J. Aust. Math. Soc. **91** (2011), 323–341
doi:10.1017/S1446788712000031

MODULES OVER QUANTUM LAURENT POLYNOMIALS

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(Received 21 February 2010; accepted 7 October 2011)

Communicated by J. Du

Abstract

We show that the Gelfand–Kirillov dimension for modules over quantum Laurent polynomials is additive with respect to tensor products over the base field. We determine the Brookes–Groves invariant associated with a tensor product of modules. We study strongly holonomic modules and show that there are nonholonomic simple modules.

2010 *Mathematics subject classification*: primary 16S35; secondary 16D30, 16D60.
Keywords and phrases: quantum polynomial, quantum torus, skew Laurent extension.

1. Introduction

Let F be a field, take nonzero scalars q_{ij} in $F \setminus \{0\}$ where $i, j \in \{1, \ldots, n\}$, and set $\mathfrak{q} = (q_{ij})$. Consider the associative F-algebra $P(\mathfrak{q})$ generated by $u_1, \ldots, u_n$ and their inverses such that

$$u_i u_j = q_{ij} u_j u_i \quad \forall i, j \in \{1, \ldots, n\}. \tag{1.1}$$

This algebra is known by various names, such as the *multiplicative analogue of the Weyl algebra*, the *quantum Laurent polynomial algebra* and the *quantum torus*. It has the structure of a twisted group algebra $F * A$ of a free abelian group A of rank n over F.

In the special case where $n = 2$, the condition (1.1) becomes $u_1 u_2 = q u_2 u_1$, where $q \in F \setminus \{0\}$. This case was first studied by Jategaonkar [15] and Lorenz [16], and it was shown that $P((q))$ shares certain curious properties with the first Weyl algebra $A_1(k)$ over a field k of characteristic zero, when q is not a root of unity in F.

McConnell and Pettit [18] first considered the case of arbitrary n. They showed that if the subgroup of the multiplicative group of F generated by the q_{ij} has the maximal possible torsion-free rank, then $P(\mathfrak{q})$ is a simple noetherian hereditary domain.

The author thanks the University of Melbourne for financial support.

Quantum Laurent polynomial algebras play a fundamental role in noncommutative geometry (see [17]). They also arise in the representation theory of torsion-free nilpotent groups as suitable localizations (see [10]).

In recent times, there has been considerable interest in the theory of these algebras and their generalizations. Their ring-theoretic properties were studied in [1, 3, 18]. Artamonov [2, 4, 5] considered projective and simple modules over general quantum polynomial rings. Brookes and Groves [12, 13] introduced a geometric invariant for $F * A$-modules modelled on the original Bieri–Strebel invariant (see [8]).

The algebras $P(\mathfrak{q})$ are precisely the twisted group algebras $F * A$ of a free finitely generated abelian group A over F. In this paper, we consider the structure of modules over the algebras $F * A$. Later in this section, we review the basic properties of these algebras. In Section 2, we then give a brief exposition of the geometric invariant $\Delta(M)$ of Brookes and Groves associated with a finitely generated $F * A$-module M. Our first main result, Theorem 3.1, determines the Brookes–Groves invariant and the Gelfand–Kirillov (GK) dimension of a tensor product of modules. More precisely, we show that if M_1 and M_2 are finitely generated $F * A_i$-modules, then for the finitely generated $F * (A_1 \oplus A_2)$-module $M_1 \otimes_F M_2$,

$$\Delta(M_1 \otimes_F M_2) = p_1^* \Delta(M_1) + p_2^* \Delta(M_2),$$

where $p_i^* : A_i^* \to (A_1 \oplus A_2)^*$ is the injection induced by the projection $p_i : A_1 \oplus A_2 \to A_i$ when $i = 1, 2$. Furthermore,

$$\text{GK-dim}(M_1 \otimes_F M_2) = \text{GK-dim}(M_1) + \text{GK-dim}(M_2).$$

Section 4 is concerned with strongly holonomic modules. These are defined analogously to holonomic $A_n(k)$-modules, where $A_n(k)$ denotes the nth Weyl algebra over a field k of characteristic zero. An A_n-module N is called *holonomic* if

$$\text{GK-dim}(N) = \tfrac{1}{2}\, \text{GK-dim}(A_n).$$

Holonomic A_n-modules form an important subclass of A_n-modules and possess some nice properties (see [9]). By [18, Section 5.1], $\text{GK-dim}(F * A) = \text{rank}(A)$ for the algebras $F * A$. We may thus call an $F * A$-module M holonomic if

$$\text{GK-dim}(M) = \tfrac{1}{2}\, \text{rank}(A).$$

Such modules are encountered in group theory (see [11]) with the additional condition that M is torsion-free as an $F * B$-module whenever B is a subgroup of A such that $F * B$ is commutative. Brookes and Groves [11] showed that, if an algebra $F * A$ with center F has a strongly holonomic module and $\text{rank}(A) = 2m$, then there is a finite index subgroup A' in A such that

$$F * A' = (F * B_1) \otimes_F \cdots \otimes_F (F * B_m),$$

and each $B_i \cong \mathbb{Z} \oplus \mathbb{Z}$ and $m = \frac{1}{2}\, \text{rank}(A)$. In Theorem 4.12, we give a new proof of this.

In [18, Section 6], the question was considered whether an algebra $F * A$ that is simple can have simple modules with distinct GK dimensions. It was shown that if $F * A$ has Krull (global) dimension one, then the GK dimension of every simple $F * A$-module is $\mathrm{rank}(A) - 1$. In fact, if an algebra $F * A$ has dimension m, where $1 \le m \le \mathrm{rank}(A)$, then the work of Brookes [10] implies that the minimum possible GK dimension of a nonzero finitely generated $F * A$-module is $\mathrm{rank}(A) - m$. The question then arises if the GK dimension of a simple $F * A$-module is always equal to this minimum, as in the dimension-one case.

In Section 5, we show that this need not be true in general. More precisely, suppose that $F * A$ has center exactly F and that A has a subgroup B such that A/B is infinite cyclic and $F * B$ is commutative. Then $F * A$ has a simple $F * B$-torsion-free module S for which $\text{GK-dim}(S) = n - 1$.

In the rest of this introductory section, we discuss the basic properties of the algebra $P(\mathfrak{q})$ and its modules. It is easily seen that the monomials $u_1^{m_1} \cdots u_n^{m_n}$, where $m_j \in \mathbb{Z}$, constitute an F-basis of $P(\mathfrak{q})$. The monomial $u_1^{m_1} \cdots u_n^{m_n}$ is denoted by $\mathbf{u}^{\mathbf{m}}$, where $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n$. We denote the set of nonzero elements of F by F^*. The facts in the next proposition were established by McConnell and Pettit.

Proposition 1.1 [18, Section 1]. *The algebra $P(\mathfrak{q})$ has the following properties.*

(i) $\mathbf{u}^{\mathbf{m}}\mathbf{u}^{\mathbf{m}'} = \prod_{j>i} q_{ji}^{m_j m'_i} \mathbf{u}^{\mathbf{m}+\mathbf{m}'}$.

(ii) $(\mathbf{u}^{\mathbf{m}})^{-1} = \mu(\mathbf{m})\mathbf{u}^{-\mathbf{m}}$, *where* $\mu(\mathbf{m}) = \prod_{j>i} q_{ji}^{m_j m_i}$.

(iii) $\alpha \in P(\mathfrak{q})$ *is a unit if and only if* $\alpha = \lambda \mathbf{u}^{\mathbf{m}}$ *for some nonzero* $\lambda \in F$.

(iv) *The group-theoretic commutator* $[\mathbf{u}^{\mathbf{a}}, \mathbf{u}^{\mathbf{b}}]$ *(that is,* $\mathbf{u}^{\mathbf{a}}\mathbf{u}^{\mathbf{b}}(\mathbf{u}^{\mathbf{a}})^{-1}(\mathbf{u}^{\mathbf{b}})^{-1}$*) lies in* F^*.

(v) *The derived subgroup of the group of units of* $P(\mathfrak{q})$ *coincides with the subgroup of* F^* *generated by the* q_{ij} *where* $1 \le i, j \le n$.

(vi) $P(\mathfrak{q})$ *is simple if and only if it has center exactly* F.

An associative F-algebra $\mathcal{A}$ is a twisted group algebra $F * A$ of a finitely generated free abelian group A over the field F if the following hold.

(i) There is an injective function $\bar{\ }: A \to \mathcal{A}$, $a \mapsto \bar{a}$, such that $\overline{A} := \mathrm{Image}(\bar{\ })$ is a basis of $\mathcal{A}$ as an F-space.

(ii) There is a function $\tau : A \times A \to F^*$ satisfying

$$\tau(a_1, a_2)\tau(a_1a_2, a_3) = \tau(a_2, a_3)\tau(a_1, a_2a_3) \quad \forall a_1, a_2, a_3 \in A \tag{1.2}$$

such that the multiplication in $\mathcal{A}$ is given by

$$\bar{a}_1\bar{a}_2 = \tau(a_1, a_2)\overline{a_1a_2} \quad \forall a_1, a_2 \in A. \tag{1.3}$$

Let A be a free abelian group with basis $\{a_1, \ldots, a_n\}$. Then there is an injection from A to $P(\mathfrak{q})$ defined by $\prod a_i^{m_i} \mapsto \prod u_1^{m_i}$, where $m_i \in \mathbb{Z}$ and $i = 1, \ldots, n$. Condition (ii) above easily follows from (1.1). Finally, the associativity of $P(\mathfrak{q})$ implies (1.2). Hence $P(\mathfrak{q})$ is a twisted group algebra $F * A$.

We note that in an algebra $F * A$, the scalars are central, so that

$$\lambda\bar{a} = \bar{a}\lambda \quad \forall \lambda \in F \ \forall a \in A.$$

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In a *crossed product* $D * A$ (see [19, Ch. 1]), where D is a division ring, the multiplication is defined as in (ii) above, but an element $d \in D$ need not be central. In fact, for all $a \in A$, there is an automorphism σ_a of D such that

$$\bar{a}d = \sigma_a(d)\bar{a} \quad \forall d \in D.$$

Given an element α of the algebra $F * A$, we may write α uniquely as $\sum_{a\in A} \kappa_a \bar{a}$, where $\kappa_a \in F$. The *support of* α *in* A is the finite subset $\{a \in A \mid \kappa_a \neq 0\}$ of A; it is written $\mathrm{supp}(\alpha)$. For a subgroup B of A, the subalgebra $\{\beta \in F * A \mid \mathrm{supp}(\beta) \subseteq B\}$ of $F * A$ is a twisted group algebra $F * B$ of B over F.

It is known (see, for example, [19, Lemma 37.8]) that if B is a subgroup of A, then $S_B = F * B \setminus \{0\}$ is an Ore subset in $F * A$. As a consequence, the subset $T_{S_B}(M)$ of M, consisting of all x such that $xs = 0$ for some $s \in S_B$, is an $F * A$-submodule of M. We say that M is S_B-torsion or $F * B$-torsion if $T_{S_B}(M) = M$ and S_B-torsion-free or $F * B$-torsion-free if $T_{S_B}(M) = 0$. We note that the right Ore localization $(F * A)S_B^{-1}$ is a crossed product $D_B * A/B$, where D_B stands for the quotient division ring of $F * B$. We shall also write $(F * A)(F * B)^{-1}$ for $(F * A)S_B^{-1}$.

Note that if $a \in A$, then $\bar{a}$ is a unit of $F * A$. Without loss of generality, we may assume that $\bar{1}$ is the identity of $F * A$. It easily follows from (1.3) that for $a_1, a_2 \in A$, the group-theoretic commutator $[\overline{a_1}, \overline{a_2}]$, given as usual by $\bar{a}_1\bar{a}_2\bar{a}_1^{-1}\bar{a}_2^{-1}$, is in F. Then the following equalities hold (see [20, Section 5.1.5]):

$$\begin{aligned} [\bar{a}_1\bar{a}_2, \bar{a}_3] &= [\bar{a}_1, \bar{a}_3][\bar{a}_2, \bar{a}_3], \\ [\bar{a}_1, \bar{a}_2\bar{a}_3] &= [\bar{a}_1, \bar{a}_2][\bar{a}_1, \bar{a}_3]. \end{aligned} \tag{1.4}$$

For a subset X of A, we define $\overline{X} = \{\bar{x} \mid x \in X\}$. Moreover, when $X_1, X_2 \subseteq A$, we define $[\overline{X_1}, \overline{X_2}] = \langle [\bar{x}_1, \bar{x}_2] \mid x_1 \in X_1, x_2 \in X_2\rangle$. It is clear that $[\overline{X_1}, \overline{X_2}]$ is a subgroup of the multiplicative group F^*.

2. The Brookes–Groves geometric invariant

We now describe a geometric invariant that was introduced in [12, 13]. It is defined for finitely generated modules over a crossed product $D * A$ of a finitely generated free abelian group A over a division ring D. Since a twisted group algebra $F * A$ is a special case of $D * A$, the definitions and theorems that follow apply to $F * A$-modules as well.

Let A be a finitely generated free abelian group and denote $\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{R})$ by A^*. Then A^* is an $\mathbb{R}$-space and $\dim(A^*) = \mathrm{rank}(A)$, where $\mathrm{rank}(A)$ is the cardinality of a basis of A. Given a basis $\mathfrak{b} = \{b_i \mid i \in I\}$ of A, there is a basis $\mathfrak{b}^* = \{b_i^* \mid i \in I\}$ dual to $\mathfrak{b}$, and this allows the construction of an isomorphism from $\mathbb{R}^{|\mathfrak{b}|}$ to A^*. We may thus speak of characters $\phi \in A^*$ as points. There is a $\mathbb{Z}$-bilinear map $\langle \cdot, \cdot \rangle : A^* \times A \to \mathbb{R}$ defined by

$$(\phi, c) \mapsto \langle \phi, c\rangle = \phi(c) \quad \forall \phi \in A^* \; \forall c \in A.$$

Given a subgroup B of A, define its annihilator $\mathrm{ann}(B)$ in A^* by

$$\mathrm{ann}(B) = \{\phi \in A^* \mid \langle \phi, B\rangle = 0\};$$

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this is a subspace, and

$$\dim(\operatorname{ann}(B)) = \operatorname{rank}(A) - \operatorname{rank}(B).$$

For a subspace V of A^*, we define $\operatorname{ann}(V)$ analogously:

$$\operatorname{ann}(V) = \{b \in A \mid \langle V, b\rangle = 0\}.$$

It is easy to show that $\operatorname{ann}(\operatorname{ann}(B)) = B$. For a point $\phi \in A^*$, we define

$$A_{\phi,0} = \{a \in A \mid \phi(a) \geq 0\},$$
$$A_{\phi,+} = \{a \in A \mid \phi(a) > 0\}.$$

Note that $A_{\phi,0}$ is a submonoid and $A_{\phi,+}$ is a subsemigroup of A. Brookes and Groves [13, Proposition 3.1] gave several equivalent definitions of the geometric invariant that are analogous to the commutative case (see [8]). The following definition was used in [10].

Definition 2.1 [13, Proposition 3.1]. Let D be a division ring and A be a free finitely generated abelian group. Let M be a finitely generated $D * A$-module with a finite generating set $\mathcal{X}$. Then the subset $\Delta(M)$ of A^* is defined by

$$\Delta(M) = \{\theta \in A^* \mid \mathcal{X}A_{\theta,0} > \mathcal{X}A_{\theta,+}\}.$$

As defined, $\Delta(M)$ seems to depend on the choice of generating set $\mathcal{X}$ for M, but it is actually independent of this choice (see [10, Section 2]).

Definition 2.2 [14, Definition 2.1]. Let M be a finitely generated $D * A$-module. For a point $\phi \in A^*$, the *trailing coefficient module* $TC_\phi(M)$ of M at ϕ is defined by

$$TC_\phi(M) = \mathcal{X}A_{\phi,0}/\mathcal{X}A_{\phi,+},$$

where $\mathcal{X}$ is a (finite) generating set for M.

Note that $TC_\phi(M)$ is a finitely generated $D * K$-module, where $K = \ker\phi$. It is immediate from Definition 2.1 that $\phi \in \Delta(M)$ if and only if $TC_\phi(M) \neq 0$. In general, $TC_\phi(M)$ need not be independent of $\mathcal{X}$. A dimension for finitely generated $D * A$-modules was introduced in [13].

Definition 2.3 [13, Definition 2.1]. Let M be a $D * A$-module. The dimension $\dim(M)$ of M is defined to be the greatest integer r such that M is not $D * B$-torsion for some subgroup B in A of rank r. Thus $0 \leq r \leq \operatorname{rank}(A)$.

It was shown in [13] that $\dim(M)$ coincides with the GK dimension of M. We shall thus mostly write GK-dim(M) for $\dim(M)$. The following useful fact was also shown in [13].

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PROPOSITION 2.4 [13, Lemma 2.2]. *Let*

$$0 \to M_1 \to M \to M_2 \to 0$$

be an exact sequence of $D * A$-modules. Then

$$\dim(M) = \max\{\dim(M_1), \dim(M_2)\}.$$

As already remarked, we may identify A^* with $\mathbb{R}^n$ and thus the subset $\Delta(M)$ of A^* with a subset of $\mathbb{R}^n$. A subset S of $\mathbb{R}^n$ is a *polyhedron* if S is a finite union of *convex polyhedra*. A *convex polyhedron* is an intersection of finitely many closed half spaces in $\mathbb{R}^n$. A polyhedron is *rational* when each of the boundaries of the half spaces that define it is rational, that is, when it is generated by rational linear combinations of the chosen dual basis. The dimension of a convex polyhedron C is the dimension of the subspace of $\mathbb{R}^n$ spanned by C. The dimension of a polyhedron is the greatest of the dimensions of its constituent convex polyhedra. In [13, Theorem 4.4], it was shown that an 'essential' subset of $\Delta(M)$ is a polyhedron of dimension equal to the GK dimension of M. It was shown in [21] that the Brookes–Groves invariant is polyhedral.

THEOREM 2.5 [21, Theorem A]. *If $D * A$ is a crossed product of a division ring D by a free finitely generated abelian group A, then $\Delta(M)$ is a closed rational polyhedral cone in* $\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{R})$ *for all finitely generated $D * A$-modules M.*

The next section gives an application of the geometric invariant to tensor products of $F * A$-modules.

3. The geometric invariant and tensor products

Given twisted group algebras $F * A_1$ and $F * A_2$, the tensor product

$$(F * A_1) \otimes_F (F * A_2)$$

of F-algebras is a twisted group algebra of $A_1 \oplus A_2$ over F. Moreover, if M_1 and M_2 are modules over $F * A_1$ and $F * A_2$, then the formula

$$(m_1 \otimes m_2)(\bar{a}_1, \bar{a}_2) = m_1\bar{a}_1 \otimes m_2\bar{a}_2 \quad \forall m_1, m_2 \in M \;\forall a_1, a_2 \in A$$

gives $M_1 \otimes_F M_2$ the structure of an $(F * A_1) \otimes_F (F * A_2)$-module. We now determine the Brookes–Groves invariant associated with a tensor product of modules.

THEOREM 3.1. *Let $p_i^* : A_i^* \to (A_1 \oplus A_2)^*$ be the injection that is induced by the projection $p_i : A_1 \oplus A_2 \to A_i$ and M_i be a finitely generated module over $F * A_i$, where $i = 1, 2$. Then for the finitely generated $F * (A_1 \oplus A_2)$-module $M_1 \otimes_F M_2$,*

$$\Delta(M_1 \otimes_F M_2) = p_1^*\Delta(M_1) + p_2^*\Delta(M_2) \tag{3.1}$$

and

$$\text{GK-dim}(M_1 \otimes_F M_2) = \text{GK-dim}(M_1) + \text{GK-dim}(M_2). \tag{3.2}$$

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PROOF. Let $M := M_1 \otimes_F M_2$. We first show that

$$p_1^*\Delta(M_1) + p_2^*\Delta(M_2) \subseteq \Delta(M). \tag{3.3}$$

We utilize the Δ-set of a module as in [13, Section 3, Definition 4]. This is given as follows: for a finitely generated $F * A$-module L and a point $\phi \in A^*$, a nontrivial ϕ-filtration of L is a family of F-subspaces L_μ of L, where $\mu \in \mathbb{R}$, that satisfies the following conditions.

(C1) $L_\nu \geq L_\mu$ whenever $\nu \leq \mu$.
(C2) $\bigcup_{\mu\in\mathbb{R}} L_\mu = L$.
(C3) $L_\mu \bar{a} = L_{\mu+\phi(a)}$ for all $a \in A$.
(C4) The subspace L_μ is a proper subspace of L for each $\mu \in \mathbb{R}$.

Then $\Delta(M)$ is defined to be the set of all $\phi \in A^*$ for which there exists a nontrivial ϕ-filtration, together with the zero of A^*. This is equivalent to Definition 2.1 (see [13, Proposition 3.1]).

Thus, to show (3.3), it suffices to show that if $\phi_i \in \Delta(M_i)$ and either ϕ_1 or ϕ_2 is nonzero, then M has a nontrivial ϕ-filtration, where $\phi = \phi_1 p_1 + \phi_2 p_2$. Suppose, for the moment, that $\phi_1 \neq 0$ and $\phi_2 \neq 0$. Since $\phi_i \in \Delta(M_i)$, there exists a nontrivial ϕ_i-filtration $\{M_i^\mu\}_{\mu\in\mathbb{R}}$ of M_i when $i = 1, 2$. We now define a ϕ-filtration on M by setting

$$M_\lambda = \sum_{\substack{\mu,\nu\in\mathbb{R}\\ \mu+\nu=\lambda}} M_1^\mu \otimes_F M_2^\nu \quad \forall \lambda \in \mathbb{R},$$

and verifying conditions (C1)–(C4).

If $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 \leq \lambda_2$ and $(\mu_2, \nu_2) \in \mathbb{R}^2$ is such that $\lambda_2 = \mu_2 + \nu_2$, then we can find $(\mu_1, \nu_1) \in \mathbb{R}^2$ such that $\mu_1 \leq \mu_2$, $\nu_1 \leq \nu_2$ and $\lambda_1 = \mu_1 + \nu_1$. But then $M_1^{\mu_1} \geq M_1^{\mu_2}$ and $M_2^{\nu_1} \geq M_2^{\nu_2}$, whence

$$M_1^{\mu_1} \otimes_F M_2^{\nu_1} \geq M_1^{\mu_2} \otimes_F M_2^{\nu_2},$$

which shows that $M_{\lambda_1} \geq M_{\lambda_2}$. Hence (C1) holds.

The elements of M may be expressed as finite sums of the decomposable elements $x_1 \otimes x_2$, where $x_i \in M_i$, so to see that $M = \bigcup_{\lambda\in\mathbb{R}} M_\lambda$, it is sufficient to show that $x_1 \otimes x_2 \in M_\lambda$ for some $\lambda \in \mathbb{R}$. But the filtrations $\{M_1^\mu\}$ and $\{M_2^\nu\}$ guarantee the existence of real numbers μ and ν such that $x_1 \in M_1^\mu$ and $x_2 \in M_2^\nu$, and then

$$x_1 \otimes x_2 \in M_1^\mu \otimes_F M_2^\nu \subseteq M_{\mu+\nu}.$$

Thus (C2) holds.

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To show (C3), we note that

$$\begin{aligned} M_\lambda(\overline{a_1}, \overline{a_2}) &= \Big(\sum_{\mu+\nu=\lambda} M_1^\mu \otimes_F M_2^\nu\Big)(\overline{a_1}, \overline{a_2}) \\ &= \sum_{\mu+\nu=\lambda} M_1^\mu \overline{a_1} \otimes_F M_2^\nu \overline{a_2} \\ &= \sum_{\mu+\nu=\lambda} M_1^{\mu+\phi_1(a_1)} \otimes_F M_2^{\nu+\phi_2(a_2)} \\ &= \sum_{\mu'+\nu'=\lambda+\phi((a_1,a_2))} M_1^{\mu'} \otimes_F M_2^{\nu'} \\ &= M_{\lambda+\phi((a_1,a_2))}. \end{aligned}$$

To show (C4), we suppose to the contrary that $M_\lambda = M$ for some $\lambda \in \mathbb{R}$. This is equivalent to asserting that $M_0 = M$. We shall show that this leads to a contradiction. By (C4), the F-vector space M_i/M_i^0 is nonzero for the nontrivial ϕ_i filtration on M_i, where $i = 1, 2$. We fix an F-basis $\mathcal{B}_i^0$ of M_i^0, and an F-basis $\mathcal{B}_i$ of M_i such that $\mathcal{B}_i^0 \subseteq \mathcal{B}_i$. Note that the inclusion $\mathcal{B}_i^0 \subseteq \mathcal{B}_i$ must be strict since M_i/M_i^0 is nonzero. Pick $u_i \in \mathcal{B}_i \setminus \mathcal{B}_i^0$. Now $\mathcal{B}_1 \otimes \mathcal{B}_2 := \{v_1 \otimes v_2 \mid v_i \in B_i\}$ is an F-basis for $M_1 \otimes_F M_2$. Moreover, the element $(u_1 \otimes u_2)$ of $\mathcal{B} := \mathcal{B}_1 \otimes \mathcal{B}_2$ does not lie in the subset

$$\mathcal{B}' := (\mathcal{B}_1^0 \otimes \mathcal{B}_2) \cup (\mathcal{B}_1 \otimes \mathcal{B}_2^0),$$

where $\mathcal{B}_1^0 \otimes \mathcal{B}_2 := \{w \otimes v \mid w \in \mathcal{B}_1^0, v \in \mathcal{B}_2\}$ and $\mathcal{B}_1 \otimes \mathcal{B}_2^0$ is defined analogously. Since $\mathcal{B}$ is a basis of M, $u_1 \otimes u_2$ is not contained in

$$M' := (M_1^0 \otimes_F M_2) + (M_1 \otimes_F M_2^0),$$

which is the F-linear span of $\mathcal{B}'$, and *a fortiori*, $u_1 \otimes u_2$ is not in

$$M_0 = \sum_{\mu\in\mathbb{R}} M_1^\mu \otimes M_2^{-\mu} = \sum_{\nu\geq 0} M_1^\nu \otimes M_2^{-\nu} + \sum_{\nu\leq 0} M_1^\nu \otimes M_2^{-\nu}.$$

Hence $M_0 \neq M$, and (C4) is established.

We have thus exhibited a nontrivial ϕ-filtration of M and so $\phi \in \Delta(M)$. It follows that if $\phi \in \Delta(M_i) \setminus \{0\}$, then $\phi = \sum_{i=1}^2 \phi_i p_i$ is in $\Delta(M)$. The case when either ϕ_1 or ϕ_2 is zero is handled similarly.

We now show the reverse inclusion of (3.3). Let $\psi \in \Delta(M)$. We define $\psi_i \in A_i^*$ by $\psi_i := \psi e_i$, where $e_i : A_i \to A_1 \oplus A_2$ is the injection of the biproduct, when $i = 1, 2$. We shall show that $\psi_i \in \Delta(M_i)$. It then follows that

$$\psi = \psi_1 p_1 + \psi_2 p_2 \in \sum_{i=1}^2 \Delta(M_i) p_i.$$

Suppose that $\psi_1 \notin \Delta(M_1)$. Let $\mathcal{X}_1$ be a finite generating set for M_1. By [13, Proposition 3.1(v)], for each $y \in \mathcal{X}_1$, there exists a nonzero $\alpha_y \in \mathrm{ann}_{F*A_1}(y)$, the

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annihilator of y in $F * A_1$, such that ψ_1 attains a unique minimum on the support $\operatorname{supp}(\alpha_y)$ of α_y in A_1. Let $\mathcal{X}_2$ be a finite $F * A$-generating set for M_2. Then

$$\mathcal{X} := \{y \otimes z \mid y \in \mathcal{X}_1, z \in \mathcal{X}_2\}$$

generates M as an $F * (A_1 \oplus A_2)$-module. Denote the image of $\alpha_y \in F * A_1$ in

$$F * (A_1 \oplus A_2) = (F * A_1) \otimes_F (F * A_2)$$

by α'_y; then

$$(y \otimes z)\alpha'_y = y\alpha_y \otimes z = 0 \otimes z = 0.$$

Furthermore, $\psi = \psi_1 p_1 + \psi_2 p_2$ has a unique minimum on $\operatorname{supp}(\alpha'_y)$. But then $\psi \notin \Delta(M)$ by [13, Proposition 3.1(v)]. This contradiction shows that $\psi_1 \in \Delta(M_1)$. Similarly, it can be shown that $\psi_2 \in \Delta(M_2)$. We have thus shown that (3.1) holds. Applying [13, Theorem 4.4], we obtain (3.2). □

4. Strongly holonomic modules

We now develop a new proof of a result of Brookes and Groves. We first give some definitions and prove some useful results.

Definition 4.1 [11, Definition 4.2]. Let M be a finitely generated $F * A$-module where $F * A$ has center exactly F. Then M is *strongly holonomic* if

$$\text{GK-dim}(M) = \tfrac{1}{2}\operatorname{rank}(A)$$

and M is torsion-free as an $F * C$-module for each commutative subalgebra $F * C$, where $C \leq A$.

Definition 4.2. A nonzero $F * A$-module N is *critical* if the GK dimension of N/L is strictly smaller than that of N whenever $0 < L < N$.

The following proposition was first shown in [13].

Proposition 4.3. *Let M be a finitely generated nonzero $F * A$-module. Then M contains a critical submodule.*

Proof. Amongst the nonzero submodules of M, choose one, N say, of minimal possible GK dimension. If N is not critical, then it has a nonzero proper submodule N_1 such that $\text{GK-dim}(N/N_1) = \text{GK-dim}(N)$. By the minimality of $\text{GK-dim}(N)$,

$$\text{GK-dim}(N_1) = \text{GK-dim}(N).$$

Applying the same argument to N_1 and so on, we obtain a chain of submodules,

$$N = N_0 \supset N_1 \supset N_2 \supset \cdots ,$$

for which $\text{GK-dim}(N_i/N_{i+1}) = \text{GK-dim}(N)$ for each i. By [18, Lemma 5.6], this chain must terminate. But this process halts only when it reaches a critical module. □

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Proposition 4.4. *Let M be a strongly holonomic $F * A$-module, where $F * A$ has center F. Then M is cyclic and has finite length. Moreover, each nonzero submodule of M is also strongly holonomic.*

Proof. We claim that if an algebra $F * A$ satisfies the conditions of Proposition 4.4, then

$$\text{GK-dim}(V) \geq \tfrac{1}{2}\,\text{rank}(A)$$

for each nonzero $F * A$-module V. Indeed, let V' be a nonzero $F * A$-module for which $\text{GK-dim}(V') < \frac{1}{2}\,\text{rank}(A)$. Then by [10, Theorem 3], there is a subgroup C of A such that $\text{rank}(C) > \frac{1}{2}\,\text{rank}(A)$ and $F * C$ is commutative. By Definition 4.1, M must be torsion-free over $F * C$. Hence $\text{GK-dim}(M) > \frac{1}{2}\,\text{rank}(A)$ by Definition 2.3. But this is contrary to the hypothesis in the proposition. Hence each nonzero subfactor of M has the same GK dimension as M. It now follows from [18, Lemma 5.6] that a strictly descending sequence of submodules of M halts after a finite number of steps. Hence M has finite length. We also note that $F * A$ is simple by Proposition 1.1(vii). It follows that M is cyclic from [6, Corollary 1.5]. □

We now introduce carrier spaces and carrier space subgroups. We recall that for a finitely generated $D * A$-module M, the subset $\Delta(M)$ is a finite union of convex polyhedra. A $D * A$-module is called *pure* when each nonzero submodule of M has GK dimension equal to that of M. It is not difficult to see that a critical module is pure, noting Proposition 2.4. It was shown in [22] that if M is *pure*, then $\Delta(M)$ is a (finite) union of convex polyhedra, each having dimension equal to the GK dimension of M. A subspace $\mathcal{V}$ of A^* is *rationally defined* if it can be generated by rational linear combinations of the elements of the chosen dual basis of A^*. A rational subspace $\mathcal{V}$ of A^* may be uniquely expressed as $\text{ann}(B)$ for a subgroup B of A such that A/B is torsion-free.

Definition 4.5. Let M be a finitely generated critical $D * A$-module with GK dimension m. Associated with the rationally defined polyhedron $\Delta(M)$, there is a finite family of m-dimensional rationally defined subspaces of A^* that occur as the linear spans of the convex polyhedra constituting $\Delta(M)$. These subspaces are called the *carrier spaces* of $\Delta(M)$.

Definition 4.6. A subgroup of A of the form $\text{ann}(\mathcal{V})$, where $\mathcal{V}$ is a carrier space of $\Delta(M)$ and M a finitely generated critical $D * A$-module, is called a *carrier space subgroup* of $\Delta(M)$.

We note that for a carrier space subgroup C of $\Delta(M)$,

$$\text{rank}(C) = \text{rank}(A) - \text{GK-dim}(M).$$

If C is a carrier space subgroup of $\Delta(M)$, then M cannot be finitely generated as an $F * C$-module by [13, Proposition 3.8]. The proof of [10, Theorem A] was based on the following important property of carrier space subgroups, which was subsequently highlighted in [11, Proposition 4.1(2)].

Lemma 4.7 [10, Theorem A]. *Let M be a critical finitely generated $F * A$-module, $\mathcal{V}$ be a carrier space of $\Delta(M)$, and $B := \mathrm{ann}(\mathcal{V})$. Then B contains a subgroup B_1 of finite index such that $F * B_1$ is commutative.*

We recall that a subgroup B of A is *isolated* in A if A/B is torsion-free.

Definition 4.8 [14]. Let M be a finitely generated $F * A$-module. Let $\mathcal{W}$ be a rational subspace of A^* and B be the isolated subgroup of A such that $\mathcal{W} = \mathrm{ann}(B)$. A point $\phi \in \mathcal{W}$ is said to be *nongeneric* for $\mathcal{W}$ and M if $TC_\phi(M)$ is not $F * C$-torsion for some infinite cyclic subgroup C of B.

The following fact was first shown in [11, Lemma 4.5]. The proof was based on a geometric characterization of nongeneric points in $\Delta(M)$.

Lemma 4.9. *Let M be a critical strongly holonomic $F * A$-module, where* $\mathrm{rank}(A) > 2$. *For each carrier space subgroup U of $\Delta(M)$, there is a subgroup W of A with the same rank as U such that $F * W$ is commutative and*

$$0 < \mathrm{rank}(U \cap W) < \mathrm{rank}(U).$$

Proof. By Lemma 4.7, U has a subgroup U' of finite index such that $F * U'$ is commutative. By Definition 4.1, M is torsion-free over $F * U'$.

We claim that M is $F * U$-torsion-free. Indeed, if the $F * U$-torsion submodule $t_U(M)$ of M were nonzero, we would be able to pick a finitely generated nonzero $F * U$-submodule N of $t_U(M)$. Assuming this, N is $F * U$-torsion and $F * U'$-torsion-free. Then

$$\text{GK-dim}(N) < \mathrm{rank}(U),$$

in view of [13, Proposition 2.6]. Moreover, $\text{GK-dim}(N) \geq \mathrm{rank}(U')$ since N is torsion-free as an $F * U'$-module, by Definition 2.3. We thus have a contradiction and so M must be $F * U$-torsion-free.

By [14, Corollary 3.7], $\mathcal{V} := \mathrm{ann}(U)$ contains a nonzero point ϕ that is nongeneric for $\mathcal{V}$ and M. By [14, Lemma 3.1], U has an infinite cyclic subgroup C such that $\phi_C \in \Delta(M \otimes_{F*A} (F * A)S^{-1})$, where ϕ_C is the character of $(A/C)^*$ induced by ϕ and $S = F * C \setminus \{0\}$. Note that $M_C := M \otimes_{F*A} (F * A)S^{-1}$ is an $(F * A)S^{-1}$-module and $(F * A)S^{-1}$ is a crossed product $D_C * A/C$, where D_C denotes the quotient division ring of $F * C$. By [14, Lemma 4.5(2)], M_C is critical.

Now ϕ_C lies in a (rationally defined) carrier space $\mathcal{V}_C := \mathrm{ann}(V/C)$ for some subgroup V of A. Set $K = \ker \phi$. Then $K/C = \ker \phi_C \geq V/C$, and so $V \leq K$. It was shown in [10, Section 2] that $D_C * V/C$ has a nonzero module that is finite-dimensional as a D_C-space.

Note that

$$\text{GK-dim}(M_C) = \text{GK-dim}(M) - \mathrm{rank}(C),$$

in view of Definition 2.3. Since $\dim \mathcal{V}_C = \text{GK-dim}(M_C)$,

$$\mathrm{rank}(V/C) = \mathrm{rank}(A/C) - \text{GK-dim}(M_C) = \mathrm{rank}(A) - \text{GK-dim}(M) = m.$$

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By [1, Corollary 3.3], V contains a subgroup W of rank m such that $F * W$ is commutative. Moreover, W is constructed in [1] so that $W \cap C = \{1\}$, whence

$$\mathrm{rank}(U \cap W) < \mathrm{rank}(U).$$

As ϕ is nonzero, $\mathrm{rank}(K) \leq 2m - 1$ and $\mathrm{rank}(U \cap W) \geq 1$ since $U, W \leq K$. □

The next lemma is a generalization of [11, Lemma 4.4].

LEMMA 4.10. *Suppose that $F * A$ has a finitely generated module M and A has a subgroup C such that A/C is torsion-free,* $\mathrm{rank}(C) = \text{GK-dim}(M)$, *and $F * C$ is commutative. Suppose, moreover, that M is not $F * C$-torsion. Then C has a virtual complement E in A such that $F * E$ is commutative. In fact given $\mathbb{Z}$-bases $\{x_1, \ldots, x_r\}$ and $\{x_1, \ldots, x_r, x_{r+1}, \ldots, x_n\}$ for C and A respectively, there exist monomials μ_j in $F * C$, where $j = r + 1, \ldots, n$, and a positive integer s such that the monomials $\mu_j \bar{x}_j^s$ commute in $F * A$.*

PROOF. Let $\bar{x}_i \bar{x}_j = q_{ij} \bar{x}_j \bar{x}_i$, where $i, j = 1, \ldots, n$ and $q_{ij} \in F^*$. We set $S = F * C \setminus \{0\}$ and denote the quotient field $(F * C)S^{-1}$ by F_S. Then $(F * A)S^{-1}$ is a crossed product

$$R = F_S * \langle x_{r+1}, \ldots, x_n \rangle.$$

By hypothesis, M is not S-torsion and so the corresponding module of fractions MS^{-1} is nonzero. Further, $\text{GK-dim}(M) = \mathrm{rank}(C)$ and so MS^{-1} is finite-dimensional as an F_S-space in view of [13, Lemma 2.3]. In [1, Section 3], we show that, if R has a module that is one-dimensional over F_S, then there exist monomials $\mu_i \in F[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_r^{\pm 1}]$ such that the monomials $\mu_i x_i$ mutually commute, where $r + 1 \leq i \leq n$. Thus we may take $E = \langle \mu_{r+1} x_{r+1}, \ldots, \mu_n x_n \rangle$ in this case. Let $s = \dim_{F_S} MS^{-1}$. The s-fold exterior power $\wedge^s (MS^{-1}_{F_S})$ is a one-dimensional module over $R' := F_S *^s \langle x_{r+1}, \ldots, x_n \rangle$ and the 2-cocycle is the sth power of the 2-cocycle of R, as observed in the remark following [1, Corollary 3.3]. Thus for R',

$$q'_{ij} = \begin{cases} q_{ij} & \text{when } i \in \{1, \ldots, r\}, j \in \{1, \ldots, n\}, \\ q_{ij}^s & \text{when } i, j \in \{r + 1, \ldots, n\}. \end{cases} \tag{4.1}$$

By [1, Proposition 3.2], the monomials $\mu_j x_j$ commute in R', that is,

$$1 = [\overline{\mu_k x_k}, \overline{\mu_l x_l}] = [\overline{\mu_k}, \overline{x_l}][\overline{x_k}, \overline{\mu_l}][\overline{x_k}, \overline{x_l}] \quad \forall k, l \in \{r + 1, \ldots, n\}.$$

In view of (4.1), $[\overline{\mu_k}, \overline{x_l}]$ and $[\overline{x_k}, \overline{\mu_l}]$ are the same in R and R', but $[\overline{x_k}, \overline{x_l}] = [\overline{x_k}, \overline{x_l}]^s$ in R'. It easily follows from this that the elements $\mu_j x_j^s$, where $j = r + 1, \ldots, n$, commute in R. □

The final lemma of this section is somewhat technical and is used in the proof of Theorem 4.12.

LEMMA 4.11. *Suppose that $F * A$ has a strongly holonomic module. If* $\mathrm{rank}(A) = 2m$, *where $m > 1$, then A has nontrivial subgroups B_1, B_2, B_3, and B_4 such that each $F * B_i$*

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is commutative and

$$[\overline{B_1}, \overline{B_2}] = [\overline{B_2}, \overline{B_3}] = [\overline{B_3}, \overline{B_4}] = \{1\},$$
$$B_1 \cap B_2 = B_3 \cap B_4 = B_1B_2 \cap B_3B_4 = \{1\},$$
$$\mathrm{rank}(B_1) + \mathrm{rank}(B_2) = \mathrm{rank}(B_2) + \mathrm{rank}(B_3) = \mathrm{rank}(B_3) + \mathrm{rank}(B_4) = m.$$

Proof. By Proposition 4.4, M contains a simple submodule that is also strongly holonomic. Hence we may assume that M is simple.

Let U be a carrier space subgroup of $\Delta(M)$. As shown in the first paragraph of the proof of Lemma 4.9, M is torsion-free as an $F * U$-module. Moreover, as noted above, $\mathrm{rank}(U) = m = \mathrm{GK\text{-}dim}(M)$. Let V be a (virtual) complement to U in A, as given by Lemma 4.10, such that $F * V$ is commutative. By Lemma 4.7, there is a finite index subgroup U_0 of U so that $F * U_0$ is commutative. But then $A_0 := U_0V$ has finite index in A. In particular, M may be regarded as a finitely generated $F * A_0$-module M_0. By [13, Lemma 2.7], $\mathrm{GK\text{-}dim}(M_0) = \mathrm{GK\text{-}dim}(M)$, and it follows that M_0 is a strongly holonomic $F * A_0$-module. For this reason, we will assume that $A = UV$ and that both $F * U$ and $F * V$ are commutative.

By Lemma 4.9, there is also a subgroup W that intersects nontrivially with U such that $\mathrm{rank}(W) = \mathrm{rank}(U) = m$. Set $B_2 := U \cap W$, and pick a subgroup B_1 in U that is maximal with respect to the condition that $B_1 \cap B_2 = \{1\}$. Let $p_V : A = U \oplus V \to V$ be the projection and p'_V be its restriction to W. Then $\ker p'_V = B_2$ and so

$$\mathrm{rank}(p'_V(W)) + \mathrm{rank}(B_2) = \mathrm{rank}(W) = m.$$

Set $B_3 := p'_V(W)$, and let B_4 be a subgroup of V that is maximal with respect to the condition that

$$B_3 \cap B_4 = \{1\}.$$

As $B_1, B_2 \leq U$ and $F * U$ is commutative, $[\overline{B_1}, \overline{B_2}] = \{1\}$ and similarly $[\overline{B_3}, \overline{B_4}] = \{1\}$. We claim that

$$[\overline{B_2}, \overline{B_3}] = \{1\}.$$

Indeed, take $u_2 \in B_2 = U \cap W$ and $v_3 \in B_3$. Now $B_3 = p'_V(W)$, hence $uv_3 \in W$ for some $u \in U$. Since $F * W$ is commutative,

$$1 = [\bar{u}\bar{v}_3, \bar{u}_2] = [\bar{u}, \bar{u}_2][\bar{v}_3, \bar{u}_2].$$

Moreover, as $F * U$ is commutative, $[\bar{u}, \bar{u}_2] = 1$ and so $[\bar{v}_3, \bar{u}_2] = 1$. □

We now give a new proof of a result of Brookes and Groves.

Theorem 4.12 [11, Theorem 4.2]. *Suppose that $F * A$ is an algebra with center F for which* $\mathrm{rank}(A) = 2m$, *and that $F * A$ has a strongly holonomic module. Then there is a finite index subgroup A' in A such that*

$$F * A' = (F * B_1) \otimes_F \cdots \otimes_F (F * B_m),$$

where each $B_i \cong \mathbb{Z} \oplus \mathbb{Z}$ and $m = \frac{1}{2}$ $\mathrm{rank}(A)$.

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Proof. We use the notation

$$F * A \overset{\text{vir}}{=} (F * A_1) \otimes_F \cdots \otimes_F (F * A_k)$$

to indicate that A has a subgroup A' of finite index such that

$$F * A' = (F * A_1) \otimes_F \cdots \otimes_F (F * A_k).$$

We shall prove the theorem by induction. If $\operatorname{rank}(A) = 2$, then there is nothing to be proved, so we assume that $\operatorname{rank}(A) = 2m$ where $m > 1$. We also assume that the theorem holds for all smaller values of m and for all fields F.

Let B_1, B_2, B_3 and B_4 be as in Lemma 4.11 and set $B := B_1B_2B_3B_4$. By the lemma, $F * B_2B_3$ is commutative and

$$\operatorname{rank}(B_2B_3) = m = \text{GK-dim}(M).$$

We fix bases in the subgroups $B_1, \ldots, B_4$ as follows:

$$\begin{aligned} B_1 &:= \langle u_{k+1}, \ldots, u_m \rangle, \\ B_2 &:= \langle u_1, \ldots, u_k \rangle, \\ B_3 &:= \langle w_{k+1}, \ldots, w_m \rangle, \\ B_4 &:= \langle w_1, \ldots, w_k \rangle. \end{aligned}$$

By Lemma 4.10 (with C taken to be B_2B_3), there are monomials $\mu_j \in F * B_2$ and $\nu_j \in F * B_3$, where $j = 1, \ldots m$, such that the monomials in

$$\{\mu_i \nu_i \bar{w}_i^s \mid i = 1, \ldots, k\} \cup \{\mu_j \nu_j \bar{u}_j^s \mid j = k+1, \ldots, m\} \tag{4.2}$$

commute mutually for some positive integer s. Set

$$\begin{aligned} \bar{w}_i' &= \nu_i \bar{w}_i^s \quad \forall i \in \{1, \ldots, k\}, \\ \bar{u}_j' &= \mu_j \bar{u}_j^s \quad \forall j \in \{k+1, \ldots, m\}. \end{aligned} \tag{4.3}$$

As noted in (4.2),

$$\begin{aligned} 1 &= [\mu_i \nu_i \overline{w_i}^s, \mu_j \nu_j \overline{u_j}^s] \\ &= [\mu_i, \mu_j \nu_j \overline{u_j}^s][\nu_i \overline{w_i}^s, \mu_j \nu_j \overline{u_j}^s] \\ &= [\mu_i, \mu_j \nu_j][\mu_i, \overline{u_j}^s][\nu_i \overline{w_i}^s, \mu_j \nu_j \overline{u_j}^s]. \end{aligned}$$

But $[\mu_i, \mu_j \nu_j] = 1$ since $F * B_2B_3$ is commutative and $[\mu_i, \overline{u_j}^s] = 1$ since $[\overline{B_1}, \overline{B_2}] = \{1\}$ by Lemma 4.11. Thus

$$\begin{aligned} 1 &= [\nu_i \overline{w_i}^s, \mu_j \nu_j \overline{u_j}^s] \\ &= [\nu_i \overline{w_i}^s, \mu_j \overline{u_j}^s][\nu_i \overline{w_i}^s, \nu_j] \\ &= [\nu_i \overline{w_i}^s, \mu_j \overline{u_j}^s][\nu_i, \nu_j][\overline{w_i}^s, \nu_j]. \end{aligned}$$

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By Lemma 4.11, $F * B_3$ is commutative and $[\overline{B_3}, \overline{B_4}] = \{1\}$. It follows that

$$[v_i, v_j] = [\overline{w_i}^s, v_j] = 1.$$

Hence in view of (4.3),

$$[\overline{w'_i}, \overline{u'_j}] = 1.$$

Set $B'_1 := \langle u'_{k+1}, \ldots, u'_m \rangle$ and $B'_4 := \langle w'_1, \ldots, W'_k \rangle$; then

$$[\overline{B'_1}, \overline{B'_4}] = \langle 1 \rangle,$$

which, in view of Lemma 4.11, gives

$$[\overline{B'_1 B_3}, \overline{B_2 B'_4}] = \langle 1 \rangle. \tag{4.4}$$

Hence

$$F * B = (F * B'_1 B_3) \otimes_F (F * B_2 B'_4). \tag{4.5}$$

By the hypothesis in the theorem, $F * A$ has center exactly F, so in view of (4.4), $F * B'_1 B_2 B_3$ has center exactly $F * B_2$. Moreover, $C := F * B'_1 B_2$ is commutative and thus M is torsion-free over C. Consequently, there is a finitely generated critical $F * B'_1 B_2 B_3$-submodule N of M such that $\text{GK-dim}(N) = m$. Localizing $F * B'_1 B_2 B_3$ at $F * B_2 \setminus \{0\}$, we obtain $F' * B'_1 B_3$, where F' is the quotient field of the integral domain $F * B_2$.

We claim that $M' := M(F * B_2)^{-1}$ is a strongly holonomic $F' * B'_1 B_3$-module. Indeed, in view of [14, Lemma 4.5(2)],

$$\text{GK-dim}(M') = \text{GK-dim}(M) - k = m - k = \tfrac{1}{2}\,\text{rank}(B'_1 B_3),$$

and M' is $F' * C$-torsion-free if $F' * C$ is commutative (see [11, Lemma 4.3]). We note that the 2-cocycle of $F' * B'_1 B_3$ is the restriction to $B'_1 B_3$ of the 2-cocycle of $F * B'_1 B_2 B_3$. Then the induction hypothesis yields that

$$F * B'_1 B_3 \overset{\text{vir}}{=} (F * C_1) \otimes_F (F * C_2) \otimes_F \cdots \otimes (F * C_{m-k}). \tag{4.6}$$

By parallel reasoning applied to $F * B_2 B_3 B'_4$, which has center $F * B_3$, we deduce that

$$F * B_2 B'_4 \overset{\text{vir}}{=} F * E_1 \otimes_F (F * E_2) \otimes_F \cdots \otimes_F (F * E_k). \tag{4.7}$$

Combining (4.5), (4.6) and (4.7) proves the theorem. □

5. Nonholonomic simple modules

We now consider the problem of the GK dimensions of simple modules over the algebras $F * A$. In particular, we wish to show there can be simple $F * A$-modules with distinct GK dimensions. We shall achieve this by embedding $F * A$ in a principal ideal domain. Given an algebra $F * A$, let B be a subgroup of A such that A/B is infinite cyclic. The localization $F * A(F * B)^{-1}$ is a crossed product $D * A/B$, where D denotes the quotient division ring $F * B(F * B)^{-1}$. Moreover, if $A/B = \langle uB \rangle$, then $D * A/B$ is a skew Laurent extension $D[\bar{u}^{\pm 1}, \sigma]$, where $\sigma(d) = \bar{u}\, d \bar{u}^{-1}$ for all $d \in D$.

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Lemma 5.1. *Suppose that $F * A$ has center exactly F and A has a subgroup B such that $F * B$ is commutative and A/B is infinite cyclic. If M is a nonzero finitely generated $F * A$-module that is finitely generated as an $F * B$-module, then M is $F * B$-torsion-free.*

Proof. Suppose to the contrary that the $F * B$-torsion submodule T of M is nonzero. Now $F * B \setminus \{0\}$ is a right Ore subset in $F * A$, so T is an $F * A$-submodule of M. Since $F * B$ is noetherian, the hypothesis in the lemma that M is finitely generated as an $F * B$-module implies that T is finitely generated as an $F * B$-module. It follows by [13, Lemma 2.7] that the GK dimensions of T as an $F * A$-module and as an $F * B$-module are equal. But T is $F * B$-torsion by definition, and so

$$\text{GK-dim}(T) < \text{rank}(B) = t - 1$$

by Definition 2.3 and [13, Proposition 2.6], where $\text{rank}(A) = t$.

We assume for clarity that $\text{GK-dim}(T) = t - 2$, for our reasoning below is equally valid for all possible values of $\text{GK-dim}(T)$ less than $t - 1$. By Definition 2.3, there is a subgroup C of B of rank $t - 2$ such that T is not $F * C$-torsion, and in view of [13, Lemma 2.6], C may be chosen such that B/C is infinite cyclic. We pick a basis $\{v_1, v_2, \ldots, v_{t-2}\}$ of C. Since $B/C \cong \mathbb{Z}$, this can be extended to a basis $\{v_1, v_2, \ldots, v_{t-2}, v_{t-1}\}$ of B. Set $S := F * C \setminus \{0\}$. We denote the right Ore localization $(F * A)S^{-1}$ as $R_0 = F(v_1, \ldots, v_{t-2})[v_{t-1}, u]$, with the localized generators $v_1, \ldots, v_{t-2}$ in parentheses. As noted above, T is not S-torsion and so the localization TS^{-1} is a nonzero R_0-module. Further, TS^{-1} is finite-dimensional as an F_1-space, where $F_1 = (F * C)S^{=1}$, in view of [13, Lemma 2.3]. By [18, Theorem 3.9],

$$[\overline{v_{t-1}}, \bar{u}]^s \prod_{i=1}^{t-2} [\overline{v_i}, \bar{u}]^{t_i} = 1,$$

where $t_i \in \mathbb{Z}$ and $s = \dim_{F_1}(TS^{-1}) > 0$. This yields

$$\left[\overline{v_{t-1}}^s \prod_{i=1}^{t-2} \overline{v_i}^{t_i}, \bar{u} \right] = 1,$$

which implies that $\overline{v_{t-1}}^s \prod_{i=1}^{t-2} \overline{v_i}^{t_i}$ is central in $F * A$, contrary to the hypothesis. It now follows that $T = 0$ as claimed. □

Definition 5.2. Let R be a principal ideal domain. An element $r \in R$ is *irreducible* if in any factorization $r = st$, where $s, t \in R$, at least one of s and t is a unit.

Proposition 5.3. *Suppose that $F * A$ has center exactly F and A has a subgroup B such that $F * B$ is commutative and A/B is infinite cyclic. Then $F * A$ has a simple module S_1 such that* $\text{GK-dim}(S_1) = 1$. *Furthermore, let $A/B = \langle uB \rangle$ and R be the right Ore localization $F * A(F * B)^{-1}$. Let r be an irreducible element in the principal ideal*

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domain R. If $\mathcal{J} := F * A \cap rR$ contains a nonzero element γ such that in the (unique) expression

$$\gamma = \sum_{i=s}^{t} \beta_i \bar{u}^i, \tag{5.1}$$

where $s, t \in \mathbb{Z}$ and $\beta_i \in F * B$, the leading elements β_s and β_t are units in $F * B$, then $S_2 := F * A/\mathcal{J}$ is a simple $F * B$-torsion-free module and $\text{GK-dim}(S_2) = n - 1$.

Proof. First, we show that S_2 is simple and $\text{GK-dim}(S_2) = n - 1$. As noted above, R is a principal ideal domain. Thus if r is irreducible in R, then rR is a maximal right ideal in R.

By [7, Lemma 3.3], $\mathcal{J} = F * A \cap rR$ is a maximal right ideal of $F * A$ if and only if for each $\beta \in F * B \setminus \{0\}$,

$$F * A = \beta F * A + \mathcal{J}. \tag{5.2}$$

We shall show that $S_2 = F * A/\mathcal{J}$ is simple by showing that (5.2) holds. Indeed, by the hypothesis of the theorem, $\mathcal{J}$ contains a nonzero element γ of the form (5.1), so by [5, Proposition 2.1], $S_2 = F * A/\mathcal{J}$ is a finitely generated $F * B$-module. Take $\beta \in F * B \setminus \{0\}$ and set $\mathcal{J}_\beta := \beta(F * A) + \mathcal{J}$ and $M_\beta := F * A/\mathcal{J}_\beta$. Then M_β is a finitely generated $F * B$-module. If $M_\beta \neq 0$, then there exists $m \in M_\beta \setminus \{0\}$, namely the coset $1 + \mathcal{J}_\beta$, such that $m\beta = 0$ for the nonzero element $\beta \in F * B$. But this is a contradiction, in view of Lemma 5.1. Hence $M_\beta = 0$, and it follows that (5.2) is satisfied and thus S_2 is simple. As already noted above, S_2 is finitely generated as an $F * B$-module and so is $F * B$-torsion-free, by Lemma 5.1. Hence by Definition 2.3, $\text{GK-dim}(S_2) \geq n - 1$. But it is impossible that $\text{GK-dim}(S_2) = n$, for this would imply that $S_2 \cong F * A$, by Definition 2.3, and it is easily seen that $F * A$ is not a simple $F * A$-module.

It remains to show that $F * A$ has a simple module S_1 with $\text{GK-dim}(S_1) = 1$. By [10, Section 2], $F * A$ has a finitely generated module T_1 with $\text{GK-dim}(T_1) = 1$. We claim that if N is a finitely generated $F * A$-module such that $\text{GK-dim}(N) = 0$, then $N = 0$. Indeed, if $N \neq 0$, then by [10, Theorem 3], A has a subgroup A' of finite index such that $F * A'$ is commutative. It is easily seen, from (1.4), that $F * A$ has center larger than F in this case, contrary to the hypothesis in Proposition 5.3.

By reasoning like that in the proof of Proposition 4.4, it follows that T_1 has finite length and so contains a simple submodule S_1 for which $\text{GK-dim}(S_1) = 1$. □

Here is our third main result, which follows from the previous proposition.

Theorem 5.4. *Suppose that $F * A$ has center exactly F and A has a subgroup B such that A/B is infinite cyclic and $F * B$ is commutative. Then $F * A$ has a simple $F * B$-torsion-free module S for which* $\text{GK-dim}(S) = n - 1$.

Example 5.5. Let t be a positive integer and K be the ordinary Laurent polynomial ring $\mathbb{Q}[u_1^{\pm 1}, \ldots, u_t^{\pm 1}]$ over $\mathbb{Q}$ in the t variables $u_1, \ldots, u_t$. Let $p_1, p_2, \ldots, p_t$ be distinct primes in $\mathbb{Z}$, and define the automorphism σ of K by $\sigma(u_i) = p_i u_i$. The skew Laurent extension $T = K[u^{\pm 1}, \sigma]$ is a quantum Laurent polynomial algebra that satisfies the

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hypothesis of Proposition 5.3. Furthermore, let $K^* := K \setminus \{0\}$ and R be the right Ore localization of T at K^*. Let r be an irreducible element of R of the form

$$r = u^k + f_1 u^{k-1} + \cdots + f_{k-1} u + g,$$

where $k \in \mathbb{Z}^+$, $f_1, \ldots, f_{k-1}, g \in K$, and g is a monomial. Clearly, r satisfies (5.1). By Proposition 5.3, the T-module $T/T \cap rR$ is simple and torsion-free over K.

Acknowledgement

This paper is mainly taken from my PhD thesis and I wish to express my sincerest gratitude to Dr. J. R. J. Groves for his supervision.

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https://doi.org/10.1017/S1446788712000031 Published online by Cambridge University Press