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ISOTROPIC IMMERSIONS WITH PARALLEL SECOND FUNDAMENTAL FORM

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Dedicated to Professor Takayuki Nôno on his retirement from Fukuoka University of Education.

ABSTRACT. The main purpose of this paper is to give a characterization of a Veronese manifold, as a generalization of a Veronese surface, in terms of isotropic immersions. This is an improvement of Itoh and Ogiue's results.

0. **Introduction.** An n-dimensional Veronese manifold is defined as a minimal immersion of an n-dimensional sphere of curvature $n/2(n+1)$ into an $(n(n+3)/2-1)$-dimensional unit sphere.

The minimal immersions of a sphere into a sphere are completely determined by do Carmo and Wallach [3], among which a Veronese manifold can be considered as the simplest one.

On the other hand, O'Neill [7] defined a notion of isotropic immersions. An *isotropic* immersion is an isometric immersion such that all its normal curvature vectors have the same length at each point. Namely, the length of the normal curvature vector is a function on the submanifold. In particular, if the function is constant, then the immersion is said to be *constant isotropic*. A Riemannian manifold of constant curvature is called a space form. The purpose of this paper is to prove the following two theorems.

THEOREM 1. *Let M be an n-dimensional space form of constant curvature c, and $\tilde{M}$ be an $(n+\frac{1}{2}n(n+1)-1)$-dimensional space form of constant curvature $\tilde{c}$. If $c<\tilde{c}$, and M is an isotropic submanifold of $\tilde{M}$, then M is immersed as a Veronese manifold into $\tilde{M}$.*

THEOREM 2. *Let M be an n-dimensional sphere of constant curvature c, and M is immersed fully as a constant isotropic submanifold in an $(n+p)$-dimensional sphere $\tilde{M}$ of constant curvature $\tilde{c}$. If p is not greater than $n(n+1)/2$, then M is one of the following:*

(i) *M is immersed as a totally umbilical hypersphere of $\tilde{M}$.*

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(ii) *M is immersed as a Veronese manifold of $\tilde{M}$.*

(iii) *M is immersed as a Veronese manifold in some totally umbilical hypersphere of $\tilde{M}$.*

REMARK. Itoh and Ogiue [4] proved Theorem 1 under the additional condition "M is immersed in $\tilde{M}$ with parallel second fundamental form". So, in our paper, we have only to show that the above condition is automatically satisfied.

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1. **Preliminaries.** Here and in the sequel, we denote by $M^m(c)$ an m-dimensional space form of constant curvature c. Let $M^n(c)$ be isometrically immersed in $\tilde{M}^{n+p}(\tilde{c})$ with metric tensor g.

Let ∇ and $\tilde{\nabla}$ be the Riemannian connections of M and $\tilde{M}$, respectively. Then the second fundamental form σ of the immersion is given by $\sigma(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y$, where X and Y are tangent vector fields on M. For a normal vector field ξ, we write $\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi$, where $-A_\xi X$ (resp. $D_X\xi$) denotes the tangential (resp. normal) component of $\tilde{\nabla}_X \xi$. We call A_ξ the associated second fundamental form with rspect to ξ so that $g(\sigma(X, Y), \xi) = g(A_\xi X, Y)$. A normal vector field ξ is said to be parallel if $D_X\xi = 0$ for any vector field X tangent to M.

We define the covariant differentiation ∇' of the second fundamental form σ with respect to the connection in (tangent bundle) + (normal bundle) as follows: $(\nabla'_X\sigma)(Y, Z) = D_X(\sigma(Y, Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z)$. The second fundamental form σ is said to be parallel if $(\nabla'_X\sigma)(Y, Z) = 0$ for all tangent vector fields X, Y and Z on M. Here, for later use, we write down Gauss, Codazzi and Ricci equations:

$$(1.1)\quad g(\sigma(Y, Z), \sigma(X, W)) - g(\sigma(X, Z), \sigma(Y, W)) = (c - \tilde{c})(g(Y, Z)g(X, W) - g(X, Z)g(Y, W)),$$

$$(1.2)\qquad (\nabla'_X\sigma)(Y, Z) = (\nabla'_Y\sigma)(X, Z),$$

$$(1.3)\qquad g(R^D(X, Y)\xi, \eta) = g([A_\xi, A_\eta](X), Y),$$

where R^D denotes the curvature tensor with respect to the induced connection D in the normal bundle.

Let $\mathfrak{H} = (1/n)(\text{trace } \sigma)$ be the mean curvature vector. The submanifold M is totally umbilic provided that $\sigma(X, Y) = g(X, Y) \cdot \mathfrak{H}$ for any tangent vector field X, Y on M. If $\mathfrak{H}$ is identically zero, the submanifold M is said to be minimal. Then it is known that the second fundamental form of the minimal immersion satisfies a differential equation, that is, we have Lemma 1.1 [2].

$$\tfrac{1}{2}\Delta \|\sigma\|^2 = \|\nabla'\sigma\|^2 + \sum_{\alpha, \beta} \text{tr}(A_\alpha A_\beta - A_\beta A_\alpha)^2 - \sum_{\alpha,\beta} (\text{tr } A_\alpha A_\beta)^2 + n\tilde{c}\, \|\sigma\|^2,$$

where Δ denotes the Laplacian.

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For a unit vector $X, \sigma(X, X)$ is called the normal curvature vector determined by X. An isometric immersion is said to be λ-*isotropic* if every normal curvature vector has the same length λ at each point.

Let $M^n(c)$ be λ-isotropically immersed in $\tilde{M}^{n+p}(\tilde{c})$ with metric tensor g. We choose a local field of orthonormal frame $\{e_1, \ldots, e_n\}$ in $M^n(c)$. Now we may easily see that $g(\sigma(e_i, e_i), \sigma(e_i, e_i)) = \lambda^2$ is equivalent to

$$
\begin{aligned}
g(\sigma(e_i, e_j), \sigma(e_k, e_l)) + g(\sigma(e_i, e_k), \sigma(e_j, e_l)) + g(\sigma(e_i, e_l), \sigma(e_j, e_k)) \\
= \lambda^2(\delta_{ij}\,\delta_{kl} + \delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk}).
\end{aligned}
\tag{1.4}
$$

On the other hand, by virtue of Gauss equation (1.1), we have

$$
g(\sigma(e_i, e_j), \sigma(e_k, e_l)) - g(\sigma(e_k, e_j), \sigma(e_i, e_l)) = (c - \tilde{c})(\delta_{ij}\,\delta_{kl} - \delta_{kj}\,\delta_{il}). \tag{1.5}
$$

Exchanging e_i and e_j, we get

$$
g(\sigma(e_i, e_j), \sigma(e_k, e_l)) - g(\sigma(e_k, e_i), \sigma(e_j, e_l)) = (c - \tilde{c})(\delta_{ij}\,\delta_{kl} - \delta_{ki}\,\delta_{jl}). \tag{1.6}
$$

Summing up (1.4), (1.5) and (1.6), we immediately obtain the following

$$
\begin{aligned}
g(\sigma(e_i, e_j), \sigma(e_k, e_l)) = (\lambda^2/3)(\delta_{ij}\,\delta_{kl} + \delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk}) \\
+ ((c - \tilde{c})/3)(2\delta_{ij}\,\delta_{kl} - \delta_{ik}\,\delta_{jl} - \delta_{il}\,\delta_{kl}).
\end{aligned}
\tag{1.7}
$$

Hence we have

$$
\begin{aligned}
g(\sigma(X, Y), \sigma(Z, U)) = g(\sigma(X, X), \sigma(X, Y)) = g(\sigma(X, X), \sigma(Y, Z)) \\
= g(\sigma(X, Y), \sigma(X, Z)) = 0 \\
\text{for orthonormal } X, Y, Z \text{ and } U.
\end{aligned}
\tag{1.8}
$$

We here note that (1.7) is essentially used in the proof of Theorem 1. Now, we recall the fundamental lemma due to O'Neill ([7]).

LEMMA 1.2. *Let* $M^n(c)$ $(n \geq 2)$ *be* λ-*isotropically* $(\lambda > 0)$ *immersed in* $\tilde{M}^{n+p}(\tilde{c})$. *Then the following inequalities hold on* $M^n(c)$:

$$
-((n+2)/2(n-1))\lambda^2 \leq c - \tilde{c} \leq \lambda^2.
$$

Let N_1 *be the first normal space of the above immersion, that is, the vector space spanned by all vectors* $\sigma(X, Y)$. *Furthermore, at each point of* $M^n(c)$, *only one of the following three cases holds:*

(1) $c - \tilde{c} = \lambda^2 \Leftrightarrow M^n(c)$ *is umbilic* $\Leftrightarrow \dim N_1 = 1$,

(2) $c - \tilde{c} = -((n+2)/2(n-1))\lambda^2 \Leftrightarrow M^n(c)$ *is minimal* $\Leftrightarrow \dim N_1 = n(n+1)/2 - 1$,

(3) $-((n+2)/2(n-1)\lambda^2 < c - \tilde{c} < \lambda^2 \Leftrightarrow \dim N_1 = n(n+1)/2$.

2. **Proof of Theorem 1.** The following calculation is quite same as in [6]. We remark that the immersion in Theorem 1 is precisely the case (2) in Lemma 1.2. Since λ is constant, differentiating (1.7) with respect to e_m, we have the following:

$$
g(\nabla'_{e_m}\sigma(e_i, e_j), \sigma(e_k, e_l)) = -g(\sigma(e_i, e_j), \nabla'_{e_m}\sigma(e_k, e_l)), \tag{2.1}
$$

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where i, j, k, l and m run over the range $\{1, 2, \ldots, n\}$. By using (2.1) and Codazzi equation (1.2) repeatedly, we get

$$\begin{aligned} g(\nabla'_{e_m}\sigma(e_i, e_j), \sigma(e_k, e_l)) &= -g(\sigma(e_i, e_j), \nabla'_{e_k}\sigma(e_m, e_l)) \\ &= g(\nabla'_{e_i}\sigma(e_k, e_j), \sigma(e_m, e_l)) = -g(\sigma(e_k, e_j), \nabla'_{e_l}\sigma(e_i, e_m)) \\ &= g(\nabla'_{e_j}\sigma(e_k, e_l), \sigma(e_i, e_m)) = -g(\sigma(e_k, e_l), \nabla'_{e_m}\sigma(e_i, e_j)). \end{aligned}$$

So we show that $g(\nabla'_{e_m}\sigma(e_i, e_j), \sigma(e_k, e_l)) = 0$. This, together with the fact that $\dim N_1 = \operatorname{codim} M$, implies that M is immersed in $\tilde{M}$ with parallel second fundamental form. Q.E.D.

3. **Proof of Theorem 2.** Of course, Lemma 1.2 is also essentially used here. First of all, we consider the case of $p \le n(n+1)/2-1$. Due to Lemma 1.2 we see that M is totally umbilic in $\tilde{M}$, if $c > \tilde{c}$. This, combined with the assumption that the immersion is full, shows that M is immersed as a totally umbilical hypersphere of $\tilde{M}$. If $c < \tilde{c}$, the case (2) in Lemma 1.2 holds everywhere on M. Therefore Theorem 1 says that M is immersed as a Veronese manifold in $\tilde{M}$, provided that $c < \tilde{c}$ and $p \le n(n+1)/2-1$.

Next we consider the case of $p = n(n+1)/2$. From the assumption that the immersion is constant isotropic, (1.7) asserts that $\dim N_1$ is constant on M. Moreover, the immersion is full. So, in consideration of Lemma 1.2, we have only to consider the following two cases. Namely, one is the case (2) of Lemma 1.2 and the other is the case (3). Now we investigate the case (3). From the assumption that λ is constant and the fact that $\dim N_1 = \operatorname{codim} M$, the same discussion as in the proof of Theorem 1 gives us that M is immersed in $\tilde{M}$ with parallel second fundamental form. It follows easily from $\nabla'\sigma = 0$ that the mean curvature vector is parallel. This, together with Smyth's result [8], implies that $M^n(c)$ is immersed in some totally umbilical hypersphere of $\tilde{M}^{n+p}(\tilde{c})$ as a minimal submanifold and hence must be a Veronese manifold. Finally we study the case (2). We remark that the case (2) does not occur for $n = 2$ (see, Calabi [1]). The rest of our paper is to show that the case (2) does not also occur for $n \ge 3$. Assume that $M^n(c)$ can be isotropically immersed fully into $\tilde{M}^{n+n(n+1)/2}\tilde{c}$ with $\dim N_1 = n(n+1)/2-1$. Now we may find that the second fundamental form of our immersion is not parallel, since our immersion is full and $\dim N_1 < \operatorname{codim} M$. For later discussion, we prepare the following result without proof (for details, see Itoh and Ogiue [4]).

$$\|\nabla'\sigma\|^2 = \frac{2n^2(n^2-1)}{n+2}(\tilde{c}-c)\left\{\frac{n}{2(n+1)}\tilde{c}-c\right\}. \tag{3.1}$$

(3.1) is induced from Lemma 1.1, (1.7) and (2) in Lemma 1.2. And hence, we have

$$\frac{n}{2(n+1)}\tilde{c} > c. \tag{3.2}$$

On the other hand, as λ is constant, the proof of Theorem 1 gives us

$$g(\nabla'_Y\sigma(Z, W), \sigma(U, V)) = 0, \tag{3.3}$$

where Y, Z, W, U and V are tangent vector fields on M. Differentiating (3.3) with respect to X, we get

$$g(\nabla'_X \nabla'_Y\sigma(Z, W), \sigma(U, V)) = -g(\nabla'_Y\sigma(Z, W), \nabla'_X\sigma(U, V)).$$

Taking the skew symmetric part in X and Y, we have

$$\begin{aligned} g(\nabla'_X\sigma(Z, W), \nabla'_Y\sigma(U, V)) - g(\nabla'_Y\sigma(Z, W), \nabla'_X\sigma(U, V)) \\ = g(\nabla'_X \nabla'_Y\sigma(Z, W) - \nabla'_Y \nabla'_X\sigma(Z, W), \sigma(U, V)). \end{aligned}$$

This, together with Ricci equation (1.3), tells us (see, Ricci formula [5])

$$\begin{aligned} g(\nabla'_X\sigma(Z, W), \nabla'_Y\sigma(U, V)) &- g(\nabla'_Y\sigma(Z, W), \nabla'_X\sigma(U, V)) \\ &= g([A_{\sigma(Z,W)}, A_{\sigma(U,V)}](X), Y) - cg(Y, Z)g(\sigma(X, W), \sigma(U, V)) \\ &\quad + cg(X, Z)g(\sigma(Y, W), \sigma(U, V)) - cg(Y, W)g(\sigma(Z, X), \sigma(U, V)) \\ &\quad + cg(X, W)g(\sigma(Z, Y), \sigma(U, V)). \end{aligned} \tag{3.4}$$

From now on, we shall induce a contradiction by using (3.4). We choose and fix a point x of M. We shall consider our problem in $T_x(M)$. It follows that the vector space spanned by all vectors $\nabla'_X\sigma(Y, Z)$ is one dimensional from $\dim N_1 = \operatorname{codim} M - 1$ and (3.3). So, we may regard $\nabla'_X\sigma(Y, Z)$ as a real number. In the following, for simplicity, we set $T(X, Y, Z)$ in place of $\nabla'_X\sigma(Y, Z)$. We here note that T is a 3-symmetric form on $T_x(M)$. Now, we denote by $\tilde{X}$ such that $T(X, X, X)$ in $\sum = \{X : \|X\| = 1\}$ gives a maximal value at $\tilde{X}$. We immediately find

$$T(\tilde{X}, \tilde{X}, Y) = 0 \quad \text{for any vector } Y \text{ orthogonal to } \tilde{X}. \tag{3.5}$$

From (3.4) and (3.5), we have

$$\begin{aligned} &T(\tilde{X}, \tilde{X}, \tilde{X})T(Y, Y, Y) \\ &\quad = g([A_{\sigma(\tilde{X},\tilde{X})}, A_{\sigma(Y,Y)}](\tilde{X}), Y) + 2cg(\sigma(\tilde{X}, Y), \sigma(Y, Y)). \end{aligned} \tag{3.6}$$

On the other hand, (1.8) suggests that the right-hand-side of (3.6) vanishes. And since $T(\tilde{X}, \tilde{X}, \tilde{X}) \neq 0$, we obtain

$$T(Y, Y, Y) = 0 \quad \text{for any } Y \text{ orthogonal to } \tilde{X}. \tag{3.7}$$

Next we shall compute $T(\tilde{X}, Y, Y)$. From (3.4) and (3.5), we find $T(\tilde{X}, Y, Y)T(Y, \tilde{X}, Y) = g([A_{\sigma(Y,Y)}, A_{\sigma(\tilde{X},Y)}](\tilde{X}), Y) - 2c\,\|\sigma(\tilde{X}, Y)\|^2$. Then, by virtue of (1.7) and (2) in Lemma 1.2, we get

$$T(\tilde{X}, Y, Y)T(Y, \tilde{X}, Y) = \frac{4n(n+1)}{(n+2)^2}(\tilde{c} - c)\left\{\frac{n}{2(n+1)}\tilde{c} - c\right\}. \tag{3.8}$$

Hence (3.2) implies that

(3.9) $T(\tilde{X}, Y, Y) \neq 0$ for any vector Y orthogonal to $\tilde{X}$.

Moreover, from (3.4), (3.7) and (2) in Lemma 1.2, the same calculation as above yields that

(3.10) $T(X, Y, Y) \neq 0$ for orthogonal X, Y and $\tilde{X}$.

Similarly we obtain $T(X, Y, Y)T(Y, \tilde{X}, Y) = 0$ for orthogonal X, Y and $\tilde{X}$. This, together with (3.9) and (3.10) gives us a contradiction. Q.E.D.

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