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AUTOMORPHIC LEFSCHETZ PROPERTIES FOR NONCOMPACT ARITHMETIC MANIFOLDS

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Abstract We prove the injectivity of Oda-type restriction maps for the cohomology of noncompact congruence quotients of symmetric spaces. This includes results for restriction between (1) congruence real hyperbolic manifolds, (2) congruence complex hyperbolic manifolds, and (3) orthogonal Shimura varieties. These results generalize results for compact congruence quotients by Bergeron and Clozel [Quelques conséquences des travaux d'Arthur pour le spectre et la topologie des variétés hyperboliques, *Invent. Math.* **192** (2013), 505–532] and Venkataramana [Cohomology of compact locally symmetric spaces, *Compos. Math.* **125** (2001), 221–253]. The proofs combine techniques of mixed Hodge theory and methods involving automorphic forms.

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Contents

1. Introduction

1.1. Main results

Let G be a connected semisimple algebraic group over $\mathbb{Q}$ and X be the symmetric space of $G(\mathbb{R})$. We write $G(\mathbb{R})^{nc}$ for the product of the noncompact factors of $G(\mathbb{R})$ and d_G for the (real) dimension of X.

For congruence subgroups $\Gamma \subset G(\mathbb{Q})$, we consider the quotients $M_\Gamma = \Gamma\backslash X$ and their cohomology groups $\mathrm{H}^i(M_\Gamma)$ with complex coefficients. The direct limit

$$\mathrm{H}^i(\mathscr{M}_G) := \mathrm{colim}_\Gamma \mathrm{H}^i(M_\Gamma)$$

is a $G(\mathbb{Q})$-module using pullback by the isomorphisms $M_{g\Gamma g^{-1}} \to M_\Gamma$ induced by g^{-1} on X.

For a semisimple subgroup $H \subset G$, let $\Gamma_H = \Gamma \cap H(\mathbb{Q})$ and $M_{H,\Gamma_H} = \Gamma_H\backslash X_H$. The totally geodesic embedding $X_H \subset X$ induces a proper map

$$M_{H,\Gamma_H} \to M_\Gamma.$$

Pullback in cohomology defines an $H(\mathbb{Q})$-equivariant map $\iota^* : \mathrm{H}^*(\mathscr{M}_G) \to \mathrm{H}^*(\mathscr{M}_H)$, and composing with the action of $G(\mathbb{Q})$ gives a map

$$\mathrm{Res} : \mathrm{H}^*(\mathscr{M}_G) \longrightarrow I_H^G \mathrm{H}^*(\mathscr{M}_H).$$

The target of Res (defined in §2.4) is a certain induced module contained in the product $\prod_{g\in G(\mathbb{Q})} \mathrm{H}^*(\mathscr{M}_H)$, so that concretely we have that $\mathrm{Res}(\alpha) \neq 0$ if and only if $\iota^*\left(g^{-1}\cdot\alpha\right) \neq 0$ for some $g \in G(\mathbb{Q})$ – that is, some Hecke translate of α restricts nontrivially to $\mathscr{M}_H$.

Theorem 1.1. *Suppose that $H \subset G$ are semisimple groups of the same $\mathbb{Q}$-rank and that $H(\mathbb{R})^{nc} \subset G(\mathbb{R})^{nc}$ is one of the following embeddings:*

(1) *$SO(1,c) \subset SO(1,d)$ (the real hyperbolic case), with neither H nor G a triality form;*
(2) *$SU(1,m) \subset SU(1,n)$ (the complex hyperbolic or ball quotient case);*
(3) *$SO(2,m) \subset SO(2,n)$ (the orthogonal Shimura variety case).*

Then the map $\mathrm{Res} : \mathrm{H}^*(\mathscr{M}_G) \longrightarrow I_H^G \mathrm{H}^*(\mathscr{M}_H)$ is injective in degrees $< d_H/2$ (and also in degree $i = d_H/2$ in the case where $SO(1,c) \subset SO(1,d)$ with c even).

This *automorphic Lefschetz property* is well known if G is anisotropic (or equivalently, M_Γ is compact): the injectivity in case (1) was proved in [12], and in cases (2) and (3) it was proved in [47] in degrees $i \leq d_H/2$. In the noncompact situation, case (1) can be proved by adapting the methods of [7, 12] with some care, and case (2) was proved in [36] (and [13]), so that the most interesting new case is (3). It includes, for example, the most basic orthogonal Shimura varieties arising from quadratic forms over $\mathbb{Q}$ of signature $(2,n)$ over

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$\mathbb{R}$ with $n \geq 4$. The treatment of the 'missing' degree $i = d_H/2$ in the noncompact $SU(1,n)$ and $SO(2,n)$ cases requires arithmetic information, and we leave it for another occasion. (Note that restriction in degree 2 from $SO(2,3)$ to $SO(2,2)$, which is not covered by Theorem 1.1, was treated by Weissauer [51] using a detailed automorphic understanding of H^2 of Siegel threefolds.)

For the inclusion $SO(1,n) \subset SU(1,n)$ we have the following, which is [12, Theorem 1.7] in the compact case. For $i < n$, the group $\mathrm{H}^i(M_\Gamma)$ carries a pure Hodge structure of weight i, and we set $\mathrm{H}^{i,0}(\mathscr{M}_G) := \mathrm{colim}_\Gamma \mathrm{H}^{i,0}(M_\Gamma)$.

Theorem 1.2. *Suppose that $H \subset G$ are of the same $\mathbb{Q}$-rank, $H(\mathbb{R})^{nc} \subset G(\mathbb{R})^{nc}$ is $SO(1,n) \subset SU(1,n)$, and H is not a triality form. Then $\mathrm{Res} : \mathrm{H}^{i,0}(\mathscr{M}_G) \longrightarrow I_H^G \mathrm{H}^*(\mathscr{M}_H)$ is injective in degrees $i \leq n/2$.*

We will discuss the proofs in §1.3.

1.2. Some history

Restriction maps between congruence quotients have been studied by numerous authors for almost 40 years, starting with the pioneering work of Oda [38]. We refer the reader to the surveys [10, 48] for a discussion of this work, and restrict ourselves here to a brief review of the history of immediate relevance to us.

The first result of this type was proved by Oda [38], who introduced the restriction map Res and proved Theorem 1.1 for $SU(1,m) \subset SU(1,n)$ in degree $i = 1$. Weissauer [51] then proved the Lefschetz property for $SO(2,2) \subset SO(2,3)$ in degree $i = 2$. Motivated by the general conjecture (of Langlands, Kottwitz, and Arthur) on the Galois representations appearing in the cohomology of Shimura varieties, Arthur [1, §9] raised the question of whether the nonprimitive cohomology of Shimura varieties can be related to smaller Shimura varieties. Harris and Li [28] applied the Burger–Sarnak [18] method to prove the Lefschetz property in degree $i = 2$ in the cases of compact complex hyperbolic and orthogonal Shimura varieties – that is, cases (2) and (3). They also conjectured injectivity in degrees $\leq d_H/2$ in these cases and showed that in case (2) it would follow from Arthur's conjectures [1] on the discrete spectrum. They also asked (when $d_G = d_H + 2$ in cases (2) and (3)) whether a linear combination of Hecke translates of the class of the divisor $M_{H,\Gamma}$ is the class of an ample divisor. Venkataramana [47] showed that this is true in cohomology rather than on the level of cycles – that is, a linear combination of translates of the cycle class in $\mathrm{H}^2(M_\Gamma)$ is the hyperplane cohomology class in the Baily–Borel projective embedding – and used this to prove the conjecture of [28] – that is, Theorem 1.1 in compact cases (2) and (3).

The automorphic approach of [28] was taken up by Bergeron and Clozel [7, 8, 11, 12], who made the remarkable discovery that Lefschetz properties hold for congruence hyperbolic manifolds (i.e., case (1) of Theorem 1.1) even though there is no complex structure available. This allows for a common approach to Lefschetz properties in different contexts, by using the Burger–Sarnak method to reduce them to uniform (in the level Γ) bounds for the nonzero eigenvalues of the Laplacian on forms on the smaller locally symmetric space. This eigenvalue bound was then deduced in case (1) in [12] from Arthur's endoscopic classification [2] of automorphic forms on orthogonal groups, completing the proof of Theorems 1.1 and 1.2 in the compact case.

In the noncompact case it is less clear what should be true, although the analogues for singular varieties of the Lefschetz theorems in [24] are suggestive. Moreover, since the cohomology of noncompact quotients is influenced by the behavior of L-functions (for example, through Eisenstein series constructions of cohomology), one expects that the question is more subtle, and this is reflected by the omission of $i = d_H/2$ in cases (2) and (3) for now.

The complex hyperbolic case of Theorem 1.1 was proved in [13, 35, 36]; here we will prove the rest of the theorem, with the case of orthogonal Shimura varieties being the main new result. In fact, our proof shows that the Lefschetz property for congruence real hyperbolic groups arises as a sort of *local* Lefschetz property at infinity for the noncompact case of orthogonal Shimura varieties. We will comment further on this later.

1.3. On the proofs

We discuss the proofs of the main theorems and some intermediate results proved along the way.

There are, roughly speaking, three types of arguments involved:

(a) automorphic arguments – mainly the Burger–Sarnak method, as in [7, 12, 28], but also rank 1 residual Eisenstein cohomology;

(b) geometric arguments – the use of cycle classes, as in [47], and mixed Hodge theory and compactifications, as in [35, 36]; and

(c) elementary arguments with Lie-algebra cohomology, as in [36].

The proof of Theorem 1.1 in the different cases uses these ingredients differently: Case (1) uses (a) and (c), case (2) uses (b) and (c), and case (3) uses (b) and (c) explicitly, but also (a) through the use of case (1). The proof of Theorem 1.2 uses mainly (a) and (c), with some mild input from (b). The use of results about weights in the topology of singular varieties in (b), which play a crucial role in our approach, constitutes the main technical novelty of this paper. This will be clear from the detailed sketch of the proof of Theorem 1.1, which we now give.

A basic role is played by the *minimal compactification* $M_\Gamma \hookrightarrow M_\Gamma^*$, which is the cusp compactification of the (real or complex) hyperbolic manifold in cases (1) and (2) and the Satake–Baily–Borel compactification in cases (2) and (3). This gives the basic exact sequence

$$0 \longrightarrow \mathrm{H}_!^k(\mathscr{M}_G) \longrightarrow \mathrm{H}^k(\mathscr{M}_G) \longrightarrow \mathrm{H}^k(i^* j_* \mathbb{C}), \tag{1.1}$$

here the interior cohomology $\mathrm{H}_!^k(\mathscr{M}_G)$ is, by definition, the image of $\mathrm{H}_c^i(\mathscr{M}_G) := \mathrm{colim}_\Gamma \mathrm{H}_c^i(M_\Gamma)$ in $\mathrm{H}^k(\mathscr{M}_G)$ and the third term is the boundary cohomology. This sequence is functorial for the inclusions $H \subset G$ considered in Theorems 1.1 and 1.2 because $M_{H,\Gamma_H} \to M_\Gamma$ extends to a morphism $M_{H,\Gamma_H}^* \to M_\Gamma^*$ of minimal compactifications. The obvious approach is to treat the interior cohomology and the contribution from the boundary separately, and this is what we do.

The Lefschetz property for interior cohomology is the following:

Theorem 1.3 (Theorem 5.1, Corollary 3.2). *The map* Res *is injective on* $\mathrm{H}^k_!(\mathscr{M}_G)$ *for* $k \le d_H/2$ *in cases (1) and (2) and for* $k < d_H/2$ *in case (3).*

The proof of this is different in the various cases. In the real hyperbolic case (1), we adapt the Burger–Sarnak approach of [7, 12, 28] to the noncompact case. This is a more or less straightforward matter of combining the method with well-known results about residual Eisenstein cohomology, but since the literature on this is less than satisfactory, we treat it in some detail. In cases (2) and (3) when there is a complex structure available, we adopt a different approach based on some mixed Hodge theory. (The complex hyperbolic case was treated in [35, 36], but the approach here is slightly different and simpler.) Theorem 1.3 in cases (2) and (3) is then a corollary of the following:

Theorem 1.4 (Theorem 3.1). *The map* Res *is injective on* $\mathrm{Gr}^W_k\mathrm{H}^k(\mathscr{M}^*_G)$ *for* $k \le d_H/2$.

This result is deduced as a corollary of a general nonvanishing criterion (Theorem 3.11) for the map on top weight quotients (for the weight filtration)

$$\mathrm{Res}: \mathrm{Gr}^W_i\mathrm{H}^i(\mathscr{M}^*_G) \to I^G_H\mathrm{Gr}^W_i\mathrm{H}^i(\mathscr{M}^*_H)$$

for a morphism between Shimura varieties, given in terms of the compact dual. This generalizes the criterion of [47] in the compact case and has other applications (see Remark 3.14). The spirit of the proof of Theorem 3.11 is that given a functorial cohomology group, some Poincaré duality, and semisimplicity, the averaging argument of [47] can be used to show that a linear combination of $G(\mathbb{Q})$-translates of the cycle class of the subvariety gives the class of the compact dual of H. The necessary ingredients are available thanks to some results in mixed Hodge theory (consequences of the weights and purity package of [6, 42], reviewed in §3.1) and the theory of Chern classes of automorphic vector bundles (results from [25, 32], reviewed in §3.2 and Appendix C). We remark that the purely automorphic (i.e., Burger–Sarnak) method cannot be made to work easily for interior cohomology in case (3) (see Remark 5.4 for details).

Having treated the interior cohomology, we deal with the cohomology at infinity. In cases (1) and (2) this is straightforward: Given sequence (1.1) and the identification of the boundary cohomology in terms of Lie-algebra cohomology, it reduces to an elementary computation with Kostant's theorem (as was already done in case (2) in [36]). The argument in case (3) of orthogonal Shimura varieties is more delicate: The boundary of M^*_Γ is more complicated, containing modular curves as well as cusps, and it is no longer true that the restriction is injective on the entire boundary cohomology. Instead, the argument is in two steps. First, using an elementary argument using Kostant's theorem as in the rank 1 cases, one extends injectivity from the interior cohomology $\mathrm{H}^i_!(\mathscr{M}_G)$ to an intermediate subspace

$$\mathrm{H}^i_!(\mathscr{M}_G) \subset \mathrm{Gr}^W_i\mathbb{H}^i_c\left(\mathscr{M}^1_G, j^1_*\mathbb{C}\right) \subset \mathrm{H}^i(\mathscr{M}_G),$$

which takes into account some contributions from the one-dimensional boundary strata (see §7.1 for the notation). Next, one extends injectivity to all of $\mathrm{H}^i(\mathscr{M}_G)$ by taking into

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account contributions from the cusps. By arguments in which weights, purity, and the description of the restriction to strata of the direct image sheaves in M_Γ^* play a crucial role, this reduces to the Lefschetz property for real hyperbolic manifolds for the pair $SO(1,m-1) \subset SO(1,n-1)$ appearing in the Levi subgroups of $SO(2,m)$ and $SO(2,n)$ – that is, to the result from case (1), although with nontrivial coefficients. This completes the sketch of the proof of Theorem 1.1.

1.4. Further remarks

Nonvanishing results for cup products in cohomology, which amount to injectivity of Res for the diagonal embedding $G \subset G \times G$, are known in the compact cases [12, 47] and for noncompact complex hyperbolic cases [36]. The nonvanishing of cup products in $\mathrm{H}_!^*(\mathscr{M}_G)$ follows from the criterion of Theorem 3.11. The extension to $\mathrm{H}^*(\mathscr{M}_G)$ should be possible using the methods used here.

Injectivity in degree $i = d_H/2$ in the complex hyperbolic case and the case of orthogonal Shimura varieties remain to be resolved. The two cases are slightly different, since in the first we have injectivity on $\mathrm{H}_!^{d_H/2}(\mathscr{M}_G)$ but not on the boundary cohomology, while in the second case we do not know injectivity on $\mathrm{H}_!^{d_H/2}(\mathscr{M}_G)$. In both cases, the classes potentially in the kernel of Res are constructed by residues of Eisenstein series, and their existence is caused by the nonvanishing of an L-value, whereas their survival under Res is also related to an L-value. We would also like to consider cases where $rank_{\mathbb{Q}}(H) < rank_{\mathbb{Q}}(G)$, to which our arguments do not necessarily apply. (Although note that what we have treated is the 'generic' case, at least over $\mathbb{Q}$ – for example, in case (3), if $m+2 \geq 6$, then we are in the situation treated here.) It appears that these questions are most naturally considered in the framework of branching laws, and we will consider them in a sequel.

Finally, the appearance of the Lefschetz property in the real hyperbolic case as a local Lefschetz property at the cusp singularities for the orthogonal Shimura variety suggests trying to reverse the logic and *deduce* the Lefschetz property in the real hyperbolic case from purely geometric facts. It seems likely that this would follow from showing that a linear combination of Hecke translates of the image of $M_{H,\Gamma_H}^* \to M_\Gamma^*$ is ample – that is, resolving the question raised in [28]. Perhaps [15] can be used profitably here.

1.5. Contents

We end the introduction with a brief discussion of the contents of the individual sections.

Section 2 introduces the congruence quotients of interest and their minimal compactifications, recalls some well-known results on their local geometry and cohomology at infinity, and introduces the restriction maps in detail.

Section 3 discusses restriction between (connected) Shimura varieties. We show using some standard mixed Hodge theory that there is a simple criterion for the injectivity of Res on the top weight quotient of $\mathrm{H}^*(\mathscr{M}_G^*)$. We apply it to $SU(1,m) \subset SU(1,n)$ and $SO(2,m) \subset SO(2,n)$ to prove injectivity on the top weight quotient and on interior cohomology in these cases.

Section 4 contains Lie-algebra cohomology computations using Kostant's theorem which are necessary to treat boundary contributions in the various cases. These are explicit elementary calculations with roots and weights.

Section 5 considers the congruence real hyperbolic case and contains the proof of case (1) of Theorem 1.1.

Section 6 considers the congruence complex hyperbolic case and contains the proof of case (2) of Theorem 1.1 and the proof of Theorem 1.2.

Section 7 considers the case of orthogonal Shimura varieties. The results of §§3, 4, and 5 are combined to prove the remaining case (3) of Theorem 1.1.

The three appendices contain some facts which are presumably well known but for which we could not find appropriate references in the literature. Appendix A contains some facts about the L^2 cohomology of arithmetic manifolds used in §§5 and 6. In fact, we only need a very special case of what is proven (Proposition A.1 in the case $SO(1,d)$ for d odd), but the facts recorded here will be useful elsewhere. Appendix B records some well-known facts about the construction of cohomology classes via residual Eisenstein series, for use in §§5 and 6. And Appendix C discusses Chern classes of automorphic vector bundles, which are used in §3.

2. Preliminaries

2.1. Congruence arithmetic quotients

The general setup we work in is as follows. Let G be a semisimple algebraic group over $\mathbb{Q}$, K the maximal compact subgroup of $G(\mathbb{R})$, and $X = G(\mathbb{R})/K$ the symmetric space. For a congruence subgroup $\Gamma \subset G(\mathbb{Q})$, the quotient

$$M_\Gamma = \Gamma \backslash X$$

is noncompact when G is $\mathbb{Q}$-isotropic. The following three cases will be the main ones of interest to us:

(i) $G(\mathbb{R})^{nc} = SO(1,d)$ for $d \geq 2$, so that X is a real hyperbolic d-space and M_Γ is a congruence hyperbolic manifold;

(ii) $G(\mathbb{R})^{nc} = SU(1,n)$ for $n \geq 2$, so that X is the complex unit n-ball and M_Γ is a congruence ball quotient (or congruence complex hyperbolic manifold);

(iii) $G(\mathbb{R})^{nc} = SO(2,n)$ for $n \geq 3$, so that

$$X = SO(2,n)/S(O(2) \times O(n)) = SO_0(2,n)/SO(2) \times SO(n)$$

and M_Γ is a Hermitian locally symmetric space which we will refer to, by an abuse of terminology, as an orthogonal Shimura variety.

In cases (ii) and (iii) the symmetric space X has an Hermitian structure, so that M_Γ is a smooth complex manifold if Γ is small enough. We will also be interested in the general case when X has an Hermitian structure; by an abuse of terminology, we will then refer to M_Γ as a Shimura variety.

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Example 2.1. The standard examples of congruence quotients of types (i)–(iii) are given by quadratic or Hermitian spaces over $\mathbb{Q}$ or number fields. For example, if (V,q) is a quadratic space over $\mathbb{Q}$ and $q_{\mathbb{R}}$ has signature $(1,d)$, then $G = SO(q)$ gives an example of (i), whereas if the signature is $(2,n)$, then it gives an example of (iii). If the number of variables is at least ≥ 5 and $q_{\mathbb{R}}$ is indefinite, then $G = SO(q)$ is necessarily $\mathbb{Q}$-isotropic and M_Γ is noncompact. More generally, if $n+2$ is odd and ≥ 5, then the only examples of type (iii) are the obvious ones – that is, they come from quadratic forms. When the number of variables is even, there are more complicated examples – for example, for $d+1 = 8$ there are triality forms of type (i).

Let $G(\mathbb{R})^c$ be the compact real form of $G(\mathbb{R})$. The compact symmetric space dual to X is

$$X^c = G(\mathbb{R})^c/K.$$

In our three cases, X^c is (i) the d-sphere, (ii) the complex projective n-space $\mathbb{P}^n$, and (iii) a quadric in $\mathbb{P}^{n+1}$. In all three cases there is a natural embedding $X \subset X^c$ and the action of $G(\mathbb{R})$ extends to the closure of X in X^c: In cases (ii) and (iii) it is the familiar $G(\mathbb{R})$-equivariant Borel embedding of X in the flag variety, and in case (i) it is clear, for example, from the upper half-space model of hyperbolic space.

2.2. Minimal compactification

In all three of our cases, there is a canonical open immersion

$$j : M_\Gamma \hookrightarrow M_\Gamma^*$$

into a compact space, which we will call the minimal compactification. For cases (i) and (ii) it is the obvious cusp compactification and also coincides with the reductive Borel–Serre compactification. For cases (ii) and (iii) and more generally, for any arithmetic quotient of an Hermitian symmetric domain, it is the Satake–Baily–Borel compactification of M_Γ as a projective variety, and we will describe it in some more detail in this generality.

The closure of X in X^c decomposes as a disjoint union of boundary components, which are (by definition) the maximal connected complex submanifolds of the closure. The stabilizer of a proper (i.e., $\neq X$) boundary component is a product of maximal parabolic subgroups of the simple factors of $G(\mathbb{R})$, and the boundary component is called rational if the stabilizer is defined over $\mathbb{Q}$, in which case it is a maximal $\mathbb{Q}$-parabolic of G. As a topological space, $M_\Gamma^* = \Gamma \backslash X^*$, where

$$X^* = \bigsqcup_{F \text{rational}} F \quad \subset X^c \tag{2.1}$$

is the union of all rational boundary components of X, equipped with the Satake topology. The action of $G(\mathbb{Q})$ on X extends to a continuous action on X^*; the stabilizer of a rational boundary component F is a maximal $\mathbb{Q}$-parabolic subgroup (in which case F is proper – i.e., $F \subset X^* - X$) or G itself (the case $F = X$). The Baily–Borel theory [5] puts an analytic structure on M_Γ^* inducing the given holomorphic structure on each stratum, and this structure is unique. Moreover, M_Γ^* has a unique structure of projective algebraic

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variety compatible with this analytic structure, and this gives a canonical quasi-projective structure on M_Γ. The decomposition (2.1) induces an algebraic stratification of M_Γ^*.

Example 2.2. If G is isotropic and $G(\mathbb{R})^{nc}$ is isogenous to $SU(1,n)$, the boundary components are points and M_Γ^* is the cusp compactification of the complex hyperbolic manifold.

Example 2.3. If $G(\mathbb{R})^{nc}$ is isogenous to $SO(2,n)$, the boundary components have complex dimension 1 (i.e., they are upper half-planes) or 0 (points). The natural filtration of X^* induces a filtration by Zariski open subsets

$$M_\Gamma \subset M_\Gamma^1 \subset M_\Gamma^*,$$

with $Z_\Gamma^1 = M_\Gamma^1 - M_\Gamma$ a disjoint union of curves and $Z_\Gamma^0 = M_\Gamma^* - M_\Gamma^1$ a finite set of cusps. (Either or both of Z_Γ^0 and Z_Γ^1 may be empty.)

The local geometry of the stratification of M_Γ^* is closely tied to the structure of parabolic subgroups, as we now review (see, e.g., [3, Sections III.4.1–III.4.2], [5, Section 3], [31, Section 6.1], or [25, Sections 7.1–7.3]). We will assume that the adjoint group G^{ad} is $\mathbb{Q}$-simple. Let P be a maximal rational parabolic subgroup. The unipotent radical W is an extension $1 \to U \to W \to V \to 1$, where U is the center of W and V is abelian. For the Lie algebras $\mathfrak{w} = \mathrm{Lie}W(\mathbb{R}), \mathfrak{u} = \mathrm{Lie}U(\mathbb{R}), \mathfrak{v} = \mathrm{Lie}V(\mathbb{R})$, we have an extension $0 \to \mathfrak{u} \to \mathfrak{w} \to \mathfrak{v} \to 0$. The action of A on $\mathfrak{u} = \mathrm{Lie}U(\mathbb{R})$ is by the square of the positive (with respect to P) generator χ of $X^*(A)$, and the action on $\mathfrak{v} = \mathrm{Lie}V(\mathbb{R})$ is by χ (if $\mathfrak{v} \neq 0$). The Levi quotient $M = P/W$ has a decomposition $M = M_\ell M_h A$, where $A \cong \mathbb{G}_m$ is the maximal $\mathbb{Q}$-split central torus in M, M_ℓ and M_h commute, (any lift of) M_h centralizes U, M_h contains no nontrivial connected $\mathbb{Q}$-anisotropic subgroup, and $M_h(\mathbb{R})$ gives an Hermitian symmetric space, which is the rational boundary component corresponding to P. (The relation with the 'five-factor decomposition' of [3, §4.1] is the following: If P is the stabilizer of F and $P = G_h(F)G_\ell(F)M(F)V(F)U(F)$ as in [3, §4.1], then $W = W(F), U = U(F), V \cong V(F), M_\ell \cong G_\ell(F)M(F)$, and $M_h A \cong G_h(F)$.) Note that if G is simply connected, the same is true of the derived group of the Levi, so $M^{der} = M_\ell^{der} \times M_h$, and hence M_h is also simply connected.

Example 2.4. Let $G = SO(q)$ for a quadratic form over $\mathbb{Q}$ of signature $(2,n)$. Assume that G has $\mathbb{Q}$-rank 2 (this is automatic for $n \geq 6$). The maximal proper $\mathbb{Q}$-parabolics of G are the stabilizers of isotropic subspaces in V, which are of dimension 1 or 2. We have the following:

1. If P is the stabilizer of an isotropic plane $I \subset V$, the unipotent radical is a nontrivial extension $1 \to \mathbb{G}_a \to W \to \mathbb{G}_a^{2(n-2)} \to 1$ and the Levi M is $GL(2) \times SO\left(I^\perp/I\right)$. Here $M_h A = GL(2)$ and $M_\ell = SO\left(I^\perp/I\right) \cong SO(n-2)$ is anisotropic over $\mathbb{R}$.
2. If P is the stabilizer of an isotropic line $I \subset V$, the unipotent radical is abelian $W \cong \mathbb{G}_a^{n-2}$ and the Levi is $M \cong \mathbb{G}_m \times SO\left(I^\perp/I\right)$. Here M_h is trivial, $M_\ell \cong SO\left(I^\perp/I\right)$ has $\mathbb{Q}$-rank 1 and $M_\ell(\mathbb{R}) = SO(1,n-1)$.

The corresponding strata of M_Γ^* are modular curves in case (1) and cusps in case (2).

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Example 2.5. It can happen that $Z_\Gamma^0 = \emptyset$ –for example, for the $\mathbb{Q}$-rank 1 inner form of $\mathrm{Sp}(4,\mathbb{R}) = Spin(2,3)$ associated with an indefinite quaternion algebra D over $\mathbb{Q}$ and a rank 2 Hermitian space over D with respect to the involution of D extending the nontrivial Galois action on the maximal subfield of D (which is real quadratic). (This is the form denoted $C_{2,1}^{(2)}$ in [46, p. 57].) In this case the stabilizer of a cusp is the inner form $D^\times$ of $GL(2)$ and the boundary of M_Γ^* is a disjoint union of Shimura curves.

Example 2.6. An example with $Z_\Gamma^1 = \emptyset$ is that of Hilbert modular surfaces, which are forms of $Spin(2,2) \cong SL(2) \times SL(2)$ with $\mathbb{Q}$-rank 1. The boundary of M_Γ^* consists of cusps.

2.3. Direct limits

Let G be a semisimple $\mathbb{Q}$-algebraic group, $X = G(\mathbb{R})/K$ the symmetric space, and $M_\Gamma = \Gamma\backslash X$ for congruence Γ. For $\Gamma' \subset \Gamma$ the covering map $M_{\Gamma'} \to M_\Gamma$ gives pullback maps in cohomology and compactly supported cohomology, so taking colimits over all congruence subgroups, we define

$$\begin{aligned} \mathrm{H}^i(\mathscr{M}_G) &:= \mathrm{colim}_\Gamma \mathrm{H}^i(M_\Gamma), \\ \mathrm{H}_c^i(\mathscr{M}_G) &:= \mathrm{colim}_\Gamma \mathrm{H}_c^i(M_\Gamma), \\ \mathrm{H}_!^i(\mathscr{M}_G) &:= \mathrm{colim}_\Gamma \mathrm{H}_!^i(M_\Gamma), \end{aligned} \tag{2.2}$$

where, as usual, $\mathrm{H}_!^i(M_\Gamma) = \mathrm{im}\left(\mathrm{H}_c^i(M_\Gamma) \to \mathrm{H}^i(M_\Gamma)\right)$ is the interior cohomology. All these are smooth $G(\mathbb{Q})$-modules, in the sense that the stabilizer of a vector is a congruence subgroup. The action of $g \in G(\mathbb{Q})$ on $\mathrm{H}^i(M_\Gamma) \subset \mathrm{H}^i(\mathscr{M}_G)$ is given by the pullback $\mathrm{H}^i(M_\Gamma) \to \mathrm{H}^i\left(M_{g\Gamma g^{-1}}\right)$ by the isomorphism $M_{g\Gamma g^{-1}} \to M_\Gamma$ induced by left translation by g^{-1} on the universal cover X. The transition maps in the colimits are injective, and $\mathrm{H}^*(M_\Gamma)$ can be recovered as the Γ-invariants in $\mathrm{H}^*(\mathscr{M}_G)$. The same remarks apply to $\mathrm{H}_c^*(\mathscr{M}_G)$ and $\mathrm{H}_!^*(\mathscr{M}_G)$.

When the symmetric space X is Hermitian or $G(\mathbb{R})^{nc}$ is isogenous to $SO(1,d)$, we also have the minimal compactification M_Γ^* as in §2.2, and we can define

$$\mathrm{H}^i(\mathscr{M}_G^*) := \mathrm{colim}_\Gamma \mathrm{H}^i\left(M_\Gamma^*\right). \tag{2.3}$$

This is a smooth $G(\mathbb{Q})$-module, and in the Hermitian case it carries a mixed (ind-)Hodge structure. In particular, it has a weight filtration with weights $\leq i$ in degree i, and the graded pieces are

$$\mathrm{Gr}_j^W \mathrm{H}^i\left(\mathscr{M}_G^*\right) = \mathrm{colim}_\Gamma \mathrm{Gr}_j^W \mathrm{H}^i\left(M_\Gamma^*\right)$$

by strictness of the weight filtration.

The inductive setup requires the use of nontrivial coefficients (at the boundary) to treat the case of trivial coefficients. A finite-dimensional algebraic representation E of $G(\mathbb{C})$ gives a local system on M_Γ which, for simplicity, we continue to denote E, and we can consider $\mathrm{H}^*(M_\Gamma, E)$, the colimit

$$\mathrm{H}^*(\mathscr{M}_G, E) = \mathrm{colim}_\Gamma \mathrm{H}^*(M_\Gamma, E),$$

and similarly $\mathrm{H}_c^*(\mathscr{M}_G,E)$ and $\mathrm{H}_!^*(\mathscr{M}_G,E)$. For minimal compactifications $j_\Gamma : M_\Gamma \hookrightarrow M_\Gamma^*$, we take the sheaf $\mathrm{H}^0(j_{\Gamma*}E)$ – this is the ordinary, underived push-forward – and define

$$\mathrm{H}^*(\mathscr{M}_G^*,E) := \mathrm{colim}_\Gamma \mathrm{H}^*\left(M_\Gamma^*, \mathrm{H}^0(j_{\Gamma*}E)\right).$$

2.4. Restriction maps

Now suppose that $H \subset G$ is an injective homomorphism of semisimple $\mathbb{Q}$-groups. Choosing (as we may) a maximal compact K in $G(\mathbb{R})$ such that $K_H = K \cap G(\mathbb{R})$ is maximal compact, and letting $\Gamma_H = \Gamma \cap H(\mathbb{Q})$ and $M_{H,\Gamma_H} = \Gamma_H \backslash H(\mathbb{R})/K_H$, we get a map $\iota : M_{H,\Gamma_H} \longrightarrow M_\Gamma$ which is well known to be proper. Thus there are induced pullback maps $\mathrm{H}^i(M_\Gamma) \to \mathrm{H}^i(M_{H,\Gamma_H})$ and $\mathrm{H}_c^i(M_\Gamma) \to \mathrm{H}_c^i(M_{H,\Gamma_H})$ (the latter because $M_{H,\Gamma_H} \to M_\Gamma$ is proper). These are compatible under the natural maps $\mathrm{H}_c^*(\cdot) \to \mathrm{H}^*(\cdot)$, forgetting supports, and hence induce $\mathrm{H}_!^i(M_\Gamma) \to \mathrm{H}_!^i(M_{H,\Gamma_H})$. In the limit over Γ we have $H(\mathbb{Q})$-equivariant maps

$$\iota^* : \mathrm{H}_{(c)}^i(\mathscr{M}_G) \to \mathrm{H}_{(c)}^i(\mathscr{M}_H)$$

in cohomology and compactly supported cohomology. There are induced homomorphisms of smooth $G(\mathbb{Q})$-modules

$$\mathrm{Res} : \mathrm{H}_{(c)}^i(\mathscr{M}_G) \longrightarrow I_H^G \, \mathrm{H}_{(c)}^i(\mathscr{M}_H),$$

where I_H^G is an induction functor such that for a smooth $H(\mathbb{Q})$-module U, $I_H^G U$ consists of functions $f : G(\mathbb{Q}) \to U$ such that $f(gh) = h^{-1} \cdot f(g)$ and f is left-invariant by a congruence subgroup of $G(\mathbb{Q})$, and the action of $g \in G(\mathbb{Q})$ is by $(g \cdot f)(x) = f\left(g^{-1}x\right)$. Then I_H^G is exact, takes smooth modules to smooth modules, and is right adjoint to restriction (these facts are completely elementary, see [35, Section 3.1]). Explicitly, Res is given by

$$\mathrm{Res}(\alpha)(g) = \iota^*\left(g^{-1} \cdot \alpha\right) = \iota^*((g\cdot)^*\alpha).$$

Note that Res restricts to a map $\mathrm{Res} : \mathrm{H}_!^i(\mathscr{M}_G) \to I_H^G \, \mathrm{H}_!^i(\mathscr{M}_H)$ on interior cohomology.

When both H and G give Hermitian symmetric spaces and K is chosen (as it may be) so that the map $H(\mathbb{R})/K_H \to G(\mathbb{R})/K$ is holomorphic, the map $M_{H,\Gamma_H} \to M_\Gamma$ extends to a morphism $M_{H,\Gamma_H}^* \to M_\Gamma^*$ of varieties of minimal compactifications. This well-known general fact (compare [44] or [27, Section 3.3]) is easily seen in our primary cases of interest using the description of M_Γ^* given in §2.2. Pullback induces an $H(\mathbb{Q})$-equivariant map $\mathrm{H}^i(\mathscr{M}_G^*) \to \mathrm{H}^i(\mathscr{M}_H^*)$, which gives a homomorphism of mixed Hodge structures

$$\mathrm{Res} : \mathrm{H}^i(\mathscr{M}_G^*) \longrightarrow I_H^G \mathrm{H}^i(\mathscr{M}_H^*)$$

by adjunction. There is a similar mapping in the real hyperbolic cases where $H(\mathbb{R})^{nc} \subset G(\mathbb{R})^{nc}$ is $SO(1,c) \subset SO(1,d)$ (up to isogeny) and in the ‘mixed’ case $SO(1,n) \subset SU(1,n)$, coming from the obvious extension of $M_{H,\Gamma_H} \to M_\Gamma$ to minimal compactifications.

Now assume that $H \subset G$ is such that the restriction of finite-dimensional representations from G to H is multiplicity-free. The situations we will treat are well known to be of this type, by classical branching laws (e.g. [22, Section 8.1.1]). Choose Borel subgroups $B_H \subset B$ and maximal tori $T_H \subset B_H$ and $T \subset B$ of $H(\mathbb{C})$ and $G(\mathbb{C})$, and for E with B-highest weight $\lambda \in X^*(T)$ let E_H be the unique summand of $E|_{H(\mathbb{C})}$ with highest weight

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$\lambda|_{T_H}$. The composition $\mathrm{H}^*(\mathscr{M}_G,E) \to \mathrm{H}^*(\mathscr{M}_H,E|_H) \to \mathrm{H}^*(\mathscr{M}_H,E_H)$ is $H(\mathbb{Q})$-invariant and induces a restriction map

$$\mathrm{Res}:\mathrm{H}^*(\mathscr{M}_G,E) \longrightarrow \mathrm{H}^*(\mathscr{M}_H,E_H)$$

with coefficients. There are similar maps for $\mathrm{H}^*_c(\mathscr{M}_G,E)$, $\mathrm{H}^*_!(\mathscr{M}_G,E)$, and $\mathrm{H}^*(\mathscr{M}^*_G,E)$.

The use of Res *for several different maps should cause no confusion, as we will always specify the domain when discussing injectivity results. We will also frequently write* $\mathrm{H}^i(\mathscr{M}),\mathrm{H}^i(\mathscr{M}^*)$, *and so on, for* $\mathrm{H}^i(\mathscr{M}_G),\mathrm{H}^i(\mathscr{M}^*_G)$, *and so on – dropping the subscript* G *when it is clear from context.*

2.5. Higher direct images in the minimal compactification

We will use a well-known description of the restriction of $j_*\mathbb{Q}_{M_\Gamma}$ to a stratum of M^*_Γ in the case of Shimura varieties. (Here by j_* we mean the push-forward on the level of derived categories – that is, Rj_* in old-fashioned notation.)

To fix notation, let $i_S: S \hookrightarrow M^*_\Gamma$ be a stratum of the minimal compactification. Choose a rational boundary component $F \twoheadrightarrow S$ and let $P = MW$ be the stabilizer of F and $M = M_h M_\ell A$ as in §2.2. For the congruence subgroup Γ, let $\Gamma_W = \Gamma \cap W(\mathbb{Q})$, $\Gamma_P = \Gamma \cap P(\mathbb{Q})$, $\Gamma_M = \Gamma_P/\Gamma_W$, $\Gamma_{M_\ell} = \Gamma_M \cap M_\ell(\mathbb{Q})$, and $\Gamma_{M_h} = \Gamma_M/\Gamma_{M_\ell}$. These are all neat arithmetic subgroups when Γ is neat.

Proposition 2.7. *For a stratum* $i_S: S \hookrightarrow M^*_\Gamma$ *we have the following:*

(1) *There is a natural isomorphism in the derived category*

$$i^*_S j_*\mathbb{Q}_{M_\Gamma} = \bigoplus_k \mathrm{H}^k\left(i^*_S j_*\mathbb{Q}_{M_\Gamma}\right)[-k] \tag{2.4}$$

of sheaves on $S = \Gamma_{M_h}\backslash F$. *The object* $\mathrm{H}^k\left(i^*_S j_*\mathbb{Q}_{M_\Gamma}\right)$ *is the local system on* S *associated with the representation of* M_h *on*

$$\mathrm{H}^k\left(i^*_S j_*\mathbb{Q}_{M_\Gamma}\right)_s \cong \bigoplus_{r+s=k} \mathrm{H}^r\left(\Gamma_{M_\ell},\mathrm{H}^s(\mathfrak{w},\mathbb{Q})\right) \tag{2.5}$$

for $s \in S$.

(2) *The weight filtration on* $\mathrm{H}^k\left(i^*_S j_*\mathbb{Q}_{M_\Gamma}\right)_s$ *is split by the action of* A *on* $\mathrm{H}^*(\mathfrak{w},\mathbb{Q})$ *– that is,* $\mathrm{Gr}^W_i \mathrm{H}^k\left(i^*_S j_*\mathbb{Q}_{M_\Gamma}\right)_s$ *is identified with the subspace on which* A *acts by* χ^{-i} *(where* $\chi \in X^*(A)$ *is such that* A *acts on the center* $\mathfrak{u}$ *of* $\mathfrak{w}$ *by* χ^2 *and on* $\mathfrak{v} = \mathfrak{w}/\mathfrak{u}$ *by* χ*; compare §2.2).*

The description (2.5) of cohomology sheaves can be found, for example, in [31, Proposition 5.6] or [23, Corollary 22.8]. The weight filtration on $\mathrm{H}^k\left(i^*_S j_*\mathbb{Q}_{M_\Gamma}\right)_s$ comes from the theory of mixed Hodge modules, and the assertion in (2) is due to Looijenga and Rapoport [31, Proposition 5.6]. (The analogue in the l-adic setting is in [39].) The existence of the decomposition (2.4) in the derived category can in fact be deduced from this, but instead one can use [19, Theorem 2.9], which proves the direct sum decomposition (2.4) in the derived category of mixed Hodge modules and the identification of the graded in the context of Shimura varieties. The isomorphisms (2.4) and (2.5) are equivariant for the actions of $(M_\ell AW)(\mathbb{Q})$ on both sides (which factor through $(M_\ell A)(\mathbb{Q})$).

Remark 2.8. The real hyperbolic case $SO(1,d)$ fits notationally into this setup by taking $M_h = \{e\}, M_\ell = SO(d-1)$, and then (1) remains true.

3. Restriction between Shimura varieties

In this section we assume that H and G both give rise to Shimura varieties, and consider the restriction $\mathrm{Res} : \mathrm{Gr}_i^W \mathrm{H}^i(\mathscr{M}_G^*) \to I_H^G \mathrm{Gr}_i^W \mathrm{H}^i(\mathscr{M}_H^*)$ on the top weight quotient for the weight filtration. We prove a criterion (Theorem 3.11) for the nonvanishing of this restriction involving the compact dual symmetric space, which is the analogue of the criterion of [47] in this situation. It implies the following:

Theorem 3.1. *If $H(\mathbb{R})^{nc} \subset G(\mathbb{R})^{nc}$ is $SU(1,m) \subset SU(1,n)$ or $SO(2,m) \subset SO(2,n)$, then* Res *is injective on* $\mathrm{Gr}_i^W \mathrm{H}^i(\mathscr{M}_G^*)$ *in degrees* $\leq m$.

The unitary case of this is contained in [35, Theorem 3.17], although the proof here is slightly different (and more direct). As a corollary, we get the following injectivity statements for interior cohomology, the first of which was proved earlier in [35] (see also [13]):

Corollary 3.2. *If $H(\mathbb{R})^{nc} \subset G(\mathbb{R})^{nc}$ is $SU(1,m) \subset SU(1,n)$, then* Res *is injective on* $\mathrm{H}_!^i(\mathscr{M}_G)$ *in degrees* $\leq m$.

If $H(\mathbb{R})^{nc} \subset G(\mathbb{R})^{nc}$ is $SO(2,m) \subset SO(2,n)$, then Res *is injective on* $\mathrm{H}_!^i(\mathscr{M}_G)$ *in degrees* $\leq m$ *if* $rk_{\mathbb{Q}}(H) \leq 1$ *and in degrees* $\leq m-1$ *if* $rk_{\mathbb{Q}}(H) = 2$.

3.1. Some cohomological facts

We will use some facts about the cohomology of (possibly) singular varieties, summarized in Proposition 3.3 and Lemma 3.5.

Recall that by [20] the rational cohomology $\mathrm{H}^*(X) = \mathrm{H}^*(X,\mathbb{Q})$ and homology $\mathrm{H}_*(X) = \mathrm{H}_*(X,\mathbb{Q})$ of a complex algebraic variety X carry rational mixed Hodge structures; in particular, they have weight filtrations. The theory of mixed Hodge modules ([42], especially §4) gives a relative version of mixed Hodge structures and allows for sheaf-theoretic arguments, mirroring the situation in l-adic cohomology over finite fields [6]. Let X be an irreducible complex variety of dimension d. Let $\mathbb{Q}_X^H$ be the canonical lift of $\mathbb{Q}_X$ to an object in the derived category of mixed Hodge modules on X – that is, $\mathbb{Q}_X^H = a_X^* \mathbb{Q}^H$, where $a_X : X \to Spec(\mathbb{C})$ and $\mathbb{Q}^H$ is the trivial Hodge structure. The rational cohomology, compactly supported cohomology, homology, and Borel–Moore homology groups of X acquire mixed Hodge structures via

$$\mathrm{H}^i(X) = \mathbb{H}^i\left(X,\mathbb{Q}_X^H\right), \qquad \mathrm{H}_c^i(X) = \mathbb{H}_c^i\left(X,\mathbb{Q}_X^H\right), \tag{3.1}$$

$$\mathrm{H}_i(X) = \mathbb{H}_c^{-i}\left(X,\mathbb{D}\mathbb{Q}_X^H\right), \qquad \mathrm{H}_i^{BM}(X) = \mathbb{H}^{-i}\left(X,\mathbb{D}\mathbb{Q}_X^H\right), \tag{3.2}$$

where $\mathbb{D}$ is the Verdier duality functor, normalized so that $\mathbb{D}\mathbb{Q}_X^H = \mathbb{Q}_X^H[2d](d)$ if X is smooth. The weights are determined by the fact that $\mathbb{Q}_X^H$ has weights ≤ 0, so, for example,

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$\mathrm{H}_c^i(X)$ has weights $\le i$ and $\mathrm{H}_i^{BM}(X)$ has weights $\ge -i$. When X is proper, which is the main case of interest to us, we have $\mathrm{H}^i(X) = \mathrm{H}_c^i(X)$ and $\mathrm{H}_i(X) = \mathrm{H}_i^{BM}(X)$, so that $\mathrm{H}^i(X)$ has weights $\le i$ and $\mathrm{H}_i(X)$ has weights $\ge -i$.

We will also use the intersection complex $IC_X^H = \left(j_{!*}\mathbb{Q}_U^H[d]\right)[-d]$, where $j : U \hookrightarrow X$ is the inclusion of an open dense smooth subset; this lifts the topological intersection complex $IC_X = (j_{!*}\mathbb{Q}_U[d])[-d]$ of X, and it is pure of weight 0. (Our notation is slightly different from [42], where $IC_X\left(\mathbb{Q}^H\right)$ is used for $j_{!*}\mathbb{Q}_U[d]$.) Replacing $\mathbb{Q}_X^H$ by IC_X^H in equation (3.2) defines rational mixed Hodge structures

$$\mathrm{IH}^i(X), \quad \mathrm{IH}_c^i(X), \quad \mathrm{IH}_i(X), \quad \mathrm{IH}_i^{BM}(X)$$

on the intersection cohomology, intersection cohomology with compact support, intersection homology, and Borel–Moore intersection homology, respectively. When X is proper, these are all pure, $\mathrm{IH}^*(X) = \mathrm{IH}_c^*(X)$ and $\mathrm{IH}_*(X) = \mathrm{IH}_*^{BM}(X)$, and the isomorphism $\mathbb{D}IC_X^H = IC_X^H[2d](d)$ extending $\mathbb{D}\mathbb{Q}_U^H = \mathbb{Q}_U^H[2d](d)$ on any smooth open subset $U \subset X$ induces duality isomorphisms $\mathrm{IH}^i(X) \cong \mathrm{IH}_{2d-i}(X)(-d)$ for all i.

Proposition 3.3. *If $f : X \to Y$ is a morphism of varieties, there are maps*

$$\begin{aligned} f^* &: \mathrm{Gr}_i^W \mathrm{H}^i(Y) \to \mathrm{Gr}_i^W \mathrm{H}^i(X), \\ f_* &: W_{-i}\mathrm{H}_i(X) \to W_{-i}\mathrm{H}_i(Y) \end{aligned} \tag{3.3}$$

for each i satisfying

$$f_*(f^*(\alpha) \cap \beta) = \alpha \cap f_*(\beta) \qquad \text{for } \alpha \in \mathrm{Gr}_i^W \mathrm{H}^i(Y), \beta \in W_{-j}\mathrm{H}_j(X). \tag{3.4}$$

If X is an irreducible proper variety of dimension d, then the following are true:

(1) *$\mathrm{H}^i(X)$ has weights $\le i$, $\mathrm{H}_i(X)$ has weights $\ge -i$, and the extreme weights are given by*

$$\begin{aligned} \mathrm{Gr}_i^W \mathrm{H}^i(X) &= \mathrm{im}\left(\mathrm{H}^i(X) \to \mathrm{IH}^i(X)\right), \\ W_{-j}\mathrm{H}_j(X) &= \mathrm{im}\left(\mathrm{IH}_j(X) \to \mathrm{H}_j(X)\right) \end{aligned} \tag{3.5}$$

for all i,j.

(2) *If $[X] \in \mathrm{H}_{2d}(X)(-d)$ is the fundamental class of X, then*

$$\cap[X] : \mathrm{Gr}_i^W \mathrm{H}^i(X) \longrightarrow W_{-(2d-i)}\mathrm{H}_{2d-i}(X)(-d) \tag{3.6}$$

is an isomorphism for all i.

(3) *If $i : Z \hookrightarrow X$ is an irreducible closed subvariety of codimension c, then the cycle class*

$$\xi_{X,Z} := (\cap[X])^{-1}(i_*[Z]) \in \mathrm{Gr}_{2c}^W \mathrm{H}^{2c}(X)(c)$$

has the property that if $\alpha \in \mathrm{H}^{2\dim Z}(X)$ with $i^*(\alpha) = \xi_{Z,pt}$, then $\alpha \cdot \xi_{X,Z} = \xi_{X,pt}$.

Proof. The statements about f^* and f_* are simply the functoriality of the weight filtration and the fact that when homology is considered as a module over the cohomology ring using the cap product, push-forward in homology is a module over pullback in

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cohomology. Statements (1) and (2) are contained in [42, §4.5], but for the reader's convenience we outline the arguments.

For the standard cohomology functor H^0 on mixed Hodge modules (which corresponds to the perverse cohomology functor ${}^p\mathrm{H}^0$ on sheaves), we have dual natural isomorphisms

$$\begin{aligned} \mathrm{Gr}_d^W \mathrm{H}^d\left(\mathbb{Q}_X^H\right) &= IC_X^H[d], \\ W_{-d}\mathrm{H}^{-d}\left(\mathbb{D}\mathbb{Q}_X^H\right) &= IC_X^H[d](d) \end{aligned} \tag{3.7}$$

(see [42, §4.5] for details). If X is proper, then these dual statements and the hypercohomology spectral sequence imply that

$$\begin{aligned} \mathrm{Gr}_i^W \mathrm{H}^i(X) &= \mathrm{im}\left(\mathrm{H}^i(X) \to \mathrm{IH}^i(X)\right), \\ W_{-j}\mathrm{H}_j(X) &= \mathrm{im}\left(\mathrm{IH}_j(X) \to \mathrm{H}_j(X)\right), \end{aligned} \tag{3.8}$$

as claimed in (1).

An irreducible variety X has a fundamental class in Borel-Moore homology,

$$[X] \in \mathrm{H}_{2d}^{BM}(X)(-d) = \mathbb{H}^0\left(X, \left(\mathbb{D}\mathbb{Q}_X^H\right)[-2d](-d)\right) = \mathrm{Hom}\left(\mathbb{Q}_X^H, \left(\mathbb{D}\mathbb{Q}_X^H\right)[-2d](-d)\right),$$

giving the fundamental class homomorphism $\mathbb{Q}_X^H \to \left(\mathbb{D}\mathbb{Q}_X^H\right)[-2d](-d)$, which is an isomorphism if X is smooth. By the identities (3.7) and standard facts about the t-structure and weights, it factors as

$$\mathbb{Q}_X^H \longrightarrow IC_X^H \longrightarrow \left(\mathbb{D}IC_X^H\right)[-2d](-d) \longrightarrow \left(\mathbb{D}\mathbb{Q}_X^H\right)[-2d](-d), \tag{3.9}$$

where the first arrow is the unique extension of the identity morphism $\mathbb{Q}_U \to \mathbb{Q}_U$ on U and the third arrow is its dual (up to a twist). The second is the Verdier duality isomorphism extending $\mathbb{Q}_U^H = \left(\mathbb{D}\mathbb{Q}_U^H\right)[-2d](-d)$, and induces Poincaré duality isomorphisms

$$\mathrm{IH}_c^i(X) = \mathbb{H}_c^i\left(X, IC_X^H\right) \cong \mathbb{H}_c^{i-2d}\left(X, \mathbb{D}IC_X^H\right)(-d) = \mathrm{IH}_{2d-i}(X)(-d) = \mathrm{IH}^{2d-i}(X)^*(-d)$$

and

$$\mathrm{IH}^i(X) = \mathbb{H}^i\left(X, IC_X^H\right) \cong \mathbb{H}^{i-2d}\left(X, \mathbb{D}IC_X^H\right)(-d) = \mathrm{IH}_{2d-i}^{BM}(X)(-d),$$

and hence a nondegenerate pairing $\mathrm{IH}^i(X) \times \mathrm{IH}_c^{2d-i}(X) \to \mathbb{Q}(-d)$. The fundamental class homomorphism induces the identity isomorphism $\mathrm{Gr}_d^W \mathrm{H}^d\left(\mathbb{Q}_X^H\right) = W_{-d}\mathrm{H}^{-d}\left(\mathbb{D}\mathbb{Q}_X^H\right)(-d)$ from equation (3.7) (as indeed it must, since it is the unique extension of $\mathbb{Q}_U^H[d] = \mathbb{D}\mathbb{Q}_U^H[d](d)$ on any smooth open $U \subset X$).

Now assume that X is proper. Then the duality of the first and third arrows in expression (3.9) implies that the cap product with the fundamental class $[X]$ induces an isomorphism

$$\cap[X] : \mathrm{Gr}_i^W \mathrm{H}^i(X) \xrightarrow{\simeq} W_{2d-i}\mathrm{H}_{2d-i}(X)(-d). \tag{3.10}$$

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This proves (2). For (3), note that if $i^*(\alpha) = \xi_{Z,pt}$, then $[Z] \cap i^*(\alpha) = 1$, so that

$$\begin{aligned} 1 &= i_*(i^*(\alpha) \cap [Z]) \\ &= \alpha \cap i_*[Z] \\ &= [X] \cap \big(\alpha \cdot (\cap [X])^{-1}(i_*[Z])\big) \\ &= [X] \cap \alpha \cdot \xi_{X,Z}, \end{aligned} \tag{3.11}$$

and thus $\alpha \cdot \xi_{X,Z} = (\cap [X])^{-1}(1) = \xi_{X,pt}$. □

Remark 3.4. The assertion in (2) is related to some facts in classical mixed Hodge theory which we will also use. If X is irreducible and proper and $Y \to X$ is a resolution of singularities, then

$$\mathrm{Gr}_i^W \mathrm{H}^i(X) = \mathrm{im}\big(\mathrm{H}^i(X) \to \mathrm{H}^i(Y)\big)$$

for all i by [20, Prop. 8.2.5]. This is equivalent to equation (3.5), because the pullback factors as $\mathrm{H}^i(X) \to \mathrm{IH}^i(X) \hookrightarrow \mathrm{H}^i(Y)$ for any inclusion $\mathrm{IH}^*(X) \subset \mathrm{H}^*(Y)$ coming from the decomposition theorem [6, 42]. Since one also has

$$\mathrm{Gr}_i^W \mathrm{H}_c^i(U) = \mathrm{im}\big(\mathrm{H}_c^i(U) \to \mathrm{IH}^i(X)\big) = \mathrm{im}\big(\mathrm{H}_c^i(U) \to \mathrm{H}^i(Y)\big),$$

where U is smooth dense open over which $Y \to X$ is an isomorphism, one sees that $\mathrm{Gr}_i^W \mathrm{H}_c^i(U) \subset \mathrm{Gr}_i^W \mathrm{H}^i(X)$ for all i.

The following purity lemma will be used later in §7:

Lemma 3.5. *Let X be a normal complex variety of dimension d with $U \subset X^1 \subset X$ a filtration by open subsets such that U is smooth and open dense in X^1, $Z^1 = X^1 - U$ is smooth of dimension 1, and $Z^0 = X - X^1$ is smooth of dimension 0. Let $j : U \hookrightarrow X$ and $i : Z^0 \hookrightarrow X$ be the inclusions. Then $\mathrm{H}^i\left(i^{0*} j_* \mathbb{Q}_U^H\right)$ has weights $\le i$ for $i \le d-2$. If $Z^1 = \emptyset$, then $\mathrm{H}^i\left(i^{0*} j_* \mathbb{Q}_U^H\right)$ has weights $\le i$ for $i \le d-1$.*

Proof. Write $j = j^0 \circ j^1$ for $j^1 : U \hookrightarrow X^1$ and $j^0 : X^1 \hookrightarrow X$. Since $IC_X = \tau_{<d} j_*^0 \tau_{<d-1} j_*^1 \mathbb{C}_U$, it follows easily that $IC_X \to j_* \mathbb{C}_U$ induces an isomorphism on cohomology sheaves in degrees $\le d-2$. The same then holds for $IC_X^H \to j_* \mathbb{Q}_U^H$. On the other hand, by pointwise purity of the intersection complex, $\mathrm{H}^i\left(IC_X^H\right)_x$ has weights $\le i$ in all degrees [6, 42]. This proves the first assertion of the lemma. If $Z^1 = \emptyset$, we have that $IC_X \to j_* \mathbb{C}_U$ induces isomorphisms on cohomology sheaves in degrees $\le d-1$, and purity proves the second assertion. □

3.2. Invariants and the compact dual

We will assume from now on that G is semisimple and $X = G(\mathbb{R})/K$ is an Hermitian symmetric domain. In addition, we assume in this subsection that G is simply connected. We return to the use of cohomology with complex coefficients and ignore Tate twists.

We will consider the $G(\mathbb{Q})$-module $\mathrm{IH}^i(\mathscr{M}^*) := \mathrm{colim}_\Gamma \mathrm{IH}^i(M_\Gamma^*)$, which is smooth and admissible. Note that equation (3.5) gives an inclusion

$$\mathrm{Gr}_*^W \mathrm{H}^*(\mathscr{M}^*) := \bigoplus_i \mathrm{Gr}_i^W \mathrm{H}^i(\mathscr{M}^*) \subset \mathrm{IH}^*(\mathscr{M}^*)$$

of $G(\mathbb{Q})$-modules.

Proposition 3.6. *The $G(\mathbb{Q})$-modules $\mathrm{Gr}^W_* \mathrm{H}^*(\mathscr{M}^*)$ and $\mathrm{IH}^*(\mathscr{M}^*)$ are semisimple, and the summand of invariants is given by*

$$\mathrm{Gr}^W_* \mathrm{H}^*(\mathscr{M}^*)^{G(\mathbb{Q})} = \mathrm{IH}^*(\mathscr{M}^*)^{G(\mathbb{Q})} = \mathrm{H}^*(X^c).$$

The embedding of $\mathrm{H}^*(X^c)$ in $\mathrm{Gr}^W_* \mathrm{H}^*(\mathscr{M}^*)$ is functorial: If $H \subset G$ gives a complex subdomain $X_H \subset X$, then the obvious diagram coming from $X^c_H \subset X^c$ and $M^*_{H,\Gamma_H} \to M^*_\Gamma$ commutes.

Proof. There is a natural isomorphism

$$\mathrm{IH}^*(\mathscr{M}^*) = \mathrm{H}^*\left(\mathfrak{g}, K, L^2_{dis}(G(\mathbb{Q})\backslash G(\mathbb{A}))\right) \tag{3.12}$$

thanks to [30, 43] and [16]. Here L^2_{dis} is the L^2 discrete spectrum; see [35, Proposition 3.8] for a detailed discussion of equation (3.12). This proves the semisimplicity statements. The inclusion of the constants in L^2 functions induces an embedding of $\mathrm{H}^*(X^c) = \mathrm{H}^*(\mathfrak{g}, K, \mathbb{C})$ in $\mathrm{IH}^*(\mathscr{M}^*)$. It follows from equation (3.12) using strong approximation and the density of $G(\mathbb{Q})$ in $G(\mathbb{R})$ (weak approximation) – see, for example, the proof [35, Proposition 3.8], which works verbatim here – that these are all the invariants.

To show that the invariants are actually in $\mathrm{Gr}^W_* \mathrm{H}^*(\mathscr{M}^*)$, we will use Chern classes of automorphic vector bundles [25, 32]. A finite-dimensional representation V of K gives a homogenous bundle $\mathscr{V}^c$ on $X^c = G(\mathbb{R})^c/K$. Restricting by the Borel embedding $X \subset X^c$ (see §2.1) and dividing by Γ gives a bundle $\mathscr{V}_\Gamma$ on $M_\Gamma = \Gamma\backslash X$ for any Γ. The bundle $\mathscr{V}_\Gamma$ does not, in general, extend to a vector bundle on M^*_Γ (although see Example 3.7 for an important exception), but Goresky and Pardon [25] defined classes $c^*_k(\mathscr{V}_\Gamma) \in \mathrm{H}^{2k}(M^*_\Gamma)$ which behave like the Chern classes of a putative extension $\mathscr{V}^*_\Gamma$ to M^*_Γ. The main property is that for $\pi : M^\Sigma_\Gamma \to M^*_\Gamma$ a smooth toroidal desingularization [3], the pullback $\pi^*(c^*_k(\mathscr{V}_\Gamma)) = c_k\left(\mathscr{V}^\Sigma_\Gamma\right)$ is the Chern class of Mumford's canonical extension $\mathscr{V}^\Sigma_\Gamma$ [27, 32]. It is a well-known consequence of Mumford's generalization of Hirzebruch proportionality that the classes $c_k\left(\mathscr{V}^\Sigma_\Gamma\right)$ generate a copy of $\mathrm{H}^*(X^c)$ in $\mathrm{H}^*\left(M^\Sigma_\Gamma\right)$. More precisely, there is an injective homomorphism $\theta : \mathrm{H}^*(X^c) \to \mathrm{H}^*\left(M^\Sigma_\Gamma\right)$ such that $\theta(c_k(\mathscr{V}^c)) = (-1)^k c_k\left(\mathscr{V}^\Sigma_\Gamma\right)$ for all $k, \mathscr{V}$ (see Lemma C.1 for a proof, following [34, Lemma 3.7.2]). They are contained in $\mathrm{Gr}^W_* \mathrm{H}^*(M^*_\Gamma) = \mathrm{im}\left(\mathrm{H}^*(M^*_\Gamma) \to \mathrm{H}^*\left(M^\Sigma_\Gamma\right)\right)$, since $\pi^*(c^*_k(\mathscr{V}_\Gamma)) = c_k\left(\mathscr{V}^\Sigma_\Gamma\right)$ by [25]. Moreover, the compatibility of the construction for different Σ (see, e.g., [27, Section 4.3.1]) shows that we have a well-defined embedding $\theta : \mathrm{H}^*(X^c) \to \mathrm{Gr}^W_* \mathrm{H}^*(M^*_\Gamma)$.

It remains to show that the classes are $G(\mathbb{Q})$-invariant and the embedding is functorial. The direct limit $\mathrm{colim}_{\Sigma,\Gamma} \mathrm{H}^*\left(M^\Sigma_\Gamma\right)$ over all pairs (Σ, Γ) where Σ is admissible for Γ is a $G(\mathbb{Q})$-module, and contains $\mathrm{Gr}^W_* \mathrm{H}^*(\mathscr{M}^*)$ as a $G(\mathbb{Q})$-submodule. Standard properties of the canonical extensions listed in [27, Section 4.3] show that the Chern classes are $G(\mathbb{Q})$-invariants in $\mathrm{colim}_{\Sigma,\Gamma} \mathrm{H}^*\left(M^\Sigma_\Gamma\right)$ and hence in $\mathrm{Gr}^W_* \mathrm{H}^*(\mathscr{M}^*)$. Finally, functoriality follows from [27, Section 4.3.4] and the definition of the map θ given in Lemma C.1. □

Example 3.7. The representation of K on the top exterior power of $\mathfrak{p}$, where $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is the Cartan decomposition given by K, gives a special automorphic bundle called the Baily–Borel bundle. This extends as a line bundle $\mathscr{L}^{bb}$ over M^*_Γ, and some power of $\mathscr{L}^{bb}$ is the $\mathscr{O}(1)$ in the Baily–Borel projective embedding (see [32, Prop. 3.4(b)]). So $\mathscr{L}^{bb}$ is

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ample and the Chern class $c_1^{bb} := c_1^*(\mathscr{L}) = c_1\left(\mathscr{L}^{bb}\right)$ fixes a generator

$$\left(c_1^{bb}\right)^n \in \mathrm{Gr}^W_{2n}\mathrm{H}^{2n}(\mathscr{M}^*) = \mathrm{IH}^{2n}(\mathscr{M}^*) \tag{3.13}$$

in top degree.

Example 3.8. If $G(\mathbb{R})^{nc} = SU(1,n)$, then $X^c = SU(1+n)/S(U(1)\times U(n)) \cong \mathbb{P}^n$. So the invariant part of $\mathrm{Gr}^W_*\mathrm{H}^*(\mathscr{M}^*)$ is $\mathbb{C}\left[c_1^{bb}\right]/\left(\left(c_1^{bb}\right)^{n+1}\right)$. (See, e.g., [35, Section 1.2] for an intrinsic description of $\mathscr{L}^{bb}$.)

Example 3.9. If $G(\mathbb{R})^{nc} = Spin(2,n)$, the compact symmetric space dual to X is

$$X^c = Spin(2+n)/Spin(2)\times_{\{\pm I\}} Spin(n) = SO(2+n)/SO(2)\times SO(n),$$

which is a quadric in $\mathbb{P}^{n+1}$. The complex cohomology ring of quadrics is well known. Let E_2 and E_n be the vector bundles on X^c corresponding to the natural representations of $SO(2)\times SO(n)$ of dimension 2 and n, respectively. When n is odd, the complex cohomology is generated by the Euler class (or first Chern class) $c_1 = c_1(E_2) \in \mathrm{H}^2(X^c)$ – that is, $\mathrm{H}^*(X^c) = \mathbb{C}[c_1]/\left(c_1^{n+1}\right)$. When $n = 2d$ is even, the complex cohomology of X^c is generated as a ring by c_1 and the Euler class $c_d = c_d(E_n) \in \mathrm{H}^{n=2d}(X^c)$, with the relations $c_1^{n+1} = 0, c_d^2 = (-1)^d c_1^{2d}$, and $c_1 c_d = 0$.

For later use, we remark that if $n = 2d \geq 4$ and $X_a^c \subset X^c$ is the inclusion of quadrics coming from $SO(2,a) \subset SO(2,n)$ for $a < n$, then $c_d|_{X_a^c} = 0$. Indeed, $0 = (c_1 c_d)|_{X_{n-1}^c} = c_1|_{X_{n-1}^c}\cdot c_d|_{X_{n-1}^c} = c_1\cdot\left(c_d|_{X_{n-1}^c}\right)$. Since $c_1\cdot$ is injective on $\mathrm{H}^*\left(X_{n-1}^c\right) = \mathbb{C}[c_1]/\left(c_1^n\right)$ in degrees $< 2(n-1)$, we must have $c_d|_{X_{n-1}^c} = 0$ if $d \geq 2$.

3.3. Cycle classes and an injectivity criterion

We will prove a criterion for the nonvanishing of Res between Shimura varieties and apply it to prove Theorem 3.1 and Corollary 3.2. Since we are only interested in cohomology with complex coefficients, we will ignore Tate twists henceforth and write $\mathrm{H}^*(X)$ for $\mathrm{H}^*(X,\mathbb{C})$.

Suppose now that $\iota : M^*_{H,\Gamma_H} \to M^*_\Gamma$ is the extension to minimal compactifications of a morphism of Shimura varieties (compare §2.4), and let $n = \dim M_\Gamma, m = \dim M_{H,\Gamma_H}$. Let

$$\xi_\Gamma := \left(\cap[M^*_\Gamma]\right)^{-1}\left(\iota_*\left[M^*_{H,\Gamma_H}\right]\right) \in \mathrm{Gr}^W_{2(n-m)}\mathrm{H}^{2(n-m)}\left(M^*_\Gamma\right)$$

be the cycle class defined earlier in Proposition 3.3, ignoring Tate twists and simplifying the notation $\left(\text{in the notation of that proposition, this would be } \xi_{M^*_\Gamma, M^*_{H,\Gamma_H}}\right)$. It is easily checked that if $\Gamma' \subset \Gamma$ is normal, then $\xi_\Gamma = |\Gamma/\Gamma'|^{-1}\sum_{\gamma\in\Gamma/\Gamma'}\gamma\cdot\xi_{\Gamma'}$, where $\xi_{\Gamma'} = \left(\cap[M^*_\Gamma]\right)^{-1}\left(\iota'_*\left(\left[M^*_{H,\Gamma'_H}\right]\right)\right)$ for $\iota' : M^*_{H,\Gamma'_H} \to M^*_{\Gamma'}$ at level Γ'.

We will also consider the closed immersion $\iota^c : X^c_H \to X^c$, which gives the cycle class

$$\xi_{X^c_H} := \left(\cap[X^c]\right)^{-1}\left(\iota^c_*\left[X^c_H\right]\right),$$

which is nonzero since X^c_H is a subvariety of the algebraic variety X^c.

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Proposition 3.10. *The $G(\mathbb{Q})$-submodule of $\mathrm{IH}^*(\mathscr{M}^*)$ generated by ξ_Γ contains the cycle class $\xi_{X_H^c} \in \mathrm{H}^{2(n-m)}(X^c)$ of X_H^c in X^c.*

Proof. The $G(\mathbb{Q})$-submodule $V \subset \mathrm{Gr}^W_{2(n-m)}\mathrm{H}^{2(n-m)}(\mathscr{M}^*) \subset \mathrm{IH}^{2(n-m)}(\mathscr{M}^*)$ generated by ξ_Γ admits a decomposition $V = V^0 \oplus V^1$, where V^1 has no invariants or coinvariants and V^0 is contained in the summand of invariants $\mathrm{H}^{2(n-m)}(X^c) \subset \mathrm{Gr}^W_{2m}\mathrm{H}^{2m}(M^*_\Gamma)$. (This is because of semisimplicity in Proposition 3.6.) Write $\xi_\Gamma = \xi^0_\Gamma + \xi^1_\Gamma$, with $\xi^i_\Gamma \in V^i$. Since V^1 has no coinvariants, $\xi^0_\Gamma \cdot \alpha = \xi_\Gamma \cdot \alpha$ for any $\alpha \in \mathrm{H}^{2m}(X^c)$, so that

$$\begin{aligned}
[M^*_\Gamma] \cap \left(\xi^0_\Gamma \cdot \alpha\right) &= [M^*_\Gamma] \cap (\xi_\Gamma \cdot \alpha) \\
&= [M^*_\Gamma] \cap \xi_\Gamma \cap \alpha \\
&= \iota_* \left[M^*_{H,\Gamma_H}\right] \cap \alpha \\
&= \iota_* \left(\left[M^*_{H,\Gamma_H}\right] \cap \iota^*(\alpha)\right).
\end{aligned} \tag{3.14}$$

On the other hand,

$$\begin{aligned}
[X^c] \cap \left(\xi_{X_H^c} \cdot \alpha\right) &= [X^c] \cap \xi_{X_H^c} \cap \alpha \\
&= \iota^c_* [X_H^c] \cap \alpha \\
&= \iota^c_* ([X_H^c] \cap \iota^{c*}(\alpha)).
\end{aligned} \tag{3.15}$$

Now the pullbacks ι^{c*} and ι^* are compatible, while under the isomorphism $\mathrm{H}^{2n}(X^c) \cong \mathrm{Gr}^W_{2n}\mathrm{H}^{2n}(M^*_\Gamma)$ the two nonzero linear forms $[M^*_\Gamma] \cap : \mathrm{Gr}^W_{2n}\mathrm{H}^{2n}(M^*_\Gamma) \to \mathrm{H}_0(M^*_\Gamma) = \mathbb{C}$ and $[X^c] \cap : \mathrm{H}^{2n}(X^c) \to \mathrm{H}_0(X^c) = \mathbb{C}$ are necessarily proportional; and the same holds for M^*_{H,Γ_H} and X_H^c. It follows from equations (3.14) and (3.15) that for $\alpha \in \mathrm{H}^{2m}(X^c)$, we have

$$[X^c] \cap \left(\xi^0_\Gamma \cdot \alpha\right) \sim [X^c] \cap \left(\xi_{X_H^c} \cdot \alpha\right),$$

where $\sim$ means up to a fixed nonzero constant independent of α. Thus $\xi^0_\Gamma \cdot \alpha \sim \xi_{X_H^c} \cdot \alpha$ for any α, and so $\xi^0_\Gamma \sim \xi_{X_H^c}$ by Poincaré duality for X^c. □

Theorem 3.11. *If $\alpha \in \mathrm{Gr}^W_i\mathrm{H}^i(\mathscr{M}^*)$ and $\mathrm{Res}(\alpha) = 0$, then $\alpha \cdot \xi_{X_H^c} = 0$.*

Proof. Suppose that $\alpha \in \mathrm{Gr}^W_i\mathrm{H}^i(M^*_\Gamma) \subset \mathrm{Gr}^W_i\mathrm{H}^i(\mathscr{M}^*)$ is such that $\mathrm{Res}(\alpha) = 0$. Set $g \in G(\mathbb{Q})$ and choose Γ' normal in Γ with $\Gamma' \subset \Gamma \cap g^{-1}\Gamma g$. Let $\gamma_1, \ldots, \gamma_r$ be the representatives for cosets of Γ' in Γ, let $p : M^*_{\Gamma'} \to M^*_\Gamma$, and let $\iota' : M^*_{H,\Gamma'_H} \to M^*_{\Gamma'}$ be the natural map at level Γ'. Then we have

$$p^{-1}\left(\iota\left(M^*_{H,\Gamma_H}\right)\right) = \bigcup_i \gamma_i \cdot \iota'\left(M^*_{H,\Gamma'_H}\right).$$

If $\mathrm{Res}(\alpha) = 0$ for $\alpha \in \mathrm{Gr}^W_i\mathrm{H}^i(M^*_\Gamma)$, then $\left(g\gamma_i^{-1}\right)^*\alpha = \left(\gamma_i^{-1}\right)^* g^*\alpha$ restricts to zero on $\iota'\left(M^*_{H,\Gamma'_H}\right)$ for each i – that is, $\iota'^*\left(\left(\gamma_i^{-1}\right)^* g^*(\alpha)\right) = 0$ for each i. Using equation (3.4), we have

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$$\begin{aligned} 0 &= \iota'_*\left(\iota'^*\left(\left(\gamma_i^{-1}\right)^* g^*\alpha\right)\cap\left[M^*_{H,\Gamma'_H}\right]\right) \\ &= \left(\gamma_i^{-1}\right)^* g^*\alpha\cap\iota'_*\left(\left[M^*_{H,\Gamma'_H}\right]\right) \\ &= [M^*_{\Gamma'}]\cap\left(\left(\gamma_i^{-1}\right)^* g^*\alpha\cdot\xi_{\Gamma'}\right) \\ &= [M^*_{\Gamma'}]\cap(g^*\alpha\cdot\gamma_i^*\xi_{\Gamma'}). \end{aligned} \tag{3.16}$$

By Proposition 3.3(2), we have that $g^*\alpha\cdot\gamma_i^*\xi_{\Gamma'}=0$. Summing over Γ/Γ' gives

$$0=g^*\alpha\cdot\xi_\Gamma=\alpha\cdot\left(g^{-1}\right)^*\xi_\Gamma.$$

Since this holds for all $g\in G(\mathbb{Q})$, Proposition 3.10 implies that $\alpha\cdot\xi_{X^c_H}=0$. □

Proof of Theorem 3.1. In the case $SU(1,m)\subset SU(1,n)$, we have $X^c=\mathbb{P}^n$, so that $\xi_{X^c_H}\sim\left(c_1^{bb}\right)^{n-m}$, where c_1^{bb} is the first Chern class of the ample Baily–Borel line bundle in formula (3.13). So $\cdot c_1^{bb}$ is injective in degrees $<n$ on $\bigoplus_i \mathrm{Gr}_i^W\mathrm{H}^i(M^*_\Gamma)\subset\mathrm{IH}^*\left(M^*_\Gamma\right)$ because of the hard Lefschetz property for intersection cohomology [6, 42], and hence $\cdot\xi_{X^c_H}$ is injective in degrees $i\le m$. Theorem 3.11 implies the injectivity of Res in degrees $\le m$.

In the case $SO(2,m)\subset SO(2,n)$, the previous argument can be applied to the simply connected covers $\tilde{H}\subset\tilde{G}$. Now we claim $\xi_{X^c_H}\sim\left(c_1^{bb}\right)^{n-m}$. If $n\ne 2m$ this is clear, since $\mathrm{Gr}^W_{2(n-m)}\mathrm{H}^{2(n-m)}\left(M^*_\Gamma\right)=\mathbb{C}\left(c_1^{bb}\right)^{n-m}$, whereas if $n=2m$, then it holds because $c_m|_{X^c_H}=0$ (see Example 3.9). It follows that $\cdot\xi_{X^c_H}$ is injective on $\bigoplus_i\mathrm{Gr}_i^W\mathrm{H}^i\left(M^*_\Gamma\right)$ in degrees $\le m$ because of the hard Lefschetz property of $c_1^{bb}\cdot$ on $\mathrm{IH}^*\left(M^*_\Gamma\right)$. Theorem 3.11 implies the injectivity of Res in degrees $\le m$. □

Proof of Corollary 3.2. The complex hyperbolic case follows easily from the observation that

$$\mathrm{H}^k_!(M_\Gamma)=\mathrm{Gr}^W_k\mathrm{H}^k_c(M_\Gamma)\qquad\text{for} k\le n$$

(see the proof of [36, Proposition 1.6]) and the fact that $\mathrm{Gr}^W_k\mathrm{H}^k_c(M_\Gamma)\subset\mathrm{Gr}^W_k\mathrm{H}^k\left(M^*_\Gamma\right)$ for all k (Remark 3.4).

Now consider the orthogonal case. First note that

$$\mathrm{H}^i_!(M_\Gamma)=\mathrm{im}\left(\mathrm{H}^i_c(M_\Gamma)\to\mathrm{IH}^i\left(M^*_\Gamma\right)\right)=\mathrm{Gr}^W_i\mathrm{H}^i_c(M_\Gamma)\qquad\text{for} i\le n-1. \tag{3.17}$$

The first equality holds because $\mathrm{IH}^i\left(M^*_\Gamma\right)\to\mathrm{H}^i(M_\Gamma)$ is injective for $i\le n-1$ (and an isomorphism for $i\le n-2$), because the boundary has dimension 1, and the second holds because $\mathrm{Gr}^W_i\mathrm{H}^i_c(M_\Gamma)\hookrightarrow\mathrm{IH}^i\left(M^*_\Gamma\right)$ for all i (see Remark 3.4). On the other hand,

$$\mathrm{Gr}^W_i\mathrm{H}^i_c(M_\Gamma)\subset\mathrm{Gr}^W_i\mathrm{H}^i\left(M^*_\Gamma\right)$$

for all i (by Remark 3.4 again). Thus $\mathrm{H}^i_!(\mathscr{M})\subset\mathrm{Gr}^W_i\mathrm{H}^i(\mathscr{M}^*)$ for $i\le n-1$, and similarly for $\mathscr{M}_H$. So the corollary follows from Theorem 3.1 in degrees $\le m-1$ in the case $rk_{\mathbb{Q}}(G)=2$. If $rk_{\mathbb{Q}}(H)=1$, then this can be improved slightly, because the singularities of M^*_{H,Γ_H} are isolated and so $\mathrm{H}^i_!(\mathscr{M})\subset\mathrm{Gr}^W_i\mathrm{H}^i(\mathscr{M}^*)$ holds for $i=m$ also. □

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Remark 3.12. The use of the embedding $\mathrm{H}^*(X^c) \subset \mathrm{Gr}^W_* \mathrm{H}^*(\mathscr{M}^*)$ (of Proposition 3.6) in the proof of Theorem 3.1 can be avoided in the unitary case and in the orthogonal case (except possibly when $n = 2m$). As noted in Example 3.7, the first Chern class of the Baily–Borel bundle can be made sense of in $\mathrm{Gr}^W_* \mathrm{H}^*(\mathscr{M}^*)$, and the invariants in $\mathrm{IH}^{2(n-m)}(M^*_\Gamma)$ are reduced to $\mathbb{C}\left(c_1^{bb}\right)^{n-m}$. So the argument can be run with $\mathrm{H}^*(X^c)$ replaced by the subring $\bigoplus_{0 \le i \le n} \mathbb{C}\left(c_1^{bb}\right)^i$.

Remark 3.13. The foregoing arguments can be modified to treat nontrivial coefficients, using the fact that the local system E on M_Γ underlies a pure polarizable variation of Hodge structure (see [31, §4] or, for a more canonical approach in the context of Shimura varieties, see [19]). This allows us to use mixed Hodge modules and arguments with weights.

Remark 3.14. The criterion of [47] in the compact case has been used in [9] to prove a number of other results about restriction using computations in the compact dual. The analogues for the top weight quotient of $\mathrm{H}^*(\mathscr{M}^*)$ in general then follow immediately using the criterion of Theorem 3.11 instead. It seems likely that (suitably formulated) they should extend to $\mathrm{H}^*(\mathscr{M})$ using the methods of later sections.

The following example shows that these bounds can sometimes be improved on:

Example 3.15. Consider the case of $SO(2,2) \subset SO(2,n)$ for $n \ge 3$, and assume $rk_{\mathbb{Q}}(H) = 2$, so that $H = SL(2) \times SL(2)$. Theorem 3.1 gives the injectivity of $\mathrm{Gr}^W_2 \mathrm{H}^2(\mathscr{M}^*) \to I^G_H \mathrm{Gr}^W_2 \mathrm{H}^2(\mathscr{M}^*_H)$. Since $\mathrm{H}^2_!(M_\Gamma) = \mathrm{IH}^2(M^*_\Gamma) = \mathrm{Gr}^W_2 \mathrm{H}^2_c(M_\Gamma)$, the map $\mathrm{Gr}^W_2 \mathrm{H}^2(M^*_\Gamma) \to \mathrm{Gr}^W_2\left(M^*_{H,\Gamma_H}\right)$ factors as

$$\mathrm{Gr}^W_2 \mathrm{H}^2_c(M_\Gamma) \to \mathrm{Gr}^W_2 \mathrm{H}^2_c(M_{H,\Gamma_H}) \to \mathrm{Gr}^W_2 \mathrm{H}^2\left(M^*_{H,\Gamma_H}\right).$$

Now $M_{H,\Gamma_H} = X_1 \times X_2$ is a product of two modular curves, so $\mathrm{Gr}^W_2 \mathrm{H}^2_c(X_1 \times X_2) = \mathrm{Gr}^W_1 \mathrm{H}^1_c(X_1) \otimes \mathrm{Gr}^W_1 \mathrm{H}^1_c(X_2)$ injects into $\mathrm{Gr}^W_2 \mathrm{H}^2(X_1^* \times X_2^*)$, and so the second map is injective. It follows that Res is always injective on $\mathrm{H}^2_!(\mathscr{M})$, improving Corollary 3.2 slightly.

Remark 3.16. In fact, when $m = 2$ and $n = 3$, Res is injective on all of $\mathrm{H}^2(\mathscr{M})$ by a result of Weissauer [51]. This is not covered by our results, since $\mathrm{H}^2_!(\mathscr{M})$ is a proper subspace of $\mathrm{H}^2(\mathscr{M})$.

Remark 3.17. The basic idea of this section is that in the presence of some functoriality, semisimplicity, and duality, one can use the averaging argument for the cycle class. This can also be applied to the reductive Borel–Serre (RBS) compactification, to get a slight generalization of Theorems 3.11 and 3.1.

The RBS compactification is a (nonalgebraic) compactification of M_Γ dominating M^*_Γ – that is, the identity of M_Γ extends to $M^{rbs}_\Gamma \to M^*_\Gamma$. The cohomology $\mathrm{H}^i\left(M^{rbs}_\Gamma\right)$ carries a mixed Hodge structure like that of a proper variety – that is, with weights $\le i$ in degree i – and the top weight quotient $\mathrm{Gr}^W_i \mathrm{H}^i\left(M^{rbs}_\Gamma\right)$ is the image of a natural map $\mathrm{H}^i\left(M^{rbs}_\Gamma\right) \to \mathrm{IH}^i(M^*_\Gamma)$. For $\iota : M_{H,\Gamma_H} \to M_\Gamma$ there is no continuous map $M^{rbs}_{H,\Gamma_H} \to M^{rbs}_\Gamma$

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extending $M_{H,\Gamma_H} \to M_\Gamma$, but nevertheless there is a natural pullback map $\mathrm{H}^*\left(M_\Gamma^{rbs}\right) \to \mathrm{H}^*\left(M_{H,\Gamma_H}^{rbs}\right)$, which is a homomorphism of mixed Hodge structures. (See, e.g., the survey [37], where these results are discussed.) Theorem 3.1 can be improved to injectivity of the induced map Res on $\mathrm{Gr}_i^W\mathrm{H}^i\left(\mathscr{M}^{rbs}\right) = \mathrm{colim}_\Gamma \mathrm{Gr}_i^W \mathrm{H}^i\left(M_\Gamma^{rbs}\right)$ for $i \le m$. Since the canonical mapping $\mathrm{H}^i(M_\Gamma^*) \to \mathrm{IH}^*(M_\Gamma^*)$ factors through $\mathrm{H}^*\left(M_\Gamma^{rbs}\right)$, it follows that $\mathrm{Gr}_i^W\mathrm{H}^i(\mathscr{M}^*) \subset \mathrm{Gr}_i^W\mathrm{H}^i\left(\mathscr{M}^{rbs}\right)$. (However, in general $\mathrm{H}^i(\mathscr{M}^*) \to \mathrm{H}^*\left(\mathscr{M}^{rbs}\right)$ is not injective.) It can be shown using methods of Eisenstein series that in the case at hand, this inclusion is proper, so we would have an improvement of Theorem 3.1.

4. Lie-algebra cohomology computations

4.1. Kostant's theorem

We recall results of [29]. Fix a complex semisimple Lie group G, a maximal torus $T \subset G$, and a Borel subgroup $B \supset T$, and let $\Phi = \Phi(T,G)$ be the root system, Φ^+ the positive roots determined by B, $\Phi^- = -\Phi^+$ the negative roots, $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$ the half-sum of positive roots, and $W = W(T,G)$ the Weyl group of T in G. Let P be a standard parabolic subgroup of G, N its unipotent radical, and $\mathfrak{n}$ its Lie algebra. The Weyl group of the Levi $L = P/N$ is a subgroup $W_L \subset W$, and we let W^P be the set of minimal length coset representatives of $W_L\backslash W$. For each $w \in W^P$, the associated set of positive roots

$$\Phi(w) = \left\{\alpha \in \Phi(T,G) : \alpha > 0, w^{-1}\alpha < 0\right\} = \Phi^+ \cap w\Phi^-,$$

which has cardinality $\ell(w)$. For a dominant $\lambda \in X^*(T)$, the weights $w(\lambda+\rho)-\rho$ for $w \in W^P$ are dominant for L and distinct. The mapping $w \mapsto \Phi(w)$ sets up a bijection between W^P and the subsets S of $\Phi(\mathfrak{n}) = \{\alpha \in \Phi(T,G) : \mathfrak{g}^\alpha \subset \mathfrak{n}\}$ for which both S and $\Phi^+ - S$ are closed under $\dot{+}$ [29, 5.10] (recall that $\alpha \dot{+} \beta$ is $\alpha+\beta$ when this is a root and empty otherwise).

Let E_λ be the irreducible finite-dimensional G-representation with highest weight $\lambda \in X^*(T)$ with respect to B. The Lie-algebra cohomology $\mathrm{H}^*(\mathfrak{n}, E_\lambda)$ is the cohomology of $\wedge^*\mathfrak{n}^* \otimes E_\lambda$ with the Lie-algebra differential. The natural P-module structure on $\wedge^*\mathfrak{n}^* \otimes E_\lambda$ descends to an $L = P/N$-module structure in cohomology. For an L-dominant weight $\mu \in X^*(T)$, let E_μ^L denote the irreducible finite-dimensional algebraic representation of L with highest weight μ. Then by [29, Theorem 5.14] there is a multiplicity-free decomposition of L-modules

$$\mathrm{H}^k(\mathfrak{n}, E_\lambda) = \bigoplus_{w\in W^P, \ell(w)=k} E^L_{w(\lambda+\rho)-\rho}. \tag{4.1}$$

Kostant also identified a highest weight vector in each summand. Let $\mathfrak{n}^-$ be the nilradical of the Lie algebra of the parabolic subgroup opposite to P. The Killing form gives isomorphisms $\mathfrak{n}^- \cong \mathfrak{n}^*$ and $\wedge^i\mathfrak{n}^- \cong \wedge^i\mathfrak{n}^*$. Choose a nonzero vector e_α in the root space $\mathfrak{g}^\alpha$ for each $\alpha \in \Phi(T, G_\mathbb{C})$, and for $w \in W^P$ define $e_w := \bigwedge_{\alpha\in\Phi(w)} e_{-\alpha} \in \wedge^{\ell(w)}\mathfrak{n}^-$. Let $v_{w\lambda} \in E_\lambda$ be a weight vector for the extremal weight $w\lambda$. Then under the identification of $\mathfrak{n}^*$ with $\mathfrak{n}^-$, the element

$$e_w \otimes v_{w\lambda} \in \wedge^{\ell(w)}\mathfrak{n}^- \otimes E_\lambda \tag{4.2}$$

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is closed in $\wedge^*\mathfrak{n}^* \otimes E_\lambda$ and its cohomology class is a highest weight vector for the summand $E^L_{w(\lambda+\rho)-\rho}$ in equation (4.1) [29, Theorem 5.14]; it is a harmonic representative for a natural Laplacian. A lowest weight vector is given by $\wedge_{\alpha\in\Phi(w)} e_{-w_0^L(\alpha)} \otimes v_{w_0^L w\lambda}$, where w_0^L is the longest element of $W_L \subset W$ [29, Remark 8.2]. In fact, taking the sum of the L-submodules of $\wedge^*\mathfrak{n}^- \otimes E_\lambda$ generated by the $e_w \otimes v_{w\lambda}$ as w runs over W^P gives (using the identification $\wedge^*\mathfrak{n}^* \cong \wedge^*\mathfrak{n}^-$) a canonical L-equivariant inclusion $\mathrm{H}^*(\mathfrak{n}, E_\lambda) \subset \wedge^*\mathfrak{n}^* \otimes E_\lambda$ inducing the identity in cohomology and compatible with products (see [29, Theorem 5.7]).

4.2. Restriction maps in $\mathfrak{n}$-cohomology

Now consider the situation where $\iota : H \to G$ is a homomorphism of real semisimple groups with finite kernel. Then for a parabolic P of G with Levi $L = P/N$, we have the parabolic $P_H = \iota^{-1}(P)$ of H with unipotent radical $N_H = \iota^{-1}(N)$ and Levi $L_H = P_H/N_H$. Let $\mathfrak{n} = \mathrm{Lie}N(\mathbb{R})$ and $\mathfrak{n}_H = \mathrm{Lie}N_H(\mathbb{R})$. For a finite-dimensional $G(\mathbb{C})$-representation E and E_H a summand of $E|_{H(\mathbb{C})}$, the restriction map

$$\mathrm{H}^*(\mathfrak{n}_\mathbb{C}, E) \longrightarrow \mathrm{H}^*\left(\mathfrak{n}_{H,\mathbb{C}}, E\right) \longrightarrow \mathrm{H}^*\left(\mathfrak{n}_{H,\mathbb{C}}, E_H\right)$$

is $L_H(\mathbb{C})$-equivariant. Consider the map

$$Res_\mathfrak{n} : \mathrm{H}^*(\mathfrak{n}_\mathbb{C}, E) \longrightarrow \prod_{m\in L(\mathbb{C})} \mathrm{H}^*\left(\mathfrak{n}_{H,\mathbb{C}}, E_H\right),$$

with coordinate indexed by $m \in L(\mathbb{C})$ given by precomposing the previous map with the adjoint action of m. Note that the kernel of $\mathrm{Res}_\mathfrak{n}$ is an $L(\mathbb{C})$-module; in particular, for each irreducible summand $E^L_{w(\lambda+\rho)-\rho}$, we have that $\mathrm{Res}_\mathfrak{n}$ is injective on $E^L_{w(\lambda+\rho)-\rho} \Longleftrightarrow \mathrm{Res}_\mathfrak{n}$ is nonzero on $E^L_{w(\lambda+\rho)-\rho} \Longleftrightarrow Res_\mathfrak{n}(e_w \otimes v_{w\lambda}) \neq 0$.

Let us assume that the restriction of finite-dimensional irreducible representations by $H(\mathbb{C}) \subset G(\mathbb{C})$ is multiplicity-free. Choose maximal tori $T_H \subset T$ and Borel subgroups $B_H \subset B$ (of $H\mathbb{C}) \subset G(\mathbb{C})$), and for an irreducible representation E with highest weight $\lambda \in X^*(T)$, let E_H be the summand of $E|_{H(\mathbb{C})}$ with highest weight $\lambda|_{T_H} \in X^*(T_H)$. The following propositions are proved by explicit elementary calculations with roots and weights using Kostant's theorem and take up the rest of this section. In each case, the restriction from $G(\mathbb{C})$ to $H(\mathbb{C})$ is multiplicity-free by classical results ([22, Section 8.1.1]).

Proposition 4.1. *Let $G = SO(d,1)$ and $H = SO(c,1)$ for $2 \leq c < d$ embedded in the standard way in G. Let $P = LN$ be a proper parabolic subgroup of G. Then* $\mathrm{Res}_\mathfrak{n}$ *is injective in degrees $i \leq c/2$ except in the case $(d,c,i) = (2k+1, 2k, k)$. In this case,* $\mathrm{H}^k(\mathfrak{n}, E)$ *has two L-irreducible summands and* $\mathrm{Res}_\mathfrak{n}$ *is injective on either one.*

Proposition 4.2. *Let $G = SU(n,1)$ and $H = SU(m,1)$ for $2 \leq m < n$ embedded in the standard way in G. Let $P = MW$ be a proper parabolic subgroup of G. Then* $\mathrm{Res}_\mathfrak{w}$ *is injective on* $\mathrm{H}^i(\mathfrak{w}, E)$ *in degrees $i < m$.*

Proposition 4.3. *Let $G = SO(2,n)$ and $H = SO(2,m)$, embedded in G in the standard way for $2 \leq m < n$, and $E = \mathbb{C}$. If $P = MW$ is the stabilizer of an isotropic plane, then* $\mathrm{Res}_\mathfrak{w}$ *is injective on* $\mathrm{H}^i(\mathfrak{w}, \mathbb{C})$ *for $i \leq m - 2$.*

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To treat $SO(1,n) \subset SU(1,n)$, we will need the following:

Proposition 4.4. *Let $G = SU(n,1)$ and $H = SO(n,1)$, embedded in the standard way in G with $n \geq 2$. Let $P = MW$ be a proper parabolic subgroup of G. Then $\mathrm{Res}_{\mathfrak{w}}$ is injective on $\mathrm{H}^{i,0}(\mathfrak{w},\mathbb{C})$ in degrees $i < n$.*

The bigrading in Proposition 4.4 refers to the Hodge structure coming from the identification of $\mathrm{H}^i(\mathfrak{w},\mathbb{C})$ with the link cohomology $\mathrm{H}^i\left(i^*_{\{x\}}j_*\mathbb{C}\right)$, where $j : M_\Gamma \hookrightarrow M^*_\Gamma$ and $i_{\{x\}} : \{x\} \hookrightarrow M^*_\Gamma$ is the inclusion of the cusp corresponding to P (see Proposition 2.7 or [36, Lemma 1.2].) It can also seen from the decomposition (4.1) [36, Remark 1.11].

The rest of this section will be taken up with the proofs of these propositions.

4.3. Proof of Proposition 4.3

To make computations we will fix some notation for roots. We may assume $\mathfrak{g} = \mathfrak{so}(2,n) = \mathfrak{so}(J)$, where

$$J = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & I_{n-2} & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix}.$$

Fix a Cartan subalgebra $\mathfrak{s}$ of $\mathfrak{so}(n-2)_\mathbb{C}$ and let $\mathfrak{t} \subset \mathfrak{g}_\mathbb{C}$ be the Cartan subalgebra defined by

$$\mathfrak{t} := \{diag(a,d,C,-d,-a) : a,d \in \mathbb{C}, C \in \mathfrak{s}\}.$$

Then $\mathfrak{t}$ is defined and maximally split over $\mathbb{R}$, and the subspace $\mathfrak{a}_\mathbb{C} \subset \mathfrak{t}$ given by $C = 0$ is the complexification of the Lie algebra $\mathfrak{a} \subset \mathfrak{g}$ of a maximal $\mathbb{R}$-split Cartan in $\mathfrak{g}$. Let $\alpha_1, \alpha_2 \in \mathfrak{t}^*$ be defined by

$$\begin{aligned} \alpha_1(diag(a,d,C,-d,-a)) &= a, \\ \alpha_2(diag(a,d,C,-d,-a)) &= d. \end{aligned} \tag{4.3}$$

The relative roots are $\Phi(\mathfrak{a}_\mathbb{C}, \mathfrak{g}_\mathbb{C}) = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1-\alpha_2), \pm(\alpha_1+\alpha_2)\}$.

Now choose for $\mathfrak{s}$ the Cartan subalgebra of block-diagonal matrices

$$\mathfrak{s} = \left\{ diag\left(\left(\begin{smallmatrix} 0 & b_1 \\ -b_1 & 0 \end{smallmatrix}\right), \ldots, \left(\begin{smallmatrix} 0 & b_k \\ -b_k & 0 \end{smallmatrix}\right)\right) : b_1, \ldots, b_k \in \mathbb{C} \right\} \tag{4.4}$$

when $n-2 = 2k$ is even and

$$\mathfrak{s} = \left\{ diag\left(\left(\begin{smallmatrix} 0 & b_1 \\ -b_1 & 0 \end{smallmatrix}\right), \ldots, \left(\begin{smallmatrix} 0 & b_k \\ -b_k & 0 \end{smallmatrix}\right), 0\right) : b_1, \ldots, b_k \in \mathbb{C} \right\} \tag{4.5}$$

when $n-2 = 2k+1$ is odd. Let $\eta_1, \ldots, \eta_k \in \mathfrak{s}^*$ be defined by

$$\eta_i\left(diag\left(\left(\begin{smallmatrix} 0 & b_1 \\ -b_1 & 0 \end{smallmatrix}\right), \ldots, \left(\begin{smallmatrix} 0 & b_k \\ -b_k & 0 \end{smallmatrix}\right)\right), (0)\right) = \sqrt{-1}b_i, \tag{4.6}$$

where (0) means the entry is omitted when $n-2$ is even and the entry is zero when $n-2$ is odd. Fix the positive system $\Phi^+(\mathfrak{s}, \mathfrak{so}(n-2)_\mathbb{C})$ with simple roots $\{\eta_i - \eta_{i+1} : 1 \leq i < k\} \sqcup$

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$\{\eta_k\}$ for $n-2=2k+1$ odd and $\{\eta_i-\eta_{i+1}:1\le i<k\}\sqcup\{\eta_k+\eta_{k-1}\}$ for $n-2=2k$ even, and take the positive system in $\Phi(\mathfrak{t},\mathfrak{g}_{\mathbb{C}})$ containing it and compatible with $\Phi^+(\mathfrak{a}_{\mathbb{C}},\mathfrak{g}_{\mathbb{C}})=\{\alpha_1,\alpha_2,\alpha_1-\alpha_2,\alpha_1+\alpha_2\}$. An explicit computation shows that the positive roots are

$$\Phi^+(\mathfrak{t},\mathfrak{g}_{\mathbb{C}})=\Phi^+(\mathfrak{s},\mathfrak{so}(n-2)_{\mathbb{C}})\sqcup\{\alpha_1\pm\eta_i,\alpha_2\pm\eta_i:1\le i\le k\}\sqcup\{\alpha_1-\alpha_2,\alpha_1+\alpha_2\}$$

for $n-2=2k$ even and

$$\Phi^+(\mathfrak{t},\mathfrak{g}_{\mathbb{C}})=\Phi^+(\mathfrak{s},\mathfrak{so}(n-2)_{\mathbb{C}})\sqcup\{\alpha_1\pm\eta_1,\alpha_2\pm\eta_i:1\le i\le k\}\sqcup\{\alpha_1,\alpha_2,\alpha_1+\alpha_2,\alpha_1-\alpha_2\}$$

for $n-2=2k+1$ odd.

The parabolic $P=MW$ in the proposition is the stabilizer of an isotropic plane, which may be assumed to be the obvious plane $\mathbb{R}e_1+\mathbb{R}e_2$ in $\mathbb{R}^{n+2}$. We will need the set $\Phi(\mathfrak{w})=\{\alpha\in\Phi(\mathfrak{t},\mathfrak{g}_{\mathbb{C}}):\mathfrak{g}_{\mathbb{C}}^{\alpha}\subset\mathfrak{w}_{\mathbb{C}}\}$. Using the foregoing description and explicit matrix descriptions, we have

$$\Phi(\mathfrak{w})=\{\alpha_1\pm\eta_i\}_{1\le i\le k}\sqcup\{\alpha_2\pm\eta_i\}_{1\le i\le k}\sqcup\{\alpha_1+\alpha_2\}$$

if $n-2=2k$ is even and

$$\Phi(\mathfrak{w})=\{\alpha_1\pm\eta_i\}_{1\le i\le k}\sqcup\{\alpha_2\pm\eta_i\}_{1\le i\le k}\sqcup\{\alpha_1,\alpha_2,\alpha_1+\alpha_2\}$$

if $n-2=2k+1$ is odd.

Now let us prove the proposition. We may assume that $\mathfrak{h}\subset\mathfrak{g}$ is given by the subspace $\mathbb{R}^{m+2}=(\mathbb{R}e_{m+1}+\cdots+\mathbb{R}e_n)^{\perp}\subset\mathbb{R}^{n+2}$ – that is, that $\mathfrak{h}=\mathfrak{so}(2,m)$ is embedded in $\mathfrak{g}=\mathfrak{so}(2,n)$ in a way that the $m+1,m+2,\dots,n$ rows and columns are zero. We will consider the cases n even and n odd separately.

First assume $n-2=2k$ is even. Then we have

$$\Phi(\mathfrak{w})=\{\alpha_1\pm\eta_i\}_{1\le i\le k}\sqcup\{\alpha_2\pm\eta_i\}_{1\le i\le k}\sqcup\{\alpha_1+\alpha_2\}.$$

For $w\in W^P$ of length $\le m-2=2k-(n-m)$, the set of roots $\Phi(w)\subset\Phi(\mathfrak{w})$ has cardinality $\le 2k-(n-m)$, so that $\Phi^+-\Phi(w)$ contains at least $n-m$ elements which belong to $\{\alpha_2\pm\eta_i\}_{1\le i\le k}$, which has cardinality $2k$. Since $\Phi^+-\Phi(w)$ is closed under $\dot{+}$ and $\alpha_1-\alpha_2\in\Phi^+-\Phi(w)$, we may choose sets I_+ and I_- in $\{1,\dots,k\}$ such that the following are true:

1. I_+ and I_- are disjoint and $|I_+\sqcup I_-|=n-m$.
2. For $i\in I_+$, we have $\{\alpha_1+\eta_i,\alpha_2+\eta_i\}\subset\Phi^+-\Phi(w)$.
3. For $i\in I_-$, we have $\{\alpha_1-\eta_i,\alpha_2-\eta_i\}\subset\Phi^+-\Phi(w)$.

(These sets are not unique, but any choice suffices for our purposes.) Let $\mathfrak{h}'$ be the copy of $\mathfrak{so}(2,m)$ given by the embedding of the subspace

$$\left(\bigoplus_{i\in I_+}\mathbb{R}e_{2i+2}\oplus\bigoplus_{i\in I_-}\mathbb{R}e_{2i+1}\right)^{\perp}\subset\mathbb{R}^{n+2};$$

that is, row and column $2i+2$ are zero for $i\in I_+$ and row and column $2i+1$ are zero for $i\in I_-$. Then the restriction of the harmonic representative $e_w=\wedge_{\alpha\in\Phi(w)}e_{-\alpha}$ in formula (4.2) to $\mathfrak{n}_{H'}$ is nonzero, and equals (up to a nonzero scalar) the harmonic representative of a similar class in $\mathfrak{n}_{H'}$. Since the subspace is conjugate to the subspace $(\mathbb{R}e_{m+1}+\cdots+\mathbb{R}e_n)^{\perp}$

by an element $m \in M(\mathbb{C}) = Spin(n-2,\mathbb{C})$, we see that $\mathfrak{h}'$ is conjugate to $\mathfrak{h}$ by $m \in M(\mathbb{C})$, and the highest weight vector e_w restricts nontrivially to $\mathrm{Ad}\left(m^{-1}\right)(\mathfrak{h})$. This proves the proposition in this case.

Next assume $n-2 = 2k+1$ is odd. Then

$$\Phi(\mathfrak{w}) = \{\alpha_1 \pm \eta_i\}_{1\le i\le k} \sqcup \{\alpha_2 \pm \eta_i\}_{1\le i\le k} \sqcup \{\alpha_1,\alpha_2,\alpha_1+\alpha_2\}.$$

For $w \in W^P$ of length $\le m-2 = 2k+1-(n-m)$, the set of roots $\Phi(w) \subset \Phi(\mathfrak{w})$ has cardinality $\le 2k+1-(n-m)$, so that $\Phi^+ - \Phi(w)$ contains at least $n-m$ elements from the set $\{\alpha_2 \pm \eta_i : 1 \le i \le k\} \sqcup \{\alpha_2\}$ of cardinality $2k+1$. As in the previous case, $\alpha_1 - \alpha_2 \in \Phi^+ - \Phi(w)$, and $\Phi(w)$ is closed under $\dot{+}$, so we may choose (not necessarily unique) sets of indices I_+ and I_- such that the following are true:

1. I_+ and I_- are disjoint, $|I_+ \sqcup I_-| = n-m-1$ if $\alpha_2 \in \Phi^+ - \Phi(w)$, and $|I_+ \sqcup I_-| = n-m$ if $\alpha_2 \notin \Phi^+ - \Phi(w)$.
2. For $i \in I_+$, we have $\{\alpha_1+\eta_i, \alpha_2+\eta_i\} \subset \Phi^+ - \Phi(w)$.
3. For $i \in I_-$, we have $\{\alpha_1-\eta_i, \alpha_2-\eta_i\} \subset \Phi^+ - \Phi(w)$.

As before, if $\alpha_2 \notin \Phi^+ - \Phi(w)$ we can define the subspace

$$\left(\bigoplus_{i\in I_+} \mathbb{R}e_{2i+2} \oplus \bigoplus_{i\in I_-} \mathbb{R}e_{2i+1}\right)^{\perp} \subset \mathbb{R}^{n+2}$$

of dimension m, and the harmonic representative e_w restricts nontrivially to the corresponding $\mathfrak{h}' = \mathfrak{so}(2,m)$ in $\mathfrak{g}$. If $\alpha_2 \in \Phi^+ - \Phi(w)$, then one adds on $\mathbb{R}e_n$ to the subspace and e_2 restricts nontrivially to the corresponding $\mathfrak{h}' = \mathfrak{so}(2,m)$. In either case, since $\mathfrak{h}'$ is conjugate to $\mathfrak{h}$ by $m \in M(\mathbb{C})$, we have proved the proposition. □

4.4. Proof of Proposition 4.1

For $E = \mathbb{C}$, the injectivity in degrees $i \le c/2$ except in the exceptional case is immediate from the fact that $\mathfrak{n}^*$ is the natural representation of the factor $SO(d-1)$ of L, and so $\mathrm{H}^i(\mathfrak{n},\mathbb{C}) = \wedge^i \mathfrak{n}^*$ are irreducible. This can be easily generalized to the case of general E, but we give a computational proof using Kostant's theorem, as we will have to verify slightly more.

We may assume $\mathfrak{g} = \mathfrak{so}(1,d) = \mathfrak{so}(J)$, where

$$J = \begin{pmatrix} & & 1 \\ & I_{d-1} & \\ 1 & & \end{pmatrix}.$$

Fix a Cartan subalgebra $\mathfrak{s}$ of $\mathfrak{so}(d-2)_{\mathbb{C}}$ and let $\mathfrak{t} \subset \mathfrak{g}_{\mathbb{C}}$ be the Cartan subalgebra defined by

$$\mathfrak{t} := \{diag(a, C, -a) : a, d \in \mathbb{C}, C \in \mathfrak{s}\}.$$

Then $\mathfrak{t}$ is defined and maximally split over $\mathbb{R}$, and the subspace $\mathfrak{a}_{\mathbb{C}} \subset \mathfrak{t}$ given by $C = 0$ is the complexification of the Lie algebra $\mathfrak{a} \subset \mathfrak{g}$ of a maximal $\mathbb{R}$-split subspace in $\mathfrak{g}$. Let

$\alpha \in \mathfrak{t}^*$ be defined by

$$\alpha(diag(a,C,-a)) = a. \tag{4.7}$$

We choose for $\mathfrak{s}$ the same Cartan subalgebra of block-diagonal matrices in $\mathfrak{so}(d-1)_{\mathbb{C}}$ specified earlier in equations (4.4) and (4.5) and use the same roots η_i and the same positive system used there. An explicit computation shows that the positive roots are

$$\Phi^+(\mathfrak{t},\mathfrak{g}_{\mathbb{C}}) = \Phi^+(\mathfrak{s},\mathfrak{so}(d-1)_{\mathbb{C}}) \sqcup \{\alpha \pm \eta_i : 1 \le i \le k\}$$

for $d-1 = 2k$ even and

$$\Phi^+(\mathfrak{t},\mathfrak{g}_{\mathbb{C}}) = \Phi^+(\mathfrak{s},\mathfrak{so}(d-1)_{\mathbb{C}}) \sqcup \{\alpha \pm \eta_i : 1 \le i \le k\} \sqcup \{\alpha\}$$

for $d-1 = 2k+1$ odd. We also have

$$\Phi(\mathfrak{n}) = \{\alpha \pm \eta_i : 1 \le i \le k\}$$

in the case $d-1 = 2k$ even and

$$\Phi(\mathfrak{n}) = \{\alpha \pm \eta_i : 1 \le i \le k\} \sqcup \{\alpha\}$$

in the case $d-1 = 2k+1$ odd.

We will list the relevant $w \in W^P$ and the sets $\Phi(w)$. We will consider the even and odd cases separately.

First assume $d-1 = 2k$ is even. Let $\alpha_1, \alpha_2, \ldots, \alpha_{k+1}$ be the set of simple roots of $\mathfrak{so}(1,d)_{\mathbb{C}}$ determined by the positive system already fixed – that is, $\alpha_1 := \alpha - \eta_1$, $\alpha_i = \eta_{i-1} - \eta_i$ for $2 \le i \le k$, and $\alpha_{k+1} = \eta_{k-1} + \eta_k$. The minimal length representatives in W^P of length $\le k$ are

$$\{s_0, s_1, \ldots, s_k, t_k\},$$

where $s_0 := 1$, $s_j := s_{\alpha_1} \cdots s_{\alpha_j}$ for $1 \le j \le k$ has length j, and $t_k = s_{k-1} s_{\alpha_{k+1}}$ has length k (compare [17, Section VI.3.1]; as usual, s_{α_i} denotes the reflection in α_i). The set $\Phi(w)$ is easily computed for these representatives: $\Phi(1) = \emptyset$ and $\Phi(s_j) = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_j\} = \{\alpha - \eta_1, \alpha_1 - \eta_2, \ldots, \alpha - \eta_j\}$ for $j \le k$, whereas $\Phi(t_k) = \{\alpha_1, \ldots, \alpha_1 + \cdots + \alpha_{k-1}, \alpha_1 + \cdots + \alpha_{k-1} + \alpha_{k+1}\} = \{\alpha - \eta_1, \alpha_1 - \eta_2, \ldots, \alpha - \eta_{k-1}, \alpha + \eta_k\}$.

Now suppose $d-1 = 2k+1$ is odd. Let $\alpha_1, \alpha_2, \ldots, \alpha_{k+1}$ be the simple roots of $\mathfrak{so}(1,d)_{\mathbb{C}}$ determined by the positive system – that is, $\alpha_1 := \alpha - \eta_1$, $\alpha_i = \eta_{i-1} - \eta_i$ for $2 \le i \le k$, and $\alpha_{k+1} = \eta_k$. The minimal length representatives in W^P of length $\le k$ are

$$\{s_0, s_1, \ldots, s_k\},$$

where $s_0 := 1$, and $s_j := s_{\alpha_1} \cdots s_{\alpha_j}$ for $1 \le j \le k$ has length j (compare [17, Section VI.4.4], where this set is denoted ${}^P W$). The set $\Phi(s_j)$ has the same description for $j \le k$ as before.

Now consider the setup of the proposition. We have $\mathfrak{h} = \mathfrak{so}(1,c)$ for $c \le d-1$, embedded in the standard way – that is, using the subspace $\mathbb{R}^{c+1} \subset \mathbb{R}^{d+1}$ spanned by $e_1, e_2, \ldots, e_c, e_{d+1}$. To show that $\mathrm{Res}_{\mathfrak{n}}$ is injective in a given degree i, it will suffice to show that the harmonic representative (i.e., L-highest weight vector) $e_w \otimes v_{w\lambda}$ restricts nontrivially in $\mathrm{H}^i(\mathfrak{n}_H, E_H)$ for each $w \in W^P$ of length i. In the case at hand, for $j < c/2$ there is a unique element in W^P of length j, namely s_j. The L-highest weight vector $e_{s_j} \otimes v_{s_j\lambda} =$

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$\bigwedge_{1\le i\le j} e_{-(\alpha-\eta_i)}\otimes v_{s_j\lambda}$ maps in $\wedge^j \mathfrak{n}_H^* \otimes E_H$ to (a nonzero multiple of) the harmonic representative $\bigwedge_{1\le i\le j} e^H_{-(\alpha-\eta_i)}\otimes v^H_{s^H_j\lambda_H}$, where $v^H_{s^H_j\lambda_H}$ is the $s^H_j(\lambda_H)$-weight vector of E_H, and hence is nonvanishing in cohomology. This proves that $\mathrm{Res}_{\mathfrak{n}}(e_w\otimes v_{w\lambda})\neq 0$, and hence that $\mathrm{Res}_{\mathfrak{n}}$ is injective for $j<c/2$. The same proof works if $j\le c/2$, as long as we are not in the exceptional case $(d,c,i)=(2k+1,2k,k)$.

In the remaining case we are considering $\mathrm{Res}_{\mathfrak{n}}$ on $\mathrm{H}^k(\mathfrak{n},E)$ for $d=2k+1, c=2k, i=k$. In this case there are two L-irreducible summands, with highest weight vectors $e_{s_k}\otimes v_{s_k\lambda}$ and $e_{t_k}\otimes v_{t_k\lambda}$, respectively. In the embedding $\mathfrak{so}(2k+1,\mathbb{C})\subset\mathfrak{so}(2k+2,\mathbb{C})$ the weight space $\mathfrak{h}^{-\alpha_k}$ is embedded diagonally in the weight spaces $\mathfrak{g}^{-\alpha_k}$ and $\mathfrak{g}^{-\alpha_{k+1}}$. Under the restriction from T to T_H we have $\alpha_k|_{T_H}=\alpha_{k+1}|_{T_H}=\eta^H_{k-1}=\alpha^H_k$. Thus the vector $e_{s_k}\otimes v_{s_k\lambda}=\bigwedge_{1\le i\le k} e_{-(\alpha-\eta_i)}\otimes v_{s_k\lambda}$ goes to (a nonzero multiple of) the vector $\bigwedge_{1\le i\le k-1} e^H_{-(\alpha-\eta_i)}\wedge e^H_{-\eta_{k-1}}\otimes v^H_{s_k\lambda}$, which is nonzero since η_{k-1} is not one of $\alpha-\eta_i$, $i\le k-1$. A similar argument applies to $e_{t_k}\otimes v_{t_k\lambda}$. □

4.5. Proof of Proposition 4.2

This was proved for $E=\mathbb{C}$ in [36, §1.6], and the elements of W^P are explicitly listed there. The proof extends to general coefficients exactly as in the previous proof. □

4.6. Proof of Proposition 4.4

A tedious computational proof is possible, but we will argue differently. Recall the notation $P=MW$ for the parabolic in $SU(1,n)$ and $P_H=P\cap H=LN$ for the parabolic in $SO(1,n)$. For $k<n$ we have a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{u}^*_{\mathbb{C}}\otimes\wedge^{k-2}\mathfrak{v}^*_{\mathbb{C}} & \longrightarrow & \wedge^k\mathfrak{v}^*_{\mathbb{C}} & \longrightarrow & \mathrm{H}^k(\mathfrak{w}_{\mathbb{C}},\mathbb{C}) & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & \wedge^k\mathfrak{n}^*_{\mathbb{C}} & = & \mathrm{H}^k(\mathfrak{n}_{\mathbb{C}},\mathbb{C}). & & \end{array} \tag{4.8}$$

Here, as in §2.2, $\mathfrak{w},\mathfrak{v},\mathfrak{u}$ are the Lie algebras of the real points of $W(\mathbb{R}),V(\mathbb{R}),U(\mathbb{R})$. The second vertical map is induced by $\mathfrak{n}\subset\mathfrak{w}$. The first vertical map is induced by the identification of $\mathfrak{v}$ with the χ-eigenspace for A in $\mathfrak{w}$; since $A\subset P_H$ and it acts by χ on $\mathfrak{u}$, we have $\mathfrak{n}\subset\mathfrak{v}$ (compare §2.2 for notation). The top row comes from the long exact sequence for the boundary divisor in the toroidal compactification, and is exact in degrees $k<n$ and is a sequence of Hodge structures (see [36, Lemma 1.3]). The bigrading on $\wedge^k\mathfrak{v}^*_{\mathbb{C}}$ comes from $\mathfrak{v}^*_{\mathbb{C}}=(\mathfrak{v}^*_{\mathbb{C}})^{1,0}+(\mathfrak{v}^*_{\mathbb{C}})^{0,1}$, given by the complex structure on $\mathfrak{v}=\mathfrak{w}/\mathfrak{u}$ (itself given by the central $U(1)\subset M(\mathbb{R})$), and the $(k,0)$-subspace $\wedge^k(\mathfrak{v}^*_{\mathbb{C}})^{1,0}$ maps isomorphically onto $\mathrm{H}^{k,0}(\mathfrak{w},\mathbb{C})$. This is because the first term in the sequence has Hodge types $(k-1,1),\dots,(1,k-1)$, because $\mathfrak{u}^*$ amounts to a Tate twist (see the proof of [36, Lemma 1.3]). Now the composition $(\mathfrak{v}^*_{\mathbb{C}})^{1,0}\hookrightarrow\mathfrak{v}^*_{\mathbb{C}}\to\mathfrak{n}^*_{\mathbb{C}}$ is an isomorphism, since the kernel of $\mathfrak{v}^*_{\mathbb{C}}\to\mathfrak{n}^*_{\mathbb{C}}$ is the complexification of a real subspace, and hence does not meet $(\mathfrak{v}^*_{\mathbb{C}})^{1,0}$. Thus $\wedge^k(\mathfrak{v}^*_{\mathbb{C}})^{1,0}\to\wedge^k\mathfrak{n}^*_{\mathbb{C}}$ is nonzero, and since $\mathrm{H}^{k,0}(\mathfrak{w},\mathbb{C})$ is M-irreducible (see [36, Remark 1.11]), $\mathrm{Res}_{\mathfrak{w}}$ is injective on $\mathrm{H}^{k,0}(\mathfrak{w}_{\mathbb{C}},\mathbb{C})$ for $k<n$. □

5. Congruence real hyperbolic manifolds

We restate the main result (Theorem 1.1) for congruence hyperbolic manifolds:

Theorem 5.1. *Suppose that $H \subset G$ are semisimple groups of the same $\mathbb{Q}$-rank such that $H(\mathbb{R})^{nc} \subset G(\mathbb{R})^{nc}$ is $SO(1,c) \subset SO(1,d)$ for $2 \le c < d$, and that neither H nor G is triality. Then $\mathrm{Res} : \mathrm{H}^i(\mathscr{M}_G, E) \to I_H^G \mathrm{H}^i(\mathscr{M}_H, E_H)$ is injective for $i \le c/2$.*

In the compact case (and $E = \mathbb{C}$), this is [12, Theorem 1.5] of Bergeron and Clozel, who, using the Burger–Sarnak method [18] following Harris and Li [28], deduced it from Arthur's endoscopic classification [2] of automorphic representations for orthogonal groups. Their arguments can be adapted to the noncompact case to prove injectivity on the interior cohomology $\mathrm{H}^i_!(\mathscr{M}, E)$ for $i \le c/2$. Since the state of the literature on this adaptation to the noncompact case is less than satisfactory, we will sketch the argument in some detail, although the ingredients are all well known. Combined with an elementary argument at infinity using Proposition 4.1, this proves the theorem in general. Before starting the proofs, we will need to recall some general facts.

Recall the classification of unitarizable $(\mathfrak{g}, K)$-modules with cohomology with coefficients in E for $G(\mathbb{R}) = SO(d,1)$ from [17, Section VI.4] or [41, Section 1.3]. (We refer to [17] for more details; this reference deals with $SO_0(d,1)$ and $E = \mathbb{C}$, but it is easy to extend to the general case using, for example, translation functors as in [17, Section VI.0].) Let $0 \le i_E \le \lfloor d/2 \rfloor$ be the minimal degree for which there exists a unitary representation V with $\mathrm{H}^*(\mathfrak{g}, K, V \otimes E) \ne \{0\}$ (so $i_{\mathbb{C}} = 0$ and $i_E = \lfloor d/2 \rfloor$ if λ is regular). For each degree $i_E \le i \le \lfloor d/2 \rfloor$, there is a unique unitary cohomological representation π_i with cohomology in degree i with respect to E, and it has cohomology in exactly degrees i and $d-i$ if d is odd or $i < d/2$, and if $d = 2k$ it has cohomology in degree k. In the case $d = 2k$, the representation π_k of $SO(1,d)$ is a discrete series representation and its restriction to $SO_0(1,d)$ is a sum of two discrete series representations. If $d = 2k+1$ is odd, the representation π_k is tempered. This completes the list of unitarizable $(\mathfrak{g}, K)$-modules with cohomology with coefficients in E. (All these depend on E, but to keep the notation simple we do not indicate this.) When we use these objects for $H(\mathbb{R})$, we will write π_i^H.

We recall some well-known facts about noncompact arithmetic quotients, for which we refer to Appendix A. There is a decomposition $L^2(\Gamma \backslash G(\mathbb{R})) = L^2_{dis}(\Gamma \backslash G(\mathbb{R})) \oplus L^2_{cts}(\Gamma \backslash G(\mathbb{R}))$ into discrete and continuous spectra and a further decomposition $L^2_{dis}(\Gamma \backslash G(\mathbb{R})) = L^2_{cusp}(\Gamma \backslash G(\mathbb{R})) \oplus L^2_{res}(\Gamma \backslash G(\mathbb{R}))$ into cuspidal and residual spectra. For $? \in \{cusp, dis, cts\}$, let

$$\mathrm{H}^*_?(\Gamma, E) = \mathrm{H}^*\left(\mathfrak{g}, K, L^2_?(\Gamma \backslash G(\mathbb{R})) \otimes E\right),$$

where the $(\mathfrak{g}, K)$-cohomology of a unitary $G(\mathbb{R})$-representation (π, V) is understood to be that of the space V^∞ of smooth vectors. The natural map

$$\mathrm{H}^*\left(\mathfrak{g}, K, L^2(\Gamma \backslash G(\mathbb{R})) \otimes E\right) \longrightarrow \mathrm{H}^*(\Gamma, E)$$

induced by $L^2(\Gamma \backslash G(\mathbb{R}))^\infty \subset C^\infty(\Gamma \backslash G(\mathbb{R}))$ is injective on $\mathrm{H}^*_{cusp}(\Gamma, E)$ and zero on $\mathrm{H}^*_{cts}(\Gamma, E)$ (see Appendix A for a proof of this well-known fact).

Lemma 5.2. *For* Γ *arithmetic in* $SO(1,d)$*, we have the following:*

(1) $\mathrm{H}^*_{cusp}(\Gamma,E) = \mathrm{H}^*_!(\Gamma,E)$.

(2) $\mathrm{H}^i_{dis}(\Gamma,E) \to \mathrm{H}^i(\Gamma,E)$ *is injective in degrees* $i \leq d/2$ *and an isomorphism for* $i \leq \lfloor d/2 \rfloor - 1$.

Proof. This follows from methods of Harder [26] recalled in Appendix B and also Rohlfs and Speh [41]. According to [41, Theorem 1.5.1], the cohomology in degrees $i < k = \lfloor d/2 \rfloor$ is all square-integrable, and by the results in [41, Section 1.4], the noncuspidal square-integrable classes are generated using residues of Eisenstein series. These classes restrict nontrivially to the boundary (see the proof of [41, Proposition 1.4.4] or Lemma B.1), so they do not belong to interior cohomology. This proves (1) of the lemma in degrees $i < k = \lfloor d/2 \rfloor$. Moreover, the restriction to the boundary is injective on the residual classes, proving (2) in degrees $i < k = \lfloor d/2 \rfloor$. Statement (1) follows in degrees $i > d - \lfloor d/2 \rfloor$ by duality. This leaves degree k when $d = 2k$ is even and degrees $k, k+1$ when $d = 2k+1$ is odd. In both cases, the only contributions to $\mathrm{H}^*_!(\Gamma,E)$ (or to $\mathrm{H}^i_{dis}(\Gamma,E)$) in these degrees are from tempered representations (namely the discrete series when $d = 2k$ is even and the tempered representation π_k when $d = 2k+1$ is odd), so they are cuspidal by a well-known observation of Wallach [50], and then (1) and (2) are clear for these contributions. □

Proposition 5.3. *Assume that neither* G *nor* H *is triality. Then* Res *is injective on* $\mathrm{H}^i_!(\mathscr{M},E)$ *for* $i \leq c/2$.

Proof. We sketch how to adapt the argument of [7, 8, 12, 28] to the noncompact case. The main points are as follows:

(1) For $i \leq c/2$, the abstract restriction of π_i to $(\mathfrak{h}, K_H)$ contains π_i^H as a direct summand and multiplicity 1 holds – that is, $\dim \mathrm{Hom}_{(\mathfrak{h},K_H)}\left(\pi_i|_H, \pi_i^H\right) = 1$. Moreover, the induced map

$$\mathrm{H}^i(\mathfrak{g}, K, \pi_i \otimes E) \to \mathrm{H}^i\left(\mathfrak{h}, K_H, \pi_i^H \otimes E_H\right) \tag{5.1}$$

is an isomorphism of one-dimensional spaces. This is proved in [7, Theorem 3.4] (see also [28, §1,§6] and [8, Théorème 5.3]). The references treat the case $E = \mathbb{C}$ but the proof works in general; alternatively, one can use translation functors as in [17, Section VI.0] to reduce the general case to this one.

(2) Let $R : C^\infty(\Gamma\backslash G(\mathbb{R})) \to C^\infty(\Gamma_H\backslash H(\mathbb{R}))$ denote restriction of functions and $R^* : H^*(\Gamma,E) \to \mathrm{H}^*(\Gamma_H, E_H)$ the induced map in cohomology. Given an irreducible summand π of $L^2_{cusp}(\Gamma\backslash G(\mathbb{R}))$ with smooth vectors $\pi^\infty = \pi_i$, the image $R(\pi_i)$ consists of bounded functions, so we may consider its closure $R(\pi)$ in $L^2(\Gamma_H\backslash H(\mathbb{R}))$. Suppose that an irreducible summand σ of $L^2(\Gamma_H\backslash H(\mathbb{R}))$ with $\sigma^\infty = \pi_i^H$ appears as a direct summand of $R(\pi)$. We claim that $R^* : \mathrm{H}^i(\Gamma,E) \to \mathrm{H}^i(\Gamma_H,E_H)$ is injective on the summand $\mathrm{H}^i(\mathfrak{g},K,\pi \otimes E)$. To see this, note that $R(\pi) = R(\pi)_{dis} \oplus R(\pi)_{cts}$, where $R(\pi)_? = R(\pi) \cap L^2_?(\Gamma\backslash G(\mathbb{R}))$, for $? = dis, cts$, and the map R^* on $\mathrm{H}^i(\mathfrak{g},K,\pi\otimes E)$ is induced by the composition

$$\pi^\infty \to R(\pi)^\infty \to R(\pi)^\infty_{dis} \to C^\infty(\Gamma_H\backslash H(\mathbb{R})).$$

(We use the fact that $\mathrm{H}^*(\mathfrak{h}, K_H, R(\pi)_{cts} \otimes E_H) \to \mathrm{H}^*(\Gamma_H, E_H)$ is zero since $R(\pi)_{cts}$ is a summand of $L^2_{cts}(\Gamma_H\backslash H(\mathbb{R}))$; see Appendix A for a proof of this fact.) The last map

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induces an injection in degree $i \le c/2$ (by Lemma 5.2(2)). The map $\pi^\infty \to R(\pi)^\infty_{dis}$ is nonzero in cohomology because the composition $\pi^\infty \to R(\pi)^\infty_{dis} \to \sigma^\infty$ is a nonzero multiple of the map $\pi_i|_H \to \pi_i^H$ in (1) (by multiplicity 1), and hence induces the nontrivial map (5.1) in cohomology. This proves that R^* is nonzero on $\mathrm{H}^*(\mathfrak{g}, K, \pi \otimes E)$, and hence injective.

(3) The Burger–Sarnak argument shows that given π as in (2), and assuming a certain isolation hypothesis on π_i^H (recalled later), we can arrange for a summand σ as in (2), perhaps after replacing H by a conjugate – that is, replacing the map R by $R_g : C^\infty(\Gamma\backslash G(\mathbb{R})) \to C^\infty\left(g\Gamma g^{-1} \cap H\backslash H(\mathbb{R})\right)$ for some $g \in G(\mathbb{Q})$. This is [28, Proposition 3.1] and the refinement [7, Proposition 3.2]. The observation of [18] is that matrix coefficients of the cuspidal representation on $\Gamma\backslash G(\mathbb{R})$ are, when restricted to $H(\mathbb{R})$, the limits, uniform on compacta, of finite sums of matrix coefficients of $H(\mathbb{R})$ appearing in the spaces $L^2\left(g^{-1}\Gamma g \cap H\backslash H(\mathbb{R})\right)$ for $g \in G(\mathbb{Q})$. A key point explained in [28] (see the remarks at the bottom of their p. 93) is that this applies to cuspidal cohomology on a noncompact quotient, because cuspidal functions are of rapid decrease, hence uniformly continuous on $\Gamma\backslash G(\mathbb{R})$, and this suffices for the argument in [18]. Thus under the isolation hypothesis, if π_i^H is weakly contained in $R(\pi)$, then there is a direct summand σ of $R_g(\pi)$ as in (2), and so by (2), R_g^* is injective on $\mathrm{H}^i(\mathfrak{g}, K, \pi \otimes E)$.

(4) The isolation hypothesis required in (3) is that π_i^H is isolated in $\left\{\pi_i^H\right\} \cup \left\{(\rho, V_\rho) \in \widehat{H}_{Aut} : d \equiv 0 \text{ on } C^i(\mathfrak{g}, K, V_\rho \otimes E)\right\}$. (We refer to [7, 28] for the precise definitions of $\widehat{H}_{Aut}$ and the relevance of this condition.) This was shown in [7] to follow from a uniform (in Γ) lower bound for the first nonzero eigenvalue of the Laplacian on i-forms on M_Γ [7]. This eigenvalue bound was later proved in [12, Theorem 1.3] using Arthur's endoscopic classification [2] of representations. (At this point triality forms must be excluded, although they do not occur if the $\mathbb{Q}$-rank is 1; see Remark 5.5.)

(5) Putting (1)–(4) together, we conclude that Res is injective on the cuspidal cohomology for $i \le c/2$, and hence, by Lemma 5.2, on $\mathrm{H}^i_!(\mathscr{M}, E)$ for $i \le c/2$. □

We will make some remarks as to the necessity of the contortions in the previous proof after finishing the proof of the theorem.

Proof of Theorem 5.1. We will use the standard commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{H}^i_!(\mathscr{M}, E) & \longrightarrow & \mathrm{H}^i(\mathscr{M}, E) & \longrightarrow & I_P^G \mathrm{H}^i(\mathfrak{n}, E) \\ & \downarrow \mathrm{Res} & & \downarrow \mathrm{Res} & & \downarrow \mathrm{Res}_\infty \\ 0 \longrightarrow & I_H^G \mathrm{H}^i_!(\mathscr{M}_H, E) & \longrightarrow & I_H^G \mathrm{H}^i(\mathscr{M}_H, E) & \longrightarrow & I_H^G I_{P_H}^H \mathrm{H}^i(\mathfrak{n}_H, E_H). \end{array} \tag{5.2}$$

This diagram comes from the properness of $M_{H,\Gamma_H} \to M_\Gamma$, or can be seen using the fact that the cusp (i.e., reductive Borel–Serre) compactification is functorial for $H \subset G$ in this case. The identification of the boundary cohomology with $I_P^G \mathrm{H}^i(\mathfrak{n}, E)$ is standard. It suffices to prove the injectivity of Res_∞ for $i \le c/2$. By the transitivity of induction, $I_H^G I_{P_H}^H = I_P^G I_{P_H}^P$, so it suffices to prove the injectivity of $\mathrm{H}^i(\mathfrak{n}, E) \to I_{P_H}^P \mathrm{H}^i(\mathfrak{n}_H, E_H)$. Now the action of $N(\mathbb{Q})$ on $\mathrm{H}^i(\mathfrak{n}, E)$ is trivial, so this factors through

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$\left(I_{P_H}^{P}\mathrm{H}^i(\mathfrak{n}_H,E_H)\right)^{N(\mathbb{Q})} = I_{L_H}^{L}\mathrm{H}^i(\mathfrak{n}_H,E_H)$, and it is enough to prove the injectivity of $\mathrm{H}^i(\mathfrak{n},E) \to I_{L_H}^{L}\mathrm{H}^i(\mathfrak{n}_H,E)$. This follows from the injectivity of $\mathrm{Res}_{\mathfrak{n}}$ in degrees $i \le c$ proved in Proposition 4.1, except possibly in the case $(d,c,i) = (2k+1,2k,k)$.

The remaining case is restriction from $SO(1,2k+1)$ to $SO(1,2k)$ in degree $i = k$. It suffices to prove that Res_∞ is injective on the image of $\mathrm{H}^k(\mathscr{M},E) \to I_P^G\mathrm{H}^k(\mathfrak{n},E)$. This map is induced by the $P(\mathbb{Q})$-map $\mathrm{H}^k(\mathscr{M},E) \to \mathrm{H}^k(\mathfrak{n},E)$ given by restriction to a deleted neighborhood of the cusp given by $P = LN$, and the image of $\mathrm{H}^k(\mathscr{M},E) \to I_P^G\mathrm{H}^k(\mathfrak{n},E)$ is $I_P^G U$, where U is the image of of $\mathrm{H}^k(\mathscr{M}) \to \mathrm{H}^k(\mathfrak{n},E)$. There is a nondegenerate duality pairing on $\mathrm{H}^k(\mathfrak{n},E)$ given by the cup product and the self-duality of E. Now U is a maximal isotropic subspace for the duality pairing on $\mathrm{H}^k(\mathfrak{n},E)$ – that is, $U^\perp = U$ – and it is also $L(\mathbb{C}) = \mathbb{C}^* \times SO(2k,\mathbb{C})$-stable. Since $\mathrm{H}^k(\mathfrak{n},E) = U_+ \oplus U_-$ is a sum of two inequivalent $SO(2k,\mathbb{C})$-modules of the same dimension (they form a single $O(2k,\mathbb{C})$-module), either $U = U_+$ or $U = U_-$, and the image $I_P^G U$ of $\mathrm{H}^k(\mathscr{M}) \to I_P^G\mathrm{H}^k(\mathfrak{n},E)$ is either $I_P^G U_+$ or $I_P^G U_-$. In the notation of §4.4, U_+ and U_- are the $SO(2k,\mathbb{C})$-modules with highest weight vectors $e_{s_k} \otimes v_{s_k\lambda}$ and $e_{t_k} \otimes v_{t_k\lambda}$, respectively. Now the restriction $\mathrm{Res}_{\mathfrak{n}} : \mathrm{H}^k(\mathfrak{n},E) \to \prod_m \mathrm{H}^k(\mathfrak{n}_H,E_H)$ induced by $\mathfrak{n}_H \subset \mathfrak{n}$ is nonzero on either summand, because it is nonzero on these highest weight vectors, as was proved in Proposition 4.1. It follows that $I_P^G\mathrm{H}^k(\mathfrak{n},E) \to I_P^G I_{P_H}^P \mathrm{H}^k(\mathfrak{n}_H,E_H)$ is injective on each of $I_P^G U_\pm$ individually, and hence on the image of $\mathrm{H}^k(\mathscr{M},E) \to I_P^G\mathrm{H}^k(\mathfrak{n},E)$, whichever of these modules it is. This completes the proof of the theorem.

Remark 5.4 (on the Burger–Sarnak method for noncompact quotients). In general, it is not clear to us that the argument of [7, 8, 12, 28] can be adapted to treat noncuspidal interior cohomology classes on a general arithmetic quotient without a better understanding of the latter, for example, using Eisenstein series. Since the argument for injectivity in cohomology treats one summand π of L^2 at a time, one needs to know that π^∞, or at least π, contains some uniformly continuous functions, the diagonal matrix coefficients of which can then be used in the Burger–Sarnak argument. If π is cuspidal, then the functions in π^∞ are of rapid decrease, hence uniformly continuous, and this suffices. It is not clear to us that the automorphic representatives of noncuspidal interior cohomology classes are uniformly continuous on $\Gamma\backslash G(\mathbb{R})$ – they are not of rapid decrease, as they would then be cuspidal – or, indeed, that the summand π^∞ contributing to such cohomology contains any uniformly continuous functions.

In the case at hand, Lemma 5.2 shows that there is no noncuspidal interior cohomology, and so this problem does not occur. For the congruence ball quotients discussed in [36], [13, §3], and the next section one, can show that the noncuspidal interior cohomology consists of nonprimitive classes (see the discussion in the next section), and the analogue of Proposition 5.3 for $SU(1,m) \subset SU(1,n)$ can be proved similarly. However, for $SO(2,n)$ the situation is more complicated, and something more is required. In any case, we will not use automorphic arguments to treat interior cohomology in the cases $SU(1,n)$ and $SO(2,n)$, since the geometric arguments of §3 are available.

Remark 5.5. A triality form over a totally real field F which becomes $SO(1,7)$ over $\mathbb{R}$ for some real embedding of F is necessarily anisotropic over $\mathbb{Q}$. This follows by looking at Tits indices (see the table on [46]; for details, see [14, p. 337f]). So in the $\mathbb{Q}$-rank 1 case of the theorem, we may ignore triality altogether.

6. Congruence complex hyperbolic manifolds

The first main result for congruence complex hyperbolic manifolds is the following:

Theorem 6.1. *Suppose that $H \subset G$ are groups of the same $\mathbb{Q}$-rank and $H(\mathbb{R})^{nc} \subset G(\mathbb{R})^{nc}$ is the inclusion $SU(1,m) \subset SU(1,n)$ with $2 \leq m < n$. Then* Res *is injective on* $\mathrm{H}^i_!(\mathscr{M}_G)$ *for $i \leq m$ and on* $\mathrm{H}^i(\mathscr{M}_G)$ *for $i < m$.*

This follows immediately from Corollary 3.2 and Proposition 4.2. This is simpler than the proof in [36].

The rest of this section consists of the proof of the following, which is Theorem 1.2:

Theorem 6.2. *Suppose that $H \subset G$ are groups of the same $\mathbb{Q}$-rank and $H(\mathbb{R})^{nc} \subset G(\mathbb{R})^{nc}$ is the inclusion $SO(1,n) \subset SU(1,n)$ with $n > 2$. Then* Res *is injective on* $\mathrm{H}^{i,0}(\mathscr{M}_G)$ *for $i \leq n/2$.*

Proof. The proof is broadly the same as that of Theorem 5.1: Given the injectivity on $\mathrm{H}^{i,0}_!(\mathscr{M})$ for $i \leq n/2$, the analogue for this situation of diagram (5.2), the strictness of the Hodge filtration, and Proposition 4.4 combine to prove the theorem.

The proof of injectivity on $\mathrm{H}^{i,0}_!(\mathscr{M})$ for $i \leq n/2$ follows the outline of the proof of Proposition 5.3, with step (1) there replaced by the following:

($1'$) For each $i < n$, there is a unique cohomological $(\mathfrak{g},K)$-module $J_{i,0}$ with cohomology in bidegree $(i,0)$. This is immediate from the classification of $(\mathfrak{g},K)$-modules with cohomology for $SU(1,n)$ [17, Sections VI.4.7–VI.4.12]. For $i \leq n/2$, the abstract restriction of $J_{i,0}$ to $(\mathfrak{h},K_H)$ contains π_i^H (the unique cohomological representation for $SO(1,n)$ with $\mathrm{H}^i\left(\mathfrak{h},K_H,\pi_i^H\right) \neq 0$; see §5) as a direct summand with multiplicity 1 – that is, $\dim \mathrm{Hom}_{(\mathfrak{h},K_H)}\left(J_{i,0}|_H,\pi_i^H\right) = 1$ – and the induced map

$$\mathrm{H}^i\left(\mathfrak{g},K,J_{i,0}\right) \longrightarrow \mathrm{H}^i\left(\mathfrak{h},K_H,\pi_i^H\right)$$

is an isomorphism of one-dimensional spaces. This is [8, Théorème 5.6].

Given ($1'$), steps (2)–(5) in the proof of Proposition 5.3 work verbatim to prove that Res is injective on $\mathrm{H}^{i,0}_{cusp}(\mathscr{M})$ for $i \leq n/2$. We are using here the agreement of the two possible Hodge structures on $\mathrm{H}^i_!(\mathscr{M})$, the first coming from geometry (hence having good properties for the boundary exact sequence and in diagram (5.2)) and the second from the inclusion $\mathrm{H}^i_!(M_\Gamma) \subset \mathrm{IH}^i\left(M_\Gamma^*\right)$ and the L^2 Hodge theory on the latter coming from equation (3.12) (hence agreeing with the Hodge types in $(\mathfrak{g},K)$-cohomology). This agreement is known by [52] because M_Γ^* has isolated singularities.

Lemma 6.3 completes the proof of the theorem. □

It remains to prove the following:

Lemma 6.3. *For Γ arithmetic in $SU(1,n)$, we have* $\mathrm{H}^{*,0}_!(M_\Gamma) = \mathrm{H}^{*,0}_{cusp}(M_\Gamma)$.

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Proof. The proof is similar to that of Lemma 5.2 for $SO(1,n)$. Suppose first that $i < n$. Since $\mathrm{IH}^i(M_\Gamma^*) = \mathrm{H}^i(M_\Gamma)$ for $i < n$, we are actually dealing with the cohomology

$$\mathrm{H}^i\left(\mathfrak{g}, K, L^2_{dis}(\Gamma\backslash G(\mathbb{R}))\right) = \bigoplus_{\pi \subset L^2_{dis}} \mathrm{H}^i(\mathfrak{g}, K, \pi),$$

where the sum is over irreducible closed summands, only finitely many of which make a nonzero contribution to the sum. For π to contribute to the $(i,0)$ summand of cohomology, we must have $\pi^\infty = J_{i,0}$. For such a summand π, the theory of Eisenstein series gives a mapping $I_{i,0} \to L^2_{dis}$ onto $\pi^\infty = J_{i,0}$. Here $I_{i,0}$ denotes the standard module of which $J_{i,0}$ is the Langlands quotient [17, Section VI.4.8]. Now the minimal degree in which $J_{i,0}$ has cohomology is i, so we are in the situation of Appendix B, and applying Lemma B.1 gives Lemma 6.3 for $i < n$. (Note that the assumption $(*)$ required in Lemma B.1 holds, since $I_{i,0}$ has cohomology in degrees $2n-i, 2n-i-1$, and $I_{i,0} \to J_{i,0}$ induces an isomorphism in degree $2n-i$; see [17, p. 133].)

The statement in degrees $i > n$ follows by duality. Finally, the equality in degree n holds because the component at infinity of a class in $\mathrm{H}^{n,0}_!(M_\Gamma)$ is of discrete series type, and hence is already cuspidal by [50]. □

Remark 6.4. The proof of the lemma shows more generally that for $i + j \le n$, we have

$$\mathrm{H}^{i,j}_!(M_\Gamma)^{prim} = \mathrm{H}^{i,j}_{cusp}(M_\Gamma)^{prim},$$

where the primitive is taken with respect to an invariant Lefschetz class. Thus the Burger–Sarnak method can be applied to prove injectivity on $\mathrm{H}^{i,j}_!(M_\Gamma)^{prim}$ for the restriction by $SU(1,m) \subset SU(1,n)$. Using the action of the Lefschetz operator, this can be used to give another proof of Theorem 6.1 for complex hyperbolic manifolds.

7. Orthogonal Shimura varieties

The main theorem in this case is the following:

Theorem 7.1. *Suppose that $H \subset G$ are of the same $\mathbb{Q}$-rank and that $H(\mathbb{R})^{nc} \subset G(\mathbb{R})^{nc}$ is the inclusion $SO(2,m) \subset SO(2,n)$ with $n > m \ge 2$. Then* Res *is injective on* $\mathrm{H}^i(\mathscr{M}_G, E)$ *for $i \le m-1$.*

We will argue as if the $\mathbb{Q}$-ranks of both G and H are 2 and indicate how the argument simplifies in the $\mathbb{Q}$-rank 1 case. The proof is by a kind of induction on the stratification of the minimal compactification, going from injectivity on interior cohomology $\mathrm{H}^i_!(\mathscr{M}_G)$ (proved earlier as Corollary 3.2) to injectivity on a larger subspace $\mathrm{Gr}^W_i \mathbb{H}^i_c(\mathscr{M}^1_G, j^1_*\mathbb{C})$ of $\mathrm{H}^i(\mathscr{M}_G)$ (defined later), which takes into account some contributions from the one-dimensional boundary strata, and then to the injectivity on all of $\mathrm{H}^i(\mathscr{M}_G)$ by taking into account some contributions from the cusps (using the Lefschetz property for real hyperbolic manifolds from §5). A similar, but simpler, argument was used in [36] for ball quotients.

To simplify the notation somewhat, we will write $\mathscr{M}$ for $\mathscr{M}_G$. (We continue to write $\mathscr{M}_H$ for the Shimura variety associated to H, of course.)

7.1. Some exact sequences related to the boundary

We will introduce some notation which will be useful. Recall the stratification of M_Γ^* discussed in §2, and denote the inclusions by

$$M_\Gamma \xhookrightarrow{j_\Gamma^1} M_\Gamma^1 \xhookrightarrow{j_\Gamma^0} M_\Gamma^*,$$

with $j_\Gamma^0 \circ j_\Gamma^1 = j_\Gamma$. This gives two cohomology long exact sequences:

1. The distinguished triangle associated with $j_{\Gamma *}^1\mathbb{C}$ on M_Γ^1 and the open-closed decomposition $M_\Gamma \xhookrightarrow{j_\Gamma^1} M_\Gamma^1 \xhookleftarrow{i_\Gamma^1} Z_\Gamma^1$ is

$$j_{\Gamma !}^1\mathbb{C} \longrightarrow j_{\Gamma *}^1\mathbb{C} \longrightarrow i_{\Gamma *}^1 i_\Gamma^{1*} j_{\Gamma *}^1\mathbb{C} \xrightarrow{+1}$$

and gives

$$\cdots \longrightarrow \mathrm{H}_c^i(M_\Gamma) \longrightarrow \mathbb{H}_c^i\left(M_\Gamma^1, j_{\Gamma *}^1\mathbb{C}\right) \longrightarrow \mathbb{H}_c^i\left(Z_\Gamma^1, i_\Gamma^{1*} j_{\Gamma *}^1\mathbb{C}\right) \longrightarrow \cdots.$$

2. The distinguished triangle associated with $j_{\Gamma *}\mathbb{C}$ and $M_\Gamma^1 \xhookrightarrow{j_\Gamma^0} M_\Gamma^* \xhookleftarrow{i_\Gamma^0} Z_\Gamma^0$ is

$$j_{\Gamma !}^0 j_{\Gamma *}^1\mathbb{C} \longrightarrow j_{\Gamma *}\mathbb{C} \longrightarrow i_{\Gamma *}^0 i_\Gamma^{0*} j_{\Gamma *}^1\mathbb{C} \xrightarrow{+1}$$

and gives

$$\cdots \longrightarrow \mathbb{H}_c^i\left(M_\Gamma^1, j_{\Gamma *}^1\mathbb{C}\right) \longrightarrow \mathrm{H}^i(M_\Gamma) \longrightarrow \mathbb{H}^i\left(Z_\Gamma^0, i_\Gamma^{0*} j_{\Gamma *}\mathbb{C}\right) \longrightarrow \cdots.$$

Both are long exact sequences of mixed Hodge structures by [42].

These exact sequences are natural with respect to passing to subgroups of Γ of finite index, and this leads us to introduce the following suggestive notation:

$$\begin{aligned}\mathbb{H}_c^i\left(\mathscr{M}^1, j_*^1\mathbb{C}\right) &:= \mathrm{colim}_\Gamma \mathbb{H}_c^i\left(M_\Gamma^1, j_{\Gamma *}^1\mathbb{C}\right),\\ \mathbb{H}_c^i\left(\mathscr{Z}^1, i^{1*} j_*^1\mathbb{C}\right) &:= \mathrm{colim}_\Gamma \mathbb{H}_c^i\left(Z_\Gamma^1, i_\Gamma^{1*} j_{\Gamma *}^1\mathbb{C}\right),\\ \mathbb{H}^i\left(\mathscr{Z}^0, i^{0*} j_*\mathbb{C}\right) &:= \mathrm{colim}_\Gamma \mathbb{H}^i\left(Z_\Gamma^0, i_\Gamma^{0*} j_{\Gamma *}\mathbb{C}\right),\end{aligned} \tag{7.1}$$

where all colimits are over congruence subgroups. These are smooth $G(\mathbb{Q})$-modules, and the map $\mathrm{H}_c^*(\mathscr{M}) \to \mathrm{H}^*(\mathscr{M})$ factors through $\mathbb{H}_c^i\left(\mathscr{M}^1, j_*^1\mathbb{C}\right) \longrightarrow \mathrm{H}^i(\mathscr{M})$. The exact sequences give exact sequences

$$\cdots \longrightarrow \mathrm{H}_c^i(\mathscr{M}) \longrightarrow \mathbb{H}_c^i\left(\mathscr{M}^1, j_*^1\mathbb{C}\right) \longrightarrow \mathbb{H}_c^i\left(\mathscr{Z}^1, i^{1*} j_*^1\mathbb{C}\right) \longrightarrow \cdots \tag{7.2}$$

and

$$\cdots \longrightarrow \mathbb{H}_c^i\left(\mathscr{M}^1, j_*^1\mathbb{C}\right) \longrightarrow \mathrm{H}^i(\mathscr{M}) \longrightarrow \mathbb{H}^i\left(\mathscr{Z}^0, i^{0*} j_*\mathbb{C}\right) \longrightarrow \cdots, \tag{7.3}$$

which are exact sequences of (colimits of) mixed Hodge structures.

We also note the following useful consequence of the purity lemma (Lemma 3.5):

$$\mathrm{H}_!^i(\mathscr{M}) = \mathrm{im}\left(\mathrm{Gr}_i^W \mathrm{H}_c^i(\mathscr{M}) \to \mathrm{Gr}_i^W \mathbb{H}_c^i\left(\mathscr{M}^1, j_*^1\mathbb{C}\right)\right) \qquad \text{for} i \le n-1. \tag{7.4}$$

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Indeed, Lemma 3.5 implies that $\mathrm{H}^i_!(M_\Gamma) = \mathrm{im}\left(\mathrm{Gr}^W_i \mathrm{H}^i_c(M_\Gamma) \to \mathrm{Gr}^W_i \mathbb{H}^i_c\left(M^1_\Gamma, j^1_*\mathbb{C}\right)\right)$ for $i \le n-1$, because $\mathrm{Gr}^W_i \mathbb{H}^i_c\left(M^1_\Gamma, j^1_{\Gamma *}\mathbb{C}\right) \subset \mathrm{Gr}^W_i \mathrm{H}^i(M_\Gamma)$ by the second exact sequence given. Since Gr^W_i commutes with the colimits, we get equation (7.4).

7.2. Proof of Theorem 7.1

Now consider the situation of $H \subset G$ and the morphism $M^*_{H,\Gamma_H} \to M^*_\Gamma$. The stratifications are compatible, in the sense that the stratification of M^*_{H,Γ_H} is the pullback of that of M^*_Γ – that is, the relevant diagrams relating strata are Cartesian. It follows that both the exact sequences are functorial – that is, there are $H(\mathbb{Q})$-module maps from each exact sequence for G to the corresponding one for H. Frobenius reciprocity gives commutative diagrams of $G(\mathbb{Q})$-modules with exact rows:

$$\begin{array}{ccccccccc}
\longrightarrow & \mathrm{H}^i_c(\mathscr{M}) & \longrightarrow & \mathbb{H}^i_c\left(\mathscr{M}^1, j^1_*\mathbb{C}\right) & \longrightarrow & \mathbb{H}^i_c\left(\mathscr{Z}^1, i^{1*}j^1_*\mathbb{C}\right) & \longrightarrow \\
& \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & I^G_H \mathrm{H}^i_c(\mathscr{M}_H) & \longrightarrow & I^G_H \mathbb{H}^i_c\left(\mathscr{M}^1_H, j^1_{H*}\mathbb{C}\right) & \longrightarrow & I^G_H \mathbb{H}^i_c\left(\mathscr{Z}^1_H, i^{1*}_H j^1_{H*}\mathbb{C}\right) & \longrightarrow
\end{array}$$

from the first sequence and the similar diagram with exact rows

$$\begin{array}{ccccccccc}
\longrightarrow & \mathbb{H}^i_c\left(\mathscr{M}^1, j^1_*\mathbb{C}\right) & \longrightarrow & \mathrm{H}^i(\mathscr{M}) & \longrightarrow & \mathbb{H}^i\left(\mathscr{Z}^0, i^{0*}j_*\mathbb{C}\right) & \longrightarrow \\
& \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & I^G_H \mathbb{H}^i_c\left(\mathscr{M}^1_H, j^1_{H*}\mathbb{C}\right) & \longrightarrow & I^G_H \mathrm{H}^i(\mathscr{M}_H) & \longrightarrow & I^G_H \mathbb{H}^i\left(\mathscr{Z}^0_H, i^{0*}_H j_{H*}\mathbb{C}\right) & \longrightarrow
\end{array}$$

from the second. Taking Gr^W_i and using equation (7.4) in the first diagram gives a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{H}^i_!(\mathscr{M}) & \longrightarrow & \mathrm{Gr}^W_i \mathbb{H}^i_c\left(\mathscr{M}^1, j^1_*\mathbb{C}\right) & \longrightarrow & \mathrm{Gr}^W_i \mathbb{H}^i_c\left(\mathscr{Z}^1, i^{1*}j^1_*\mathbb{C}\right) \\
& & \downarrow{\scriptstyle \mathrm{Res}_!} & & \downarrow{\scriptstyle \mathrm{Res}^1} & & \downarrow{\scriptstyle \mathrm{Res}^1_\infty} \\
0 & \longrightarrow & I^G_H \mathrm{H}^i_!(\mathscr{M}_H) & \longrightarrow & I^G_H \mathrm{Gr}^W_i \mathbb{H}^i_c\left(\mathscr{M}^1_H, j^1_{H*}\mathbb{C}\right) & \longrightarrow & I^G_H \mathrm{Gr}^W_i \mathbb{H}^i_c\left(\mathscr{Z}^1_H, i^{1*}_H j^1_{H*}\mathbb{C}\right),
\end{array} \tag{7.5}$$

in which the upper row is exact for $i \le n-1$ and the lower row is exact for $i \le m-1$. (We have used the purity of $\mathrm{H}^i_!(\mathscr{M})$.) Similarly, taking Gr^W_i and using equation (7.4) in the second diagram gives a commutative diagram with exact rows for $i \le m-1$:

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$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{Gr}_i^W \mathbb{H}_c^i\left(\mathscr{M}^1, j_*^1\mathbb{C}\right) & \longrightarrow & \mathrm{H}^i(\mathscr{M}) & \longrightarrow & \mathrm{Gr}_i^W \mathbb{H}^i\left(\mathscr{Z}^0, i^{0*}j_*\mathbb{C}\right) \\ & \downarrow \mathrm{Res}^1 & & \downarrow \mathrm{Res} & & \downarrow \mathrm{Res}_\infty^0 \\ 0 \longrightarrow & I_H^G \mathrm{Gr}_i^W \mathbb{H}_c^i\left(\mathscr{M}_H^1, j_{H*}^1\mathbb{C}\right) & \longrightarrow & I_H^G \mathrm{Gr}_i^W \mathrm{H}^i(\mathscr{M}_H) & \longrightarrow & I_H^G \mathrm{Gr}_i^W \mathbb{H}^i\left(\mathscr{Z}_H^0, i_H^{0*} j_{H*}\mathbb{C}\right). \end{array} \tag{7.6}$$

We have used the purity of $\mathrm{H}^i(\mathscr{M})$ in degrees $\leq m-1$, which follows from the fact that $\mathrm{IH}^i(M_\Gamma^*) = \mathrm{H}^i(M_\Gamma)$ in degrees $i \leq n-2$.

We see from these diagrams that Theorem 7.1, namely the injectivity of Res on $\mathrm{H}^i(\mathscr{M})$ in degrees $\leq m-1$, follows from the conjunction of Corollary 3.2 (injectivity on interior cohomology in degrees $\leq m-1$) and the following two statements:

Proposition 7.2. *The map* Res_∞^1 *in diagram (7.5) is injective in degrees* $i \leq m-1$.

Proposition 7.3. *The map* Res_∞^0 *in diagram (7.6) is injective in degrees* $i \leq m-1$.

The rest of this subsection will be taken up with the proofs of these two propositions. The first will use Proposition 4.3, and the second will use the Lefschetz property for real hyperbolic manifolds in Theorem 5.1.

Proof of Proposition 7.2. Recall that Proposition 7.2 asserts the injectivity of

$$\mathrm{Res}_\infty^1 : \mathrm{Gr}_i^W \mathbb{H}_c^i\left(\mathscr{Z}^1, i^{1*}j_*^1\mathbb{C}\right) \longrightarrow I_H^G \mathrm{Gr}_i^W \mathbb{H}_c^i\left(\mathscr{Z}_H^1, i_H^{1*} j_{H*}^1\mathbb{C}\right)$$

in degrees $i \leq m-1$. We will deduce from Proposition 4.3 the a priori stronger assertion that the map

$$\mathbb{H}_c^i\left(\mathscr{Z}^1, i^{1*}j_*^1\mathbb{C}\right) \longrightarrow I_H^G \mathbb{H}_c^i\left(\mathscr{Z}_H^1, i_H^{1*} j_{H*}^1\mathbb{C}\right) \tag{7.7}$$

is injective in this range; since this is a morphism of mixed Hodge structures, the statement about the ith graded follows. By definition,

$$\mathbb{H}_c^i\left(\mathscr{Z}^1, i^{1*}j_*^1\mathbb{C}\right) = \mathrm{colim}_\Gamma \mathbb{H}_c^i\left(Z_\Gamma^1, i_\Gamma^{1*} j_{\Gamma*}^1\mathbb{C}\right).$$

Choose a rational boundary component F of dimension 1 and let $P = MW$ be its stabilizer, which is the maximal parabolic stabilizing an isotropic plane in V. The stratum of M_Γ^* given by F is $S_\Gamma := \Gamma_{M_h}\backslash F$, and it is a component of Z_Γ^1; let $i_{S_\Gamma} : S_\Gamma \hookrightarrow M_\Gamma^1$ be the inclusion. Then we have natural identifications

$$\begin{aligned} \mathbb{H}_c^i\left(\mathscr{Z}^1, i^{1*}j_*^1\mathbb{C}\right) &= I_P^G\left(\mathrm{colim}_\Gamma \mathbb{H}_c^i(S_\Gamma, i_{S_\Gamma}^* j_{\Gamma*}\mathbb{C}_{M_\Gamma})\right) \\ &= \bigoplus_k I_P^G(\mathrm{colim}_\Gamma \mathbb{H}_c^{i-k}(S_\Gamma, \mathrm{H}^k(i_{S_\Gamma}^* j_{\Gamma*}\mathbb{C}_{M_\Gamma}))) \\ &= \bigoplus_k I_P^G(\mathrm{colim}_\Gamma \mathrm{H}_c^{i-k}(S_\Gamma, \mathrm{H}^k(\mathfrak{w}, \mathbb{C}))) \\ &= \bigoplus_k I_P^G \mathrm{H}_c^{i-k}(\mathscr{M}_M, \mathrm{H}^k(\mathfrak{w}, \mathbb{C})). \end{aligned} \tag{7.8}$$

Here the first equality is an elementary argument keeping track of the cusps (see [35, Lemma 3.3] for a similar argument for ball quotients) and the second is given by Proposition 2.7. We have also used the fact that $M_\ell(\mathbb{R})$ is compact, so that $\Gamma_{M_\ell A} = \{e\}$ for neat Γ. There is a similar expression for the $H(\mathbb{Q})$-module $\mathbb{H}^i_c\left(\mathscr{Z}^1_H, i^{1*}_H j^1_{H*}\mathbb{C}\right)$ and hence for the target of expression (7.7), namely

$$I^G_H \mathbb{H}^i_c\left(\mathscr{Z}^1_H, i^{1*}_H j^1_{H*}\mathbb{C}\right) = \bigoplus_k I^G_H I^H_{P_H}\left(\operatorname{colim}_{\Gamma_H} \mathrm{H}^{i-k}_c\left(S_{\Gamma_H}, \mathrm{H}^k(\mathfrak{w}_H, \mathbb{C})\right)\right).$$

By the transitivity of induction $I^G_H I^H_{P_H} = I^G_P I^P_{P_H}$ we are reduced to showing that

$$\operatorname{colim}_\Gamma \mathrm{H}^{i-k}_c\left(S_\Gamma, \mathrm{H}^k(\mathfrak{n}, \mathbb{C})\right) \longrightarrow I^P_{P_H} \operatorname{colim}_{\Gamma_H} \mathrm{H}^{i-k}_c\left(S_{\Gamma_H}, \mathrm{H}^k(\mathfrak{n}_H, \mathbb{C})\right)$$

is injective for $i \leq m-1$. Now the action of $W(\mathbb{Q})$ on the left-hand side is trivial, and

$$\left(I^P_{P_H} \mathrm{H}^{i-k}_c\left(S_\Gamma, \mathrm{H}^k(\mathfrak{n}_H, \mathbb{C})\right)\right)^{W(\mathbb{Q})} = I^M_{M_H} \mathrm{H}^{i-k}_c\left(S_{\Gamma_H}, \mathrm{H}^k(\mathfrak{n}_H, \mathbb{C})\right),$$

so we must show that

$$\operatorname{colim}_\Gamma \mathrm{H}^{i-k}_c\left(S_\Gamma, \mathrm{H}^k(\mathfrak{n}, \mathbb{C})\right) \longrightarrow I^M_{M_H} \operatorname{colim}_{\Gamma_H} \mathrm{H}^{i-k}_c\left(S_{\Gamma_H}, \mathrm{H}^k(\mathfrak{n}_H, \mathbb{C})\right)$$

is injective for $i \leq m-1$. But now note that $S_\Gamma = S_{\Gamma_H}$ for neat Γ (M and M_H differ only in the compact factor), and the case $i = k$ does not occur (because $\mathrm{H}^0_c(S_\Gamma, V) = 0$ for any V), so this follows from Proposition 4.3. □

Proof of Proposition 7.3. Recall that the proposition asserts the injectivity of

$$\mathrm{Res}^0_\infty : \mathrm{Gr}^W_i \mathbb{H}^i\left(\mathscr{Z}^0, i^{0*} j_*\mathbb{C}\right) \longrightarrow I^G_H \mathrm{Gr}^W_i \mathbb{H}^i\left(\mathscr{Z}^0_H, i^{0*}_H j_{H*}\mathbb{C}\right)$$

in degrees $i \leq m-1$. We will reduce this to the Lefschetz property for congruence hyperbolic manifolds – that is, Theorem 5.1.

Let $P = MW$ be the stabilizer of an isotropic line I in V, and F be the associated rational boundary component. Let $i_{s_\Gamma} : \{s_\Gamma\} \hookrightarrow M^1_\Gamma$ be the stratum given by F in M^*_Γ. Then by Proposition 2.7 there is an isomorphism in the derived category

$$i^*_{S_\Gamma} j_{\Gamma*}\mathbb{C} = \bigoplus_k \mathrm{H}^k\left(i^*_{S_\Gamma} j_{\Gamma*}\mathbb{C}\right)[-k],$$

and moreover,

$$\mathrm{H}^i\left(i^*_{s_\Gamma} j_{\Gamma*}\mathbb{C}\right) = \bigoplus_k \mathrm{H}^{i-k}\left(\Gamma_M, \wedge^k \mathfrak{u}^*_\mathbb{C}\right),$$

where we have used the facts that $\Gamma_M = \Gamma_{M_\ell}$ (assuming Γ is neat) is in $SO(1,n-1)$ and $\mathfrak{w} = \mathfrak{u}$ is abelian, so that $\mathrm{H}^*(\mathfrak{w}, \mathbb{C}) = \wedge^* \mathfrak{u}^*$. By Proposition 2.7, the k-summand is pure of weight $2k$, so that

$$\mathrm{Gr}^W_i \mathrm{H}^i\left(i^*_{s_\Gamma} j_{\Gamma*}\mathbb{C}\right) = \mathrm{H}^{i/2}\left(\Gamma_M, \wedge^{i/2} \mathfrak{u}^*_\mathbb{C}\right)$$

if i is even and zero if i is odd.

We then have

$$\begin{aligned}\mathrm{Gr}_i^W \mathbb{H}^i\left(\mathscr{Z}^0, i^{0*} j_* \mathbb{C}\right) &= I_P^G \mathrm{colim}\mathrm{H}^{i/2}\left(\Gamma_M, \wedge^{i/2}\mathfrak{u}_{\mathbb{C}}^*\right)\\ &= I_P^G \mathrm{H}^{i/2}\left(\mathscr{M}_M, \wedge^{i/2}\mathfrak{u}_{\mathbb{C}}^*\right)\end{aligned}$$

for i even and zero for i odd. This discussion applies also to $\mathrm{Gr}_i^W \mathbb{H}^i\left(\mathscr{Z}_H^0, i_H^{0*} j_{H*}\mathbb{C}\right)$, giving

$$\begin{aligned}\mathrm{Gr}_i^W \mathbb{H}^i\left(\mathscr{Z}^0, i^{0*} j_* \mathbb{C}\right) &= I_{P_H}^H \mathrm{colim}\mathrm{H}^{i/2}\left(\Gamma_{M_H}, \wedge^{i/2}\mathfrak{u}_{H,\mathbb{C}}^*\right)\\ &= I_P^G \mathrm{H}^{i/2}\left(\mathscr{M}_{M_H}, \wedge^{i/2}\mathfrak{u}_{H,\mathbb{C}}^*\right)\end{aligned}$$

for i even and zero for i odd. By the transitivity of induction, the injectivity of Res_∞^0 in degree i is reduced to that of

$$\mathrm{H}^{i/2}\left(\mathscr{M}_M, \wedge^{i/2}\mathfrak{u}_{\mathbb{C}}^*\right) \longrightarrow I_{P_H}^P \mathrm{H}^{i/2}\left(\mathscr{M}_{M_H}, \wedge^{i/2}\mathfrak{u}_{H,\mathbb{C}}^*\right).$$

The action of $W(\mathbb{Q}) = U(\mathbb{Q})$ on the source is trivial, so this factors through the $U(\mathbb{Q})$-invariants of the target – that is,

$$\left(I_{P_H}^P \mathrm{colim}\mathrm{H}^{i/2}\left(\Gamma_{M_H}, \wedge^{i/2}\mathfrak{u}_{H,\mathbb{C}}^*\right)\right)^{U(\mathbb{Q})} = I_{M_H}^M \mathrm{colim}\mathrm{H}^{i/2}\left(\Gamma_{M_H}, \wedge^{i/2}\mathfrak{u}_{H,\mathbb{C}}^*\right).$$

We are thus reduced to the injectivity of

$$\mathrm{colim}\mathrm{H}^{i/2}\left(\Gamma_M, \wedge^{i/2}\mathfrak{u}_{\mathbb{C}}^*\right) \longrightarrow I_{M_H}^M \mathrm{colim}\mathrm{H}^{i/2}\left(\Gamma_{M_H}, \wedge^{i/2}\mathfrak{u}_{H,\mathbb{C}}^*\right).$$

Suppose $i \leq m-1$. Then $\wedge^{i/2}\mathfrak{u}_{\mathbb{C}}^*$ is irreducible and $\wedge^{i/2}\mathfrak{u}_{H,\mathbb{C}}^*$ is the M_H-summand containing the M-highest weight vector, so the injectivity of the previous map follows from the Lefschetz property in Theorem 5.1. (The subgroup M_H is never triality.) This concludes the proof. □

This concludes the proof of Theorem 7.1.

Appendix A. Some facts about the L^2 cohomology of arithmetic manifolds

Let G be a semisimple algebraic group over $\mathbb{Q}$, K a maximal compact subgroup of $G(\mathbb{R})$, $X = G(\mathbb{R})/K$, and $\Gamma \subset G(\mathbb{Q})$ a congruence subgroup. (Everything in this appendix should apply to any arithmetic subgroup, but we will use results from [21], which is written in an adelic context, so we make this assumption at the outset.)

The L^2 cohomology of Γ with coefficients in a finite-dimensional algebraic representation E of $G(\mathbb{C})$ is

$$\mathrm{H}_{(2)}^*(\Gamma, E) = \mathrm{H}^*\left(\mathfrak{g}, K, L^2(\Gamma\backslash G(\mathbb{R})) \otimes E\right).$$

(The $(\mathfrak{g}, K)$-cohomology of a $G(\mathbb{R})$-representation (π, V) is, by definition, that of the space V^∞ of smooth vectors.) This is not the usual definition, which requires looking at the complex of L^2 differential forms with L^2 weak differential (or smooth L^2 forms with L^2 differential), but they are known to agree [16, Prop. 5.4]. The decompositions

$$\begin{aligned}L^2(\Gamma\backslash G(\mathbb{R})) &= L_{dis}^2(\Gamma\backslash G(\mathbb{R})) \oplus L_{cts}^2(\Gamma\backslash G(\mathbb{R}))\\ &= L_{cusp}^2(\Gamma\backslash G(\mathbb{R})) \oplus L_{res}^2(\Gamma\backslash G(\mathbb{R})) \oplus L_{cts}^2(\Gamma\backslash G(\mathbb{R}))\end{aligned}$$

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of the L^2 spectrum into cuspidal, residual, and continuous parts (given by Langlands' theory of Eisenstein series) induce decompositions

$$\begin{aligned}\mathrm{H}^*_{(2)}(\Gamma,E) &= \mathrm{H}^*_{dis}(\Gamma,E)\oplus \mathrm{H}^*_{cts}(\Gamma,E)\\ &= \mathrm{H}^*_{cusp}(\Gamma,E)\oplus \mathrm{H}^*_{res}(\Gamma,E)\oplus \mathrm{H}^*_{cts}(\Gamma,E),\end{aligned}$$

where $\mathrm{H}^*_?(\Gamma,E) = \mathrm{H}^*\left(\mathfrak{g},K,L^2_?(\Gamma\backslash G(\mathbb{R}))\otimes E\right)$. The summand $\mathrm{H}^*_{dis}(\Gamma,E)$ is identified with the (finite-dimensional) space of E-valued L^2 harmonic forms for the invariant metric (a result due to Borel and Garland; see [16, Prop. 4.4(i)]), whereas the summand $\mathrm{H}^*_{cts}(\Gamma,E)$ either vanishes (e.g., when G is equal-rank) or is infinite-dimensional [16]. The inclusion $L^2(\Gamma\backslash G(\mathbb{R}))^\infty \subset C^\infty(\Gamma\backslash G(\mathbb{R}))$ induces a natural map

$$\mathrm{H}^*_{(2)}(\Gamma,E)\longrightarrow \mathrm{H}^*(\Gamma,E)$$

which is injective on cuspidal cohomology. Proposition A.1 is well known to experts, but for lack of a suitable reference we give a proof, which is a simple matter of applying results of [21] and [16]. It goes beyond the existing literature [16] only in the cases where $G(\mathbb{R})$ does not have a discrete series – for example, $GL(n)$ – and we need to use it in the main body of the paper for the case $SO(1,d)$, d odd.

Proposition A.1. *The map* $\mathrm{H}^*_{(2)}(\Gamma,E)\to \mathrm{H}^*(\Gamma,E)$ *is zero on* $\mathrm{H}^*_{cts}(\Gamma,E)$.

Proof. Following Franke [21], let $S_1(\Gamma\backslash G(\mathbb{R}))\subset L^2(\Gamma\backslash G(\mathbb{R}))$ be the submodule of smooth functions which are uniformly in L^2 – that is, smooth functions f such that $Df\in L^2$ for all $D\in U(\mathfrak{g})$. The inclusion $S_1\subset L^2$ induces isomorphisms in cohomology (by [21, Theorem 3]), so $S_1\subset C^\infty$ induces the map in question. It factors as

$$S_1(\Gamma\backslash G(\mathbb{R}))\subset S_{\log}(\Gamma\backslash G(\mathbb{R}))\subset C^\infty(\Gamma\backslash G(\mathbb{R})),$$

where $S_{\log} = S_{\log}(\Gamma\backslash G(\mathbb{R}))$ is the space of functions which are uniformly L^2 up to logarithmic terms (see [21, §5] or [49, Section 6.1]). It suffices to show that $\mathrm{H}^*_{cts}(\Gamma,E)$, as a summand of $\mathrm{H}^*(\mathfrak{g},K,S_1\otimes E)$, goes to zero in $\mathrm{H}^*(\mathfrak{g},K,S_{\log}\otimes E)$. We will do this using further reductions to the bounded spectra $S_{1,b}\subset S_1$ and $S_{\log,b}\subset S_{\log}$ with respect to the Casimir operator, a notion introduced in [21, Section 5.1] (compare [49, Section 6.3]). We will show the following:

1. $S_{1,b}\subset L^2$ induces an isomorphism $\mathrm{H}^*(\mathfrak{g},K,S_{1,b}\otimes E) = \mathrm{H}^*_{(2)}(\Gamma,E)$.
2. $S_{\log,b}\subset S_{\log}$ induces an isomorphism in cohomology.
3. $S_{1,b}\subset S_{\log,b}$ induces zero on $\mathrm{H}^*_{cts}(\Gamma,E)$.

This will prove the proposition.

(1) To show that $S_{1,b}\subset L^2$ induces an isomorphism in cohomology, we use Langlands spectral decomposition of L^2. There are compatible direct sum decompositions

$$S_{1,b} = \bigoplus_{\{R\}} S_{1,b,\{R\}} \subset L^2 = \bigoplus_{\{R\}} L^2_{\{R\}}$$

indexed by associate classes of parabolic subgroups. The $R=G$ summands are $L^{2,\infty}_{dis}$ and L^2_{dis}. (In the usual statement of $\{R\}$-decompositions, the $R=G$ summand is the cuspidal

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part; we are using the obvious rearrangement.) The inclusion $L^{2,\infty}_{dis} \subset L^2_{dis}$ induces an isomorphism by definition, so we must show the same for $S_{1,b,\{P\}} \subset L^2_{\{P\}}$ for proper P. For $P = MAN$, we have (compare [21, Theorem 11])

$$S_{1,b,\{P\}} = \left(\mathrm{Ind}_P^G L^2_c(i\mathfrak{a}^*) \otimes A_2^M\right)^{W_M},$$

where A_2^M is the space of L^2 automorphic forms on $\Gamma_M \backslash M(\mathbb{R})$, and the Weyl group W_M of M acts by intertwining operators. The $\{P\}$-summand of L^2 has the following description (see, e.g., [16, Section 4.3]). For each irreducible summand V of $L^2_{dis}(\Gamma_M\backslash M(\mathbb{R}))$, we have the direct integral

$$E_{P,V} = \int_{\mathfrak{a}^*}^{\oplus} \mathrm{Ind}_P^G \mathbb{C}_{\rho+i\mu} \otimes V.$$

Then $L^2_{\{P\}}$ is the invariants under the action of W_M by intertwining operators on the (countable) Hilbert-space direct sum of $E_{P,V}$ as V varies over all irreducible summands. One could, equivalently, restrict to a subset of V modulo W_M-equivalence and take the sum of direct integrals like this one over the positive Weyl chamber $\mathfrak{a}^{*+}$, which is the formulation in [16, Section 4.3].

Now $\mathrm{H}^*(\mathfrak{g}, K, E_{P,V} \otimes E) = \{0\}$ unless $E_{P,V}$ shares K-types with $\wedge^*(\mathfrak{g}/\mathfrak{k}) \otimes E^*$ and the Casimir acts by the correct scalar, so it follows that there is a finite set $\{V_i\}_{i\in I}$ of V such that the cohomology becomes a finite sum

$$\mathrm{H}^*\left(\mathfrak{g}, K, L^2_{\{P\}} \otimes E\right) = \left(\bigoplus\nolimits_{i\in I} \mathrm{H}^*(\mathfrak{g}, K, E_{P,V_i} \otimes E)\right)^{W_M}$$

(see [16, Prop. 4.4(ii)]). The same applies to $S_{1,b,\{P\}}$ – that is, we may replace A_2^M by $\bigoplus_{i\in I} A_2^M \cap V_i^\infty$ and get the same cohomology. The computation of a single summand $\mathrm{H}^*(\mathfrak{g}, K, E_{P,V_i} \otimes E)$ is contained in [16, Theorem 3.4], and the answer is similar to the usual computation for induced representations in [17, Theorem III.3.3]: There is a unique $s \in W^P$ such that

$$\mathrm{H}^*(\mathfrak{g}, K, E_{P,V_i} \otimes E) = \mathrm{H}^{*-\ell(s)}\left(\mathfrak{m}, K_M, V_i \otimes E^M_{s(\lambda+\rho)-\rho}\right) \otimes \mathrm{H}^*\left(\mathfrak{a}, \int_{\mathfrak{a}^*}^{\oplus} \mathbb{C}_{i\mu} d\mu\right),$$

where $E^M_{s(\lambda+\rho)-\rho}$ is the restriction to M of $E^{MA}_{s(\lambda+\rho)-\rho}$ (notation as in Kostant's theorem in §4.1). The parallel computation for $\mathrm{Ind}_P^G L^2_c(i\mathfrak{a}^*) \otimes V_i$ (by the same arguments as in the proof of [16, Theorem 3.4]) gives the same expression, with $\int_{\mathfrak{a}^*}^{\oplus} \mathbb{C}_{i\mu} d\mu$ replaced by $L^2_c(i\mathfrak{a}^*)$. The assertion that $S_{1,b,\{P\}} \subset L^2_{\{P\}}$ induces an isomorphism now boils down to the assertion that (for each P) the inclusion of $L^2_c(i\mathfrak{a}^*) = \mathrm{colim}_{\Omega \subset \mathfrak{a}^*} \int_{\Omega}^{\oplus} \mathbb{C}_{i\mu} d\mu$ (the colimit taken over compact Ω) into the direct integral $\int_{\mathfrak{a}^*}^{\oplus} \mathbb{C}_{i\mu} d\mu$ is an isomorphism in $\mathfrak{a}$-cohomology. This elementary fact follows, for example, from [16], which shows that this is already true of $\int_{\Omega}^{\oplus} \mathbb{C}_{i\mu} d\mu \subset \int_{\mathfrak{a}^*}^{\oplus} \mathbb{C}_{i\mu} d\mu$ if $0 \in \Omega$.

(2) To show that the inclusion $S_{\log,b} \subset S_{\log}$ induces an isomorphism in cohomology, we simply combine [21, Theorem 10] and the spectral sequence in [21, Theorem 7]. (This may not be the simplest or most natural proof, but it is certainly the shortest to write down here!)

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(3) We are reduced to considering the inclusion $S_{1,b} \subset S_{\log,b}$. Now $S_{\log,b}$ has a spectral decomposition analogous to that of $S_{1,b}$, in which the $\{P\}$-summand for $P = MAN$ is

$$\left(\mathrm{Ind}_P^G D_c'(i\mathfrak{a}^*) \otimes A_2^M\right)^{W_M},$$

where $D_c'(i\mathfrak{a}^*)$ is the space of compactly supported distributions (compare [21, Theorem 12]). The $P = G$ summand is still $L^{2,\infty}$. So the fact that $S_{1,b,\{P\}} \subset S_{\log,b,\{P\}}$ induces zero for P proper boils down to the fact that the inclusion of $\mathfrak{a}$-modules $L_c^2(i\mathfrak{a}^*) \subset D_c'(i\mathfrak{a}^*)$ induces zero in $\mathfrak{a}$-cohomology. This is immediate: The first has cohomology in degrees in $[1, \dim\mathfrak{a}]$ (by [16, Prop. 3.2], as already remarked), and the latter has cohomology only in degree 0 (e.g., by [21, Lemma 1]; this reduces to the fact that the complex $D_c'(\mathbb{R}) \xrightarrow{x\cdot} D_c'(\mathbb{R})$ has cohomology only in degree 0). This concludes the proof of the proposition. □

The preceding proof used three results [21, Theorems 10, 12, and 13]) whose proofs constitute the technical heart of Franke's work. It is possible that they can be avoided, but the method of proof gives rather more, as we now show. The results which follow are not used in the body of the paper, but will be useful elsewhere. The following is a corollary of the proof of the proposition:

Corollary A.2. *If E is rationally defined (i.e., has a rational structure preserved by $G(\mathbb{Q})$) then the square-integrable cohomology, which is (by definition) the image of $\mathrm{H}^*_{(2)}(\Gamma, E) \to \mathrm{H}^*(\Gamma, E)$, is a rational subspace.*

Proof. By [33, Theorem A], the cohomology of $S_{\log}$ is isomorphic to the lower middle weighted cohomology of [23], and this has a rational structure ([23, Chapter IV]) compatible with the map to $\mathrm{H}^*(\Gamma, E)$. □

In fact, we can refine this statement somewhat using the same methods. Recall that there is a subspace $S_{-\log}(\Gamma\backslash G(\mathbb{R})) \subset S_{\log}(\Gamma\backslash G(\mathbb{R}))$ defined by using a condition dual to the one defining $S_{\log}$ (see [21, §5] or [49, Section 6.1]).

Proposition A.3. *The image of $\mathrm{H}^*(\mathfrak{g}, K, S_{-\log} \otimes E) \to \mathrm{H}^*(\mathfrak{g}, K, S_{\log} \otimes E)$ is identified with $\mathrm{H}^*_{dis}(\Gamma, E)$ or, equivalently, with the space of E-valued L^2 harmonic forms.*

Proof. This was proved in [33, Theorem B] under the assumption that G is equal-rank, in which case the map in the proposition is an isomorphism and both groups compute the L^2 cohomology. In general, we argue as follows. By [21, Theorem 10] and the spectral sequence of [21, Theorem 7], we know that $S_{\pm\log,b} \subset S_{\pm\log}$ induce isomorphisms in $(\mathfrak{g}, K)$-cohomology, so it is enough to consider the inclusion $S_{-\log,b} \subset S_{\log,b}$. For these spaces there are compatible decompositions

$$S_{-\log,b} = \bigoplus_{\{P\}} S_{-\log,b,\{P\}} \subset S_{\log,b} = \bigoplus_{\{P\}} S_{\log,b,\{P\}}$$

indexed by associate classes of parabolic subgroups. The $\{P\}$-summand for $S_{-\log,P}$ is

$$\left(\mathrm{Ind}_P^G C_c^\infty(i\mathfrak{a}^*) \otimes A_2^M\right)^{W_M}.$$

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The $\{P\}$-summand for $S_{\log,b}$ was recalled in the proof of the previous proposition and amounts to replacing $C_c^\infty(i\mathfrak{a}^*)$ by $D_c'(i\mathfrak{a}^*)$ in this expression; the map $S_{-\log,b,\{P\}} \subset S_{\log,b,\{P\}}$ is induced by $C_c^\infty(i\mathfrak{a}^*) \subset D_c'(i\mathfrak{a}^*)$. The map in $\mathfrak{a}$-cohomology induced by this inclusion is zero, since $\mathrm{H}^*(\mathfrak{a}, C_c^\infty(i\mathfrak{a}^*))$ is concentrated in degree $\dim \mathfrak{a}$ while $\mathrm{H}^*(\mathfrak{a}, D_c'(i\mathfrak{a}^*))$ is concentrated in degree 0. By the standard computations of cohomology for induced representations (recalled earlier in the proof of the previous proposition), we see that $S_{-\log,b,\{P\}} \subset S_{\log,b,\{P\}}$ is zero in cohomology for $P \neq G$. Since the $P = G$ summands are both identified with $L^{2,\infty}_{dis}$, the first statement follows. □

The previous corollary is refined by the following:

Corollary A.4. *If E is rationally defined, then the space of square-integrable E-valued harmonic forms on $\Gamma\backslash X$ has a canonical (Betti) rational structure.*

Proof. By [33, Theorem A], the groups in the proposition are the upper and lower middle weighted cohomology groups, which have natural $\mathbb{Q}$-structures ([23, Chapter IV]). □

Remark A.5. In contrast to the corollary, the space of cuspidal harmonic forms, which is simply the cuspidal cohomology, should not be expected to be Betti rational in general, for example, for Sp(4). Of course, it is well known to be Betti rational in the case of $GL(n)$ [21, 7.6] and some related cases – for example, for $SO(1,d)$ it follows from Lemma 5.2.

Remark A.6. When $X = G(\mathbb{R})/K$ has an Hermitian structure, something much stronger than the corollary is true thanks to equation (3.12), namely that the space of L^2 harmonic forms is part of a mixed realization over the number field of definition of $\Gamma\backslash X$ (the reflex field, if we work in the context of Shimura varieties). In particular, it has both Betti and de Rham rational structures.

Appendix B. Residual Eisenstein cohomology in corank 1

We summarize here some very well-known facts (essentially going back to [26]) on the construction of cohomology via residual Eisenstein series from cuspidal data on maximal parabolic subgroups. In the body of the paper these are applied to the rank 1 groups $SO(1,n)$ and $SU(1,n)$.

For a $G(\mathbb{R})$-representation V, the smooth vectors are denoted V^∞ and $\mathrm{H}^*(\mathfrak{g},K,V)$ is the $(\mathfrak{g},K)$-cohomology of V^∞. For a $(\mathfrak{g},K)$-module or $G(\mathbb{R})$-representation V, let

$$d_{\min}(V) = \min\left\{i : \mathrm{H}^i(\mathfrak{g},K,V) \neq 0\right\}, \qquad d_{\max}(V) = \max\left\{i : \mathrm{H}^i(\mathfrak{g},K,V) \neq 0\right\},$$

assuming these make sense and are finite.

Let $P = LN \subset G$ be a maximal parabolic subgroup and A the $\mathbb{Q}$-split part of the center of L. Let $\sigma \subset L^2_{cusp}\left(\Gamma_L A(\mathbb{R})^0\backslash L(\mathbb{R})\right)$ be a cuspidal automorphic form on L. For $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, let $I_\lambda = \mathrm{Ind}_P^G \sigma \otimes \mathbb{C}_\lambda$ be the (normalized) induced representation. The theory of Eisenstein series produces a $(\mathfrak{g},K)$-homomorphism to the space of automorphic forms,

$$\mathbf{E} : I_\lambda^\infty \to \mathscr{A}(\Gamma\backslash G(\mathbb{R})),$$

which is meromorphic in $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$. If λ is a pole of $\mathbf{E}$, meaning that the Eisenstein series $E(\phi,\lambda)$ has a pole for generic ϕ in the space of σ, then P is self-associate and the pole is real and simple if $Re(\lambda) \in (\mathfrak{a}^*)^+$. Taking the residue at such a λ defines a residual Eisenstein operator

$$\mathbf{E}^* : I_\lambda^\infty \to \mathscr{A}(\Gamma \backslash G(\mathbb{R})).$$

By [21], the $(\mathfrak{g},K)$-cohomology of $\mathscr{A}(\Gamma\backslash G(\mathbb{R})) \otimes E$ is $\mathrm{H}^*(\Gamma,E)$.

Lemma B.1. *Suppose that P,σ are as discussed and that $\mathbf{E}$ has a pole at λ with $Re(\lambda) \in (\mathfrak{a}^*)^+$, so that λ is real. Suppose that the Langlands quotient J_λ of I_λ is cohomological with respect to a coefficient representation E, and further that*

(∗) $d_{\max}(I_\lambda \otimes E) = d_{\max}(J_\lambda \otimes E)$ *and* $\mathrm{H}^{d_{\max}(I_\lambda\otimes E)}(\mathfrak{g},K,I_\lambda \otimes E) \to \mathrm{H}^{d_{\max}(J_\lambda\otimes E)}(\mathfrak{g},K,J_\lambda \otimes E)$ *is an isomorphism*

(see Remark B.2). Then the map in cohomology induced by $\mathbf{E}^*(I_\lambda^\infty) \subset \mathscr{A}$ in degree $d_{\min}(J_\lambda \otimes E)$ is injective and the nonzero classes in the subspace

$$\mathrm{H}^{d_{\min}(J_\lambda\otimes E)}(\mathfrak{g},K,\mathbf{E}^*(I_\lambda^\infty)\otimes E) \subset \mathrm{H}^{d_{\min}(J_\lambda\otimes E)}(\Gamma,E)$$

restrict nontrivially to the boundary – that is, do not belong to $\mathrm{H}_!^*(\Gamma,E)$.

Proof. We write I,J for I_λ,J_λ and ignore the coefficients E, as they are not relevant. The residue of a cuspidal Eisenstein series at a point of the positive Weyl chamber is well known to be square-integrable, so that $\mathbf{E}^*(I^\infty) \subset \mathscr{A} \cap L^2_{dis}$, and as an abstract representation, $\mathbf{E}^*(I^\infty)$ is the Langlands quotient J^∞. Taking the constant term of automorphic forms along P defines a mapping

$$I^\infty \xrightarrow{\mathbf{E}^*} \mathscr{A} \longrightarrow I^{*,\infty},$$

where $I^* = \mathrm{Ind}_P^G \sigma^* \otimes \mathbb{C}_{-\lambda}$ is the contragredient of I. (The usual expression for the constant term defines maps $I_\lambda \to \mathscr{A} \to I_\lambda \oplus I_\lambda^*$ for generic λ, but for the residual operator at a pole only the second term is nonvanishing.) The composite is a nonzero multiple of the standard interwining operator $I \to I^*$, the image of which is the Langlands quotient J, and the factoring is exactly $I^\infty \to J^\infty \subset I^{*,\infty}$.

Now by assumption $d_{\max}(I) = d_{\max}(J)$, and so by duality ([17, Proposition I.7.6] for the irreducible unitary module J and [17, Theorem III.3.3 and Proposition I.7.6] for I) we have

$$d_{\min}(J) = \dim X - d_{\max}(J) = \dim G(\mathbb{R})/K - d_{\max}(I) = d_{\min}(I^*)$$

and $\mathrm{H}^{d_{\min}(J)}(\mathfrak{g},K,J) \cong \mathrm{H}^{d_{\min}(J)}(\mathfrak{g},K,I^*)$. Moreover, for a class in $\mathrm{H}^{d_{\min}(J)}(\mathfrak{g},K,\mathbf{E}^*(I))$ in $\mathrm{H}^*(\Gamma,\mathbb{C})$, the induced mapping

$$C^*(\mathfrak{g},K,\mathbf{E}^*(I^\infty)) \to C^*(\mathfrak{g},K,I^{*,\infty})$$

gives, via the identification of $\mathrm{H}^*(\mathfrak{g},K,I^{*,\infty})$ with a summand of the cohomology of the P-boundary, the restriction of the class to the P-boundary. (This is contained in [26] in a differential-geometric language and in [45] in representation-theoretic terms.) The restriction is therefore nonzero, because $\mathrm{H}^{d_{\min}(J)}(\mathfrak{g},K,J) \cong \mathrm{H}^{d_{\min}(J)}(\mathfrak{g},K,I^*)$, and so $\mathrm{H}^{d_{\min}(J)}(\mathfrak{g},K,\mathbf{E}^*(I)) \to \mathrm{H}^*(\Gamma,\mathbb{C})$ is injective. The classes in this subspace survive on restriction to the boundary, so they are not in interior cohomology. □

Remark B.2. In fact, $(*)$ always holds for a unitary cohomological Langlands quotient, but rather than prove this general fact, we just check it in the cases of interest where the lemma is applied.

Appendix C. Chern classes of automorphic vector bundles

We are in the situation of §3.2: G is semisimple and simply connected, $X = G(\mathbb{R})/K$ is Hermitian, and $M_\Gamma = \Gamma\backslash X$. Fix a smooth toroidal compactification $M_\Gamma \hookrightarrow M_\Gamma^\Sigma$ in which the boundary is a simple normal crossings divisor [3]. Let $\mathrm{Rep}(H)$ denote the category of finite-dimensional representations of a compact group H. With E in $\mathrm{Rep}(K)$ are associated the homogeneous bundle $\mathscr{E}^c$ on X^c, the automorphic vector bundle $\mathscr{E}_\Gamma$ on $M_\Gamma = \Gamma\backslash X$, and the canonical extension $\mathscr{E}_\Gamma^\Sigma$ of $\mathscr{E}_\Gamma$ to M_Γ^Σ. The following statement is well known to experts; we borrow the proof from [34, Lemma 3.7.2]):

Lemma C.1. *There is an injective ring homomorphism* $\theta : \mathrm{H}^*(X^c, \mathbb{Q}) \to \mathrm{H}^*\left(M_\Gamma^\Sigma, \mathbb{Q}\right)$ *with* $\theta(c_k(\mathscr{E}^c)) = (-1)^k c_k\left(\mathscr{E}_\Gamma^\Sigma\right)$ *for all* $E \in \mathrm{Rep}(K)$.

Proof. Following a suggestion of N. Fakhruddin we will use K-theory to prove this. Let $K^0(-)$ denote the topological K-theory of a space and $ch : K^0(-) \to \mathrm{H}^*(-,\mathbb{Q})$ the Chern character homomorphism. We write $R(H)$ for the representation ring of a compact group H – that is, the Grothendieck group of the category $\mathrm{Rep}(H)$.

We first show that the ring homomorphism $R(K) \to \mathrm{H}^*\left(M_\Gamma^\Sigma, \mathbb{Q}\right)$ defined by $V \mapsto ch\left(\mathscr{V}_\Gamma^\Sigma\right)$ and extended $\mathbb{Q}$-linearly defines a ring homomorphism

$$\kappa : K^0(X^c) \otimes \mathbb{Q} \to \mathrm{H}^*\left(M_\Gamma^\Sigma, \mathbb{Q}\right). \tag{C.1}$$

Since $X^c = G(\mathbb{R})^c/K$ with $G(\mathbb{R})^c$ simply connected (it is the maximal compact of the simply connected group $G(\mathbb{C})$) and $K \subset G(\mathbb{R})^c$ is a subgroup of maximal rank, the construction $V \mapsto \mathscr{V}^c$ gives an isomorphism

$$R(K) \otimes_{R(G(\mathbb{R})^c)} \mathbb{Z} \to K^0(X^c),$$

where $\mathbb{Z}$ is an $R(G(\mathbb{R})^c)$-module via the dimension homomorphism (by [40, Theorem 3]). Since the left-hand side is the quotient of $R(K)$ by the ideal generated by $\ker(\dim : R(G(\mathbb{R})^c) \to \mathbb{Z})$ and ch is a ring homomorphism, to show that κ is well defined it suffices to show that $ch\left(\mathscr{E}^\Sigma\right) = \dim E$ if $E \in \mathrm{Rep}(G(\mathbb{R})^c)$. The degree 0 term of the Chern character of a bundle is its rank, so it suffices to check that $c_k\left(\mathscr{E}_\Gamma^\Sigma\right) = 0$ for $k > 0$ for such E. Now Mumford showed that the kth Chern form of the invariant or Nomizu connection defines a current on M_Γ^Σ which represents $\left(\text{up to a factor of } \left(2\pi\sqrt{-1}\right)^k\right)$ the Chern class $c_k\left(\mathscr{E}_\Gamma^\Sigma\right)$ ([32, Theorem 3.1 and Theorem 1.4]). But if E is a $G(\mathbb{R})^c$-representation, the curvature 2-form of the Nomizu connection vanishes identically (see, e.g., [25, Proposition 5.3]), hence so do its Chern forms for $k > 0$. Thus $c_k\left(\mathscr{E}_\Gamma^\Sigma\right) = 0$ for $k > 0$ and $ch\left(\mathscr{E}_\Gamma^\Sigma\right) = \dim E$, and we have κ as in formula (C.1). It is a ring homomorphism because canonical extension is compatible with the tensor product [27, Section 4.2] and the Chern character is multiplicative.

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Since X^c is a flag variety, it has only even-degree cohomology, so the Chern character gives an isomorphism $ch : K^0(X^c) \otimes \mathbb{Q} \to \mathrm{H}^*(X^c,\mathbb{Q})$ (compare [4, 2.4]). Now define

$$\theta := \kappa \circ ch^{-1} \circ \sigma,$$

where $\sigma : \mathrm{H}^*(X^c,\mathbb{Q}) \to \mathrm{H}^*(X^c,\mathbb{Q})$ is defined by $\sigma(\alpha) = (-1)^{\deg(\alpha)/2}\alpha$. Since $\mathrm{H}^*(X^c,\mathbb{Q})$ is concentrated in even degrees, this makes sense, and σ is a ring homomorphism. Note that $\theta(\sigma(ch(\mathscr{E}^c)) = ch\left(\mathscr{E}_\Gamma^\Sigma\right)$, from which it follows that $\theta(c_k(\mathscr{E}^c)) = (-1)^k c_k\left(\mathscr{E}_\Gamma^\Sigma\right)$. This implies that θ is injective (i.e., nonzero) in top degree $2n = 2\dim_{\mathbb{C}} X$: Choose a nonzero monomial $c_{k_1}(\mathscr{E}_1^c)\cdots c_{k_r}(\mathscr{E}_r^c)$ with $\sum_i k_i = n$; it spans $\mathrm{H}^{2n}(X^c,\mathbb{Q})$. By Mumford's version of proportionality [32, Theorem 3.2],

$$\left[M_\Gamma^\Sigma\right] \cap \theta\left(c_{k_1}(\mathscr{E}_1^c)\cdots c_{k_r}(\mathscr{E}_r^c)\right) = (-1)^n \cdot C \cdot [X^c] \cap c_{k_1}(\mathscr{E}_1^c)\cdots c_{k_r}(\mathscr{E}_r^c) \neq 0,$$

where C is a nonzero constant independent of $k_i,\mathscr{E}_i$. It follows that θ is injective: For nonzero $\alpha \in \mathrm{H}^i(X^c)$, choose $\beta \in \mathrm{H}^{2n-i}(X^c)$ such that $\alpha\cdot\beta \neq 0$. Then $0 \neq \theta(\alpha\cdot\beta) = \theta(\alpha)\cdot\theta(\beta)$, so that $\theta(\alpha) \neq 0$. □

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