in } A_{\mathfrak{a},\mathfrak{b}}, \quad U, V = 0 \text{ on } \partial A_{\mathfrak{a},\mathfrak{b}}.$$

In particular, $(U, V) \in K$ by the maximum principle for scalar equations. We denote $\Phi(\cdot) = F(0, \cdot)$. Our goal is to show that Φ has a positive fixed point (u, v).

Let us verify the hypotheses in proposition 4.1.

(i) We take $(u, v) \in K$ such that $\|(u, v)\| = r$, for some $r > 0$ to be chosen, and $(u, v) = \theta\Phi(u, v)$, $\theta \in (0, 1]$. In particular, $\|u\|_\infty, \|v\|_\infty \leqslant r$. As before, we set $\tilde{u} = \kappa u$ and write

$$\begin{cases} -\mathcal{M}^\pm(D^2\tilde{u}) = \theta\kappa v^p \leqslant \kappa v^p \\ -\mathcal{M}^\pm(D^2v) = \theta u^{q-\frac{1}{p}}\, u^{\frac{1}{p}} \leqslant \kappa^{-\frac{1}{p}}\, \|u\|_\infty^{\frac{pq-1}{p}}\, (\tilde{u})^{\frac{1}{p}} = \kappa(\tilde{u})^{\frac{1}{p}} \end{cases}$$

since $pq > 1$, as long as $\kappa := \|u\|_{L^\infty(A_{\mathfrak{a},\mathfrak{b}})}^{\frac{pq-1}{p+1}} \leqslant r^{\frac{pq-1}{p+1}}$. Then we choose $r > 0$ small enough such that $r^{\frac{pq-1}{p+1}} < \lambda_1^+(\mathcal{M}^\pm, \mathcal{M}^\pm, A_{\mathfrak{a},\mathfrak{b}})$. Since $u, v = 0$ on $\partial A_{\mathfrak{a},\mathfrak{b}}$, then by the maximum principle for the Lane-Emden system for fully nonlinear operators with weights in [**13**] we get $u, v \leqslant 0$ in $A_{\mathfrak{a},\mathfrak{b}}$. Since $(u, v) \in K$ then $u, v \equiv 0$ in $A_{\mathfrak{a},\mathfrak{b}}$, but this contradicts the fact that $\|(u, v)\| = r > 0$.

(ii) Case 1: $t \geqslant t_0$

If F_t has a fixed point (u_t, v_t) then $\tilde{u}_t = \kappa u_t$ and v_t solve

$$\begin{cases} -\mathcal{M}^\pm(D^2\tilde{u}_t) = \kappa(v_t + t)^p \geqslant \kappa v_t^p \\ -\mathcal{M}^\pm(D^2v_t) = (u_t + t)^{q-\frac{1}{p}}\, (u_t + t)^{\frac{1}{p}} \geqslant t_0^{\frac{pq-1}{p}}\, \kappa^{-\frac{1}{p}} (\tilde{u}_t)^{\frac{1}{p}} = \kappa(\tilde{u}_t)^{\frac{1}{p}} \end{cases}$$

with $\tilde{u}_t, v_t > 0$ in $A_{\mathfrak{a},\mathfrak{b}}$, where

$$\kappa = t_0^{\frac{pq-1}{p+1}}.$$

Now, the definition of first eigenvalue $\lambda_1^+(A_{\mathfrak{a},\mathfrak{b}}) = \lambda_1^+(\mathcal{M}^\pm, \mathcal{M}^\pm, A_{\mathfrak{a},\mathfrak{b}})$ for the fully nonlinear weighted Lane-Emden system in [**13**] yields

$$\kappa \leqslant \lambda_1^+(A_{\mathfrak{a},\mathfrak{b}}).$$

Thus we choose t_0 large enough such that $\kappa = 2\lambda_1^+(A_{\mathfrak{a},\mathfrak{b}})$ in order to derive a contradiction.

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Case 2: $t \leqslant t_0$

In this case we infer that lemma 3.1 immediately produces a priori bounds for the fixed points of $F(t, \cdot)$ in bounded intervals of t, that is, for each fixed $t_0 > 0$ it will give $\|(u_t, v_t)\| \leqslant C(t_0)$ for all solutions $u = u_t$, $v = v_t$ of $F(t, u, v) = (u, v)$ with $t \in [0, t_0]$. Indeed, we define the function

$$w_t(r) := \frac{1}{p+1}\,|v_t + t|^{p+1}(r) + \frac{1}{q+1}\,|u_t + t|^{q+1}(r) \quad \text{for } t \geqslant 0.$$

Then Step 1, Step 2 hold for $t > 0$ exactly as in the case $t = 0$. Moreover, the symmetry result in theorem 2.2 applies as well (and so Step 5) since we maintain the zero boundary condition $u_t = v_t = 0$ on $\partial A_{\mathfrak{a},\mathfrak{b}}$. On the other hand, a positive solution (u_t, v_t) of

$$-\mathcal{M}^{\pm}(D^2 u_t) = (v_t + t)^p, \quad -\mathcal{M}^{\pm}(D^2 v_t) = (u_t + t)^q \text{ in } A_{\mathfrak{a},\mathfrak{b}}, \quad u_t, v_t = 0 \text{ on } \partial A_{\mathfrak{a},\mathfrak{b}}$$

produces a positive solution $(\tilde{u}_t, \tilde{v}_t)$, with $\tilde{u}_t = u_t + t$ and $\tilde{v}_t = v_t + t$, of

$$-\mathcal{M}^{\pm}(D^2 \tilde{u}_t) = \tilde{v}_t^p, \quad -\mathcal{M}^{\pm}(D^2 \tilde{v}_t) = \tilde{u}_t^q \text{ in } A_{\mathfrak{a},\mathfrak{b}}, \quad \tilde{u}_t, \tilde{v}_t = t \text{ on } \partial A_{\mathfrak{a},\mathfrak{b}}.$$

Thus, the proof of Step 4 in lemma 3.1 is unchangeable for $(\tilde{u}_t, \tilde{v}_t)$ in place of (u, v), since we only used in such a proof that the solution is nonnegative at $\mathfrak{b}$.

Therefore, it is enough to choose $R = 2C(t_0)$ in order to conclude that F_t does not have fixed points satisfying $\|(u_t, v_t)\| = R$ whenever $t \leqslant t_0$. The complementary case $\|(u_t, v_t)\| = R$ with $t \geqslant t_0$ is automatically fulfilled by Case 1. □

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