Nagoya Math. J., **229** (2018), 99–112
DOI 10.1017/nmj.2016.63

# LEVI-UMBILICAL REAL HYPERSURFACES IN A COMPLEX SPACE FORM

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**Abstract.** We give a classification of Levi-umbilical real hypersurfaces in a complex space form $\widetilde{M}_n(c)$, $n \geqslant 3$, whose Levi form is proportional to the induced metric by a nonzero constant. In a complex projective plane $\mathbb{CP}^2$, we give a local construction of such hypersurfaces and moreover, we give new examples of Levi-flat real hypersurfaces in $\mathbb{CP}^2$.

## §1. Introduction

Let $M$ be a $(2n-1)$-dimensional manifold and $TM$ be its tangent bundle. A *CR-structure* on $M$ is a complex rank $(n-1)$ subbundle $\mathcal{H} \subset \mathbb{C}TM = TM \otimes \mathbb{C}$ satisfying

$$\text{(i)} \qquad \mathcal{H} \cap \bar{\mathcal{H}} = \{0\},$$
$$\text{(ii)} \qquad [\mathcal{H}, \mathcal{H}] \subset \mathcal{H} \text{ (integrability)},$$

where $\bar{\mathcal{H}}$ denotes the complex conjugation of $\mathcal{H}$.

Then there exists a unique subbundle $D = \mathrm{Re}\{\mathcal{H} \oplus \bar{\mathcal{H}}\}$, called the *Levi subbundle* (maximally holomorphic subbundle) of $(M, \mathcal{H})$, and a unique bundle map $J$ such that $J^2 = -I$ and $\mathcal{H} = \{X - iJX | X \in D\}$. We call $(D, J)$ the real representation of $\mathcal{H}$. Let $E \subset T^*M$ be the conormal bundle of $D$. If $M$ is an oriented CR-manifold then $E$ is a trivial bundle, hence admits globally defined a nowhere zero section $\eta$, that is, a real one-form on $M$ such that $\mathrm{Ker}(\eta) = D$. For $(D, J)$ we define the Levi form by

$$L : D \times D \to \mathcal{F}(M), \qquad L(X, Y) = d\eta(X, JY)$$

Received October 15, 2015. Revised November 3, 2016. Accepted November 4, 2016.
2010 Mathematics subject classification. 53B20, 53C15, 53C40.
J. T. Cho was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology (2016R1D1A1B03930756). M. Kimura was supported by JSPS KAKENHI Grant Number JP16K05119.

where $\mathcal{F}(M)$ denotes the algebra of differential functions on $M$. If the Levi form is nondegenerate (positive or negative definite, resp.), then the CR-structure is called a *nondegenerate* (*strongly pseudo-convex*, resp.) *pseudo-Hermitian CR-structure*.

Now, let $\widetilde{M}^n$ be an $n$-dimensional Kähler manifold and let $M^{2n-1}$ be a real hypersurface in $\widetilde{M}$. Then $M$ is called *Levi-flat* if the Levi form vanishes. In the present paper, we introduce the so-called *Levi-umbilicity*. If the Levi form $L$ is proportional to the induced metric $g$ by a nonzero constant $k$, then $M$ is said to be *Levi-umbilical*.

A complex $n$-dimensional complete and simply connected Kähler manifold of constant holomorphic sectional curvature $c$ is called a complex space form, which is denoted by $\widetilde{M}_n(c)$. A complex space form consists of a complex projective space $\mathbb{CP}^n$, a complex Euclidean space $\mathbb{CE}^n$ or a complex hyperbolic space $\mathbb{CH}^n$, according as $c > 0$, $c = 0$ or $c < 0$. Recently, Siu [14] proved the nonexistence of compact smooth Levi-flat hypersurfaces in $\mathbb{CP}^n$ of dimensions $\geqslant 3$. When $n = 2$, Ohsawa [13] proved the nonexistence of compact real analytic Levi-flat hypersurfaces in $\mathbb{CP}^2$. Here, it is remarkable that the assumption of compactness has a crucial role. Indeed, there are noncomplete examples which are realized as ruled hypersurfaces and Levi-flat in $\mathbb{CP}^n$ (see Section 3). We also find that there does not exist a Levi-flat Hopf hypersurface in $\mathbb{CP}^n$ or $\mathbb{CH}^n$ (cf. [6]). In the present paper, we give noncompact examples of Levi-flat real hypersurfaces which are not ruled hypersurfaces in $\mathbb{CP}^2$ (see Section 5).

On the other hand, Takagi [16], [17] classified the homogeneous real hypersurfaces in $\mathbb{CP}^n$ into six types. Cecil and Ryan [4] extensively studied a real hypersurface whose structure vector $\xi$ is a principal curvature vector, which is realized as tubes over certain submanifolds in $\mathbb{CP}^n$, by using its focal map. A real hypersurface of a complex space form is said to be a *Hopf hypersurface* if its structure vector is a principal curvature vector. By making use of those results and the mentioned work of Takagi, Makoto Kimura [8] proved the classification theorem for Hopf hypersurfaces of $\mathbb{CP}^n$ whose all principal curvatures are constant. For the case $\mathbb{CH}^n$, Berndt [2] proved the classification theorem for Hopf hypersurfaces whose all principal curvatures are constant.

The main purpose of the present paper is to give a classification of Levi-umbilical real hypersurfaces in a complex space form.

THEOREM 1. *If a real hypersurface $M$ of a complex space form $\widetilde{M}_n(c)$ is Levi-umbilical, then $n=2$ or $M$ is a Hopf hypersurface. Moreover, in case that $M$ is connected, complete and $n \geqslant 3$, we have the following.*

(I) *If $\widetilde{M}_n(c) = \mathbb{CP}^n$, then $M$ is congruent to one of the following:*

(1) *a geodesic hypersphere, that is, a tube of radius $r$ over $\mathbb{CP}^{n-1}$, where $0 < r < \frac{\pi}{2}$,*
(2) *a tube of radius $r$ over a complex quadric $\mathbb{CQ}^{n-1}$, where $0 < r < \frac{\pi}{4}$.*

(II) *If $\widetilde{M}_n(c) = \mathbb{CH}^n$, then $M$ is congruent to one of the following:*

(1) *a horosphere in $\mathbb{CH}^n$,*
(2) *a geodesic hypersphere or a tube of radius $r \in \mathbb{R}_+$ over a totally geodesic $\mathbb{CH}^{n-1}$,*
(3) *a tube of radius $r \in \mathbb{R}_+$ over a totally real hyperbolic space $\mathbb{RH}^n$.*

(III) *If $\widetilde{M}_n(c) = \mathbb{CE}^n$, then $M$ is locally congruent to one of the following:*

(1) *a sphere $S^{2n-1}(r)$ of radius $r \in \mathbb{R}_+$,*
(2) *a generalized cylinder $S^{n-1}(r) \times \mathbb{E}^n$ of radius $r \in \mathbb{R}_+$.*

In Section 5, we give a construction of Levi-umbilical non-Hopf hypersurfaces in $\mathbb{CP}^2$.

## §2. Almost contact metric structures and the associated CR-structures

In this paper, all manifolds are assumed to be connected and of class $C^\infty$. First, we give a brief review of several fundamental concepts and formulas which we need later on. An odd-dimensional differentiable manifold $M$ has *an almost contact structure* if it admits a (1,1)-tensor field $\phi$, a vector field $\xi$ and a 1-form $\eta$ satisfying

$$\phi^2 = -I + \eta \otimes \xi, \ \ \eta(\xi) = 1. \tag{1}$$

Then we can find always a compatible Riemannian metric, namely which satisfies

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2}$$

for all vector fields on $M$. We call $(\eta, \phi, \xi, g)$ an *almost contact metric structure* of $M$ and $M = (M; \eta, \phi, \xi, g)$ an *almost contact metric manifold.*

The *fundamental 2-form* $\Phi$ is defined by $\Phi(X,Y)=g(\phi X,Y)$. If $M$ satisfies in addition $d\eta=\Phi$, then $M$ is called a *contact metric manifold*, where $d$ is the exterior differential operator. From (1) and (2) we easily get

$$\phi\xi=0, \qquad \eta\circ\phi=0, \qquad \eta(X)=g(X,\xi). \tag{3}$$

The tangent space $T_pM$ of $M$ at each point $p\in M$ is decomposed as $T_pM=D_p\oplus\{\xi\}_p$(direct sum), where we denote $D_p=\{v\in T_pM|\eta(v)=0\}$. Then $D:p\to D_p$ defines a distribution orthogonal to $\xi$. For an almost contact metric manifold $M$, one may define naturally an almost complex structure on the product manifold $M\times\mathbb{R}$, where $\mathbb{R}$ denotes the real line. If the almost complex structure is integrable, $M$ is said to be *normal*. The integrability condition for the almost complex structure is the vanishing of the tensor $[\phi,\phi]+2d\eta\otimes\xi$, where $[\phi,\phi]$ denotes the Nijenhuis torsion of $\phi$. For more details about the general theory of almost contact metric manifolds, we refer to [3].

On the other hand, for an almost contact metric manifold $M$, the restriction $J=\phi|D$ of $\phi$ to $D$ defines an almost complex structure in $D$. As soon as $M$ satisfies

$$[JX,JY]-[X,Y]\in D \text{ (or } [JX,Y]+[X,JY]\in D) \tag{4}$$

and

$$[J,J](X,Y)=0 \tag{5}$$

for all $X,Y\in D$, where $[J,J]$ is the Nijenhuis torsion of $J$, then the pair $(\eta,J)$ is called an (integrable) CR-structure associated with the almost contact metric structure $(\eta,\phi,\xi,g)$. For example, a normal almost contact metric manifold has an integrable CR-structure [7]. In addition, the associated Levi form $L$ defined by $L(X,Y)=d\eta(X,JY)$, $X,Y\in D$, is nondegenerate (positive or negative definite, resp.), then $(\eta,J)$ is called a nondegenerate (strongly pseudo-convex, resp.) pseudo-Hermitian CR-structure. We may refer to [5], [7], [18] about CR-structures associated with (almost) contact metric structures.

## §3. Real hypersurfaces in a complex space form

Let $M$ be an immersed real hypersurface of a Kähler manifold $\widetilde{M}=(\widetilde{M};\tilde{J},\tilde{g})$ and $N$ a local unit normal vector in a neighborhood of each point. By $\tilde{\nabla}$, $\sigma$ we denote the Levi-Civita connection in $\widetilde{M}$ and the second fundamental form associated with the shape operator $A$ with respect to $N$, respectively. Then the Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y)N, \qquad \tilde{\nabla}_X N = -AX$$

for any vector fields $X$ and $Y$ tangent to $M$. Here, we note that $\sigma(X, Y) = g(AX, Y)$, where $g$ denotes the Riemannian metric of $M$ induced from $\tilde{g}$. An eigenvector (resp. eigenvalue) of the shape operator $A$ is called a principal curvature vector (resp. principal curvature). For any vector field $X$ tangent to $M$, we put

$$\tilde{J}X = \phi X + \eta(X)N, \qquad \tilde{J}N = -\xi. \tag{6}$$

We easily see that the structure $(\eta, \phi, \xi, g)$ is an almost contact metric structure on $M$, that is, satisfies (1) and (2). From the condition $\tilde{\nabla}\tilde{J} = 0$, the relations (6) and by making use of the Gauss and Weingarten formulas, we have

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{7}$$

$$\nabla_X \xi = \phi AX. \tag{8}$$

From now, let $\widetilde{M}_n(c)$ be a complex space form of constant holomorphic sectional curvature $c$. Then, from the Codazzi equation, we have

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}. \tag{9}$$

By using (7) and (8), we see that a real hypersurface in a Kähler manifold always satisfies (4) and (5), the integrability condition of the associated CR-structure. From (8) we find that $M$ is *Levi-flat* if and only if

$$g((\phi A + A\phi)X, Y) = 0, \quad X, Y \perp \xi, \tag{10}$$

and $M$ is *Levi-umbilical* if and only if there exists *nonzero constant* $k \in \mathbb{R}$ such that

$$g((\phi A + A\phi)X, Y) = kg(\phi X, Y), \quad X, Y \perp \xi. \tag{11}$$

Here we recall ruled real hypersurfaces in $\mathbb{CP}^n$ or $\mathbb{CH}^n$. Such a space is *a foliated real hypersurface whose leaves are complex hyperplanes* $\mathbb{CP}^{n-1}$

*or* $\mathbb{CH}^{n-1}$, *respectively in* $\mathbb{CP}^n$ *or* $\mathbb{CH}^n$. That is, let $\gamma : I \to \widetilde{M}_n(c)$ be a regular curve in $\widetilde{M}_n(c)$ ($\mathbb{CP}^n$ or $\mathbb{CH}^n$). Then for each $t \in I$, let $M^{(t)}_{n-1}(c)$ be a totally geodesic complex hypersurface which is orthogonal to holomorphic plane Span$\{\dot\gamma, J\dot\gamma\}$. We have a ruled real hypersurface $M = \bigcup_{t\in I} M^{(t)}_{n-1}(c)$. A ruled real hypersurface is non-Hopf and particularly it is noncomplete real hypersurface in $\mathbb{CP}^n$ (see, [10] for the case $\mathbb{CP}^n$ and see [1] for the case $\mathbb{CH}^n$, respectively). The shape operator $A$ is written by the following form:

$$
\begin{aligned}
A\xi &= \mu\xi + \nu V \;\; (\nu \neq 0),\\
AV &= \nu\xi,\\
AX &= 0 \quad \text{for any } X \perp \xi, V,
\end{aligned}
\tag{12}
$$

where $V$ is a unit vector orthogonal to $\xi$, and $\mu$, $\nu$ are differentiable functions on $M$. Then, we easily see that *ruled real hypersurfaces in* $\mathbb{CP}^n$ *or in* $\mathbb{CH}^n$ *are Levi-flat.*

## §4. Proof of Theorem 1

In this section, we prove Theorem 1. Let $M$ be a Levi-umbilical real hypersurface in a complex space form $\widetilde{M}_n(c)$. If we differentiate (11) covariantly, then we have

$$
\begin{aligned}
&g\left((\nabla_X A)\phi Y + A(\nabla_X\phi)Y + A\phi\nabla_X Y + (\nabla_X\phi)AY\right.\\
&\qquad \left. + \phi(\nabla_X A)Y + \phi A\nabla_X Y, Z\right) + g((A\phi+\phi A)Y, \nabla_X Z)\\
&\quad = k(g((\nabla_X\phi)Y, Z) + g(\phi\nabla_X Y, Z) + g(\phi Y, \nabla_X Z)),
\end{aligned}
\tag{13}
$$

for any vector fields $X, Y, Z \perp \xi$. Use (7) to get

$$
\begin{aligned}
&g((\nabla_X A)\phi Y + \phi(\nabla_X A)Y, Z)\\
&\quad + g(\eta(Y)A^2X - g(AX, Y)A\xi + \eta(AY)AX - g(AX, AY)\xi, Z)\\
&\quad + g((A\phi+\phi A)\nabla_X Y, Z) - g((A\phi+\phi A)\nabla_X Z, Y)\\
&= k(g(\phi\nabla_X Y, Z) - g(\phi\nabla_X Z, Y)).
\end{aligned}
\tag{14}
$$

We decompose $\nabla_X Y = \nabla_X Y^\perp + \eta(\nabla_X Y)\xi$, where $\nabla_X Y^\perp$ denotes the part of $\nabla_X Y$ orthogonal to $\xi$. Using (8) and (11), (14) becomes

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$$
\begin{aligned}
&g((\nabla_X A)\phi Y + \phi(\nabla_X A)Y, Z) \\
&\quad + g(\eta(Y)A^2X - g(AX, Y)A\xi + \eta(AY)AX - g(AX, AY)\xi, Z) \\
&\quad + \eta(\nabla_X Y)g(U, Z) - \eta(\nabla_X Z)g(U, Y) = 0,
\end{aligned} \tag{15}
$$

where we have put $U = \nabla_\xi \xi$. Use (8) to obtain

$$
\begin{aligned}
&g((\nabla_X A)Z, \phi Y) - g((\nabla_X A)Y, \phi Z) \\
&\quad = g(\phi AX, Y)g(U, Z) - g(\phi AX, Z)g(U, Y) \\
&\qquad + \eta(Z)g(A^2X, Y) - \eta(Y)g(A^2X, Z) \\
&\qquad + \eta(AZ)g(AX, Y) - \eta(AY)g(AX, Z).
\end{aligned} \tag{16}
$$

Taking the cyclic sum of (16) for $X, Y, Z$, using (9) we have

$$
\begin{aligned}
&g((A\phi + \phi A)X, Y)g(U, Z) + g((A\phi + \phi A)Y, Z)g(U, X) \\
&\quad + g((A\phi + \phi A)Z, X)g(U, Y) = 0.
\end{aligned} \tag{17}
$$

Using (11) in (17) again, we have

$$
k(g(\phi X, Y)g(U, Z) + g(\phi Y, Z)g(U, X) + g(\phi Z, X)g(U, Y)) = 0. \tag{18}
$$

If we put $Z = U$ in (18), then we have

$$
k(g(\phi X, Y)\|U\|^2 + g(\phi Y, U)g(U, X) + g(\phi U, X)g(U, Y)) = 0. \tag{19}
$$

Replace $Y$ by $\phi X$ in (19), then it turns to

$$
k(g(X, X)\|U\|^2 - g(X, U)g(U, X) + g(\phi U, X)g(U, \phi X)) = 0. \tag{20}
$$

For an adapted orthonormal basis $\{e_i, \xi\}$, $i = 1, \cdots, 2n-2$, we put $X = e_i$ and taking the sum for $i = 1, \cdots, 2n-2$, then since $k \neq 0$ we have

$$
(n-2)\|U\|^2 = 0.
$$

From this, we find that $n = 2$ or $M$ is a Hopf hypersurface, that is, $A\xi = \mu\xi$, where we have used (8). Now, we assume that $n \geqslant 3$. Then Levi-umbilicity condition (11) yields that $\phi A + A\phi = k\phi$, $k \neq 0$. Due to results of [11] (in case of $\mathbb{CP}^n$), [19], [15] (in case of $\mathbb{CH}^n$), and [12] (in case of $\mathbb{CE}^n$) we find the following.

(I) If $\widetilde{M}_n(c) = \mathbb{CP}^n$, then $M$ is locally congruent to one of the following:

(1) a geodesic hypersphere, that is, a tube of radius $r$ over $P_{n-1}\mathbb{C}$, where $0 < r < \frac{\pi}{2}$,
(2) a tube of radius $r$ over a complex quadric $\mathbb{CQ}^{n-1}$, where $0 < r < \frac{\pi}{4}$.

(II) If $\widetilde{M}_n(c) = \mathbb{CH}^n$, then $M$ is locally congruent to one of the following:

(1) a horosphere in $\mathbb{CH}^n$,
(2) a geodesic hypersphere or a tube of radius $r \in \mathbb{R}_+$ over a totally geodesic $\mathbb{CH}^{n-1}$,
(3) a tube of radius $r \in \mathbb{R}_+$ over a totally real hyperbolic space $\mathbb{RH}^n$.

(III) If $\widetilde{M}_n(c) = \mathbb{CE}^n$, then $M$ is locally congruent to one of the following:

(1) a sphere $S^{2n-1}(r)$ of radius $r \in \mathbb{R}_+$,
(2) a generalized cylinder $S^{n-1}(r) \times \mathbb{E}^n$ of radius $r \in \mathbb{R}_+$.

Then, we have Theorem 1. ∎

## §5. Three-dimensional Levi-umbilical hypersurfaces in $\mathbb{CP}^2$

In this section, we give a construction of 3-dimensional Levi-flat or Levi-umbilical real hypersurfaces in $\mathbb{CP}^2$. First, we prepare

Lemma 2. *Let $M^{2n-1}$ $(n \geqslant 2)$ be a Levi-flat hypersurface in a Kähler manifold $\widetilde{M}^n$. Then* trace $A = \eta(A\xi)$ *on $M$. The converse holds when $n = 2$.*

Lemma 3. *Let $M^{2n-1}$ $(n \geqslant 2)$ be a Levi-umbilical hypersurface in a Kähler manifold $\widetilde{M}^n$. Then* trace $A - \eta(A\xi)$ *is a nonzero constant on $M$. The converse holds when $n = 2$.*

Now, according to [9], we construct Levi-flat or Levi-umbilical hypersurfaces respectively in $\mathbb{CP}^2$. We denote $S^n$ as the unit sphere of which the center is the origin in $\mathbb{R}^{n+1}$. We consider the following submanifolds of $\mathbb{C}^3$:

$$\begin{aligned}\mathbb{C}^3 &\supset S^5\\ &\supset \sin rS^3 \times \cos rS^1\\ &\supset \sin r(\sin\theta S^1 \times \cos\theta S^1) \times \cos rS^1,\end{aligned}$$

where $0 < r, \theta < \pi/2$. Let $\gamma : I \to (0, \pi/2) \times (0, \pi/2)$, $\gamma(s) = (r(s), \theta(s))$ be a (nonconstant) curve defined on an interval $I$. We put

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$$\widetilde{M}_\gamma := \bigcup_{s\in I} \sin r(s)(\sin\theta(s)S^1 \times \cos\theta(s)S^1) \times \cos r(s)S^1 \quad \text{and} \quad M_\gamma := \pi(\widetilde{M}_\gamma), \tag{21}$$

where $\pi : S^5 \to \mathbb{CP}^2$ is the Hopf fibration. Then $\widetilde{M}_\gamma$ is a hypersurface of $S^5$, and since $\widetilde{M}_\gamma$ is invariant under the $S^1$-action, $M_\gamma$ is a real hypersurface of $\mathbb{CP}^2$. Note that $M_\gamma$ is foliated by flat Lagrangian torus $T^2$ in $\mathbb{CP}^2$.

Let $x, y, z \in S^1 \subset \mathbb{C}$ and denote

$$\tilde{x} = \sin r \sin\theta x, \qquad \tilde{y} = \sin r \cos\theta y, \qquad \tilde{z} = \cos r z,$$

where $0 < r, \theta < \pi/2$. Then the position vector $\Psi$ of $\widetilde{M}_\gamma$ is given by

$$\Psi = \Psi(r(s), \theta(s)) = (\tilde{x}, \tilde{y}, \tilde{z}) = (\sin r \sin\theta x, \sin r \cos\theta y, \cos r z)$$

and unit normal vectors $N_1$ and $N_2$ of 3-dimensional submanifold

$$\sin r(\sin\theta S^1 \times \cos\theta S^1) \times \cos r S^1$$

in $S^5$ at $\Psi$ are given as

$$N_1 := \frac{\partial\Psi}{\partial r} = (\cot r\tilde{x}, \cot r\tilde{y}, -\tan r\tilde{z}) = (\cos r \sin\theta x, \cos r \cos\theta y, -\sin r z)$$

and

$$N_2 := \frac{1}{\sin r}\frac{\partial\Psi}{\partial\theta} = \left(\frac{\cot\theta}{\sin r}\tilde{x}, -\frac{\tan\theta}{\sin r}\tilde{y}, 0\right) = (\cos\theta x, -\sin\theta y, 0).$$

Put $\dot{\Psi} = \frac{d}{ds}\Psi(r(s), \theta(s))$. Then we have

$$\dot{\Psi} = \dot{r}N_1 + \dot{\theta}\sin r N_2 \qquad \left(\dot{r} = \frac{dr}{ds}, \dot{\theta} = \frac{d\theta}{ds}\right).$$

By taking an arc-length parameterization, we may put $(\dot{r})^2 + (\dot{\theta})^2 \sin^2 r = 1$ and

$$\dot{r} = \cos\alpha, \qquad \dot{\theta} = \frac{\sin\alpha}{\sin r}. \tag{22}$$

Hence $\dot{\Psi} = \cos\alpha N_1 + \sin\alpha N_2$. Let

$$\widetilde{N} = -\sin\alpha N_1 + \cos\alpha N_2.$$

Then $\widetilde{N}$ is a unit normal vector field of $\widetilde{M}_\gamma$ in $S^5$. Since $\widetilde{N}$ is $S^1$-invariant, $N := \pi_*(\widetilde{N})$ is a unit normal vector field of $M_\gamma$ in $\mathbb{CP}^2$. We have

$$\begin{aligned}\dot{\Psi} &= \left(\left(\cos\alpha\cot r + \sin\alpha\frac{\cot\theta}{\sin r}\right)\tilde{x},\right.\\ &\qquad \left.\left(\cos\alpha\cot r - \sin\alpha\frac{\tan\theta}{\sin r}\right)\tilde{y}, -\cos\alpha\tan r\tilde{z}\right)\\ &= ((\cos\alpha\cos r\sin\theta + \sin\alpha\cos\theta)\,x,\\ &\qquad (\cos\alpha\cos r\cos\theta - \sin\alpha\sin\theta)\,y, -\cos\alpha\sin rz)\,,\end{aligned}$$

and

$$\begin{aligned}\widetilde{N} &= \left(\left(-\sin\alpha\cot r + \cos\alpha\frac{\cot\theta}{\sin r}\right)\tilde{x},\right.\\ &\qquad \left.\left(-\sin\alpha\cot r - \cos\alpha\frac{\tan\theta}{\sin r}\right)\tilde{y}, \sin\alpha\tan r\tilde{z}\right)\\ &= ((-\sin\alpha\cos r\sin\theta + \cos\alpha\cos\theta)\,x,\\ &\qquad (-\sin\alpha\cos r\cos\theta - \cos\alpha\sin\theta)y, \sin\alpha\sin rz)\,.\end{aligned}$$

The tangent space of $\widetilde{M}_\gamma$ at $\Psi$ is spanned by the following *orthonormal* vectors:

$$i\Psi, \qquad i\widetilde{N}, \qquad i\dot{\Psi} \qquad \text{and} \qquad \dot{\Psi}.$$

Here $i\Psi$ is a unit vertical vector of the Hopf fibration $\pi : S^5 \to \mathbb{CP}^2$ and the others are horizontal.

Let $D$ and $\widetilde{A}$ be the flat connection of $\mathbb{C}^3$ and the shape operator of the hypersurface $\widetilde{M}_\gamma$ in $S^5$, respectively. Then by the Weingarten formula, we have

$$\widetilde{A}W = -D_W\widetilde{N} \quad \text{for } W \in T_\Psi(\widetilde{M}_\gamma).$$

Covariant differentiation of $\widetilde{N}$ for $\dot{\Psi}$ is given by

$$\begin{aligned}D_{\dot{\Psi}}\widetilde{N} &= \frac{\partial}{\partial s}\widetilde{N} = -\dot{\alpha}\dot{\Psi}\\ &\quad + \Big((-\sin\alpha(-\dot{r}\sin r\sin\theta + \dot{\theta}\cos r\cos\theta) - \dot{\theta}\cos\alpha\sin\theta)x,\\ &\quad (-\sin\alpha(-\dot{r}\sin r\cos\theta - \dot{\theta}\cos r\sin\theta) - \dot{\theta}\cos\alpha\cos\theta)y, \dot{r}\sin\alpha\cos rz\Big)\\ &= -\dot{\alpha}\dot{\Psi} + \Big((\sin\alpha\cos\alpha\sin r\sin\theta - \frac{\sin\alpha}{\sin r}(\sin\alpha\cos r\cos\theta + \cos\alpha\sin\theta))x,\end{aligned}$$

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$$(\sin\alpha\cos\alpha\sin r\cos\theta+\frac{\sin\alpha}{\sin r}(\sin\alpha\cos r\sin\theta-\cos\alpha\cos\theta))y,$$
$$\cos\alpha\sin\alpha\cos rz\Big)$$
$$=-(\dot\alpha+\sin\alpha\cot r)\dot\Psi.$$

Hence we obtain

$$\widetilde{A}\dot\Psi=(\dot\alpha+\sin\alpha\cot r)\dot\Psi. \tag{23}$$

Also we have

$$\widetilde{A}(i\Psi)=-i\widetilde{N},$$
$$\widetilde{A}(i\widetilde{N})=-i\Psi+\mu i\widetilde{N}+\nu i\dot\Psi,$$
$$\widetilde{A}(i\dot\Psi)=\nu i\widetilde{N}+\lambda i\dot\Psi,$$

where

$$\mu:=-\langle D_{i\widetilde{N}}\widetilde{N},i\widetilde{N}\rangle,\qquad \nu:=-\langle D_{i\widetilde{N}}\widetilde{N},i\dot\Psi\rangle,\qquad \lambda:=-\langle D_{i\dot\Psi}\widetilde{N},i\dot\Psi\rangle.$$

Computations (2.8) of [9] yield:

$$\mu=\sin^3\alpha(\cot r-\tan r)+3\sin\alpha\cos^2\alpha\cot r-\frac{\cos^3\alpha}{\sin r}(\cot\theta-\tan\theta), \tag{24}$$

$$\nu=\cos\alpha\left(\sin^2\alpha(\cot r+\tan r)-\frac{\cos\alpha\sin\alpha}{\sin r}(\cot\theta-\tan\theta)-\cos^2\alpha\cot r\right), \tag{25}$$

$$\lambda=\sin\alpha\left(\sin^2\alpha\cot r-\frac{\cos\alpha\sin\alpha}{\sin r}(\cot\theta-\tan\theta)-\cos^2\alpha(\cot r+\tan r)\right). \tag{26}$$

Let $U=-\pi_*(i\dot\Psi)$. Then $\phi U=\pi_*(\dot\Psi)$. Also we have $\xi=-JN=-\pi_*(i\widetilde{N})$. Then the shape operator $A$ of $M_\gamma$ in $\mathbb{CP}^2$ with respect to $N$ is given by

$$A\xi=\mu\xi+\nu U,\qquad AU=\nu\xi+\lambda U,\qquad A\phi U=(\dot\alpha+\sin\alpha\cot r)\phi U. \tag{27}$$

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Hence with respect to $M_\gamma$ in $\mathbb{CP}^2$, we have

$$\begin{aligned} \operatorname{trace} A - \eta(A\xi) &= \dot{\alpha} + \sin\alpha \cot r + \lambda \\ &= \dot{\alpha} + \sin\alpha\bigg((1+\sin^2\alpha)\cot r - \frac{\cos\alpha\sin\alpha}{\sin r}(\cot\theta - \tan\theta) \\ &\quad - \cos^2\alpha(\cot r + \tan r)\bigg). \end{aligned} \tag{28}$$

Proposition 4. *Let $(r(s), \theta(s), \alpha(s))$ be a solution of the system of nonlinear ODE,*

$$\begin{aligned} &\dot{r} = \cos\alpha, \qquad \dot{\theta} = \frac{\sin\alpha}{\sin r}, \\ &\dot{\alpha} + \sin\alpha\bigg((1+\sin^2\alpha)\cot r - \frac{\cos\alpha\sin\alpha}{\sin r}(\cot\theta - \tan\theta) \\ &\quad - \cos^2\alpha(\cot r + \tan r)\bigg) = 0, \end{aligned} \tag{29}$$

*such that the initial condition satisfying $0 < r(0), \theta(0) < \pi/2$. Then the real hypersurface $M_\gamma$ in $\mathbb{CP}^2$, defined by (21) is Levi-flat.*

A special solution of (29) is given by

$$\theta = \text{constant}, \qquad \alpha \equiv 0 \mod \pi.$$

In this case, we have $\dot{\alpha} + \sin\alpha\cot r = \lambda = 0$ and $M_\gamma$ is a *ruled real hypersurface.*

Proposition 5. *Let $k$ be a nonzero constant and let $(r(s), \theta(s), \alpha(s))$ be a solution of the system of nonlinear ODE,*

$$\begin{aligned} &\dot{r} = \cos\alpha, \qquad \dot{\theta} = \frac{\sin\alpha}{\sin r}, \\ &\dot{\alpha} + \sin\alpha\bigg((1+\sin^2\alpha)\cot r - \frac{\cos\alpha\sin\alpha}{\sin r}(\cot\theta - \tan\theta) \\ &\quad - \cos^2\alpha(\cot r + \tan r)\bigg) = k, \end{aligned} \tag{30}$$

*such that the initial condition satisfying $0 < r(0), \theta(0) < \pi/2$. Then the real hypersurface $M_\gamma$ in $\mathbb{CP}^2$, defined by (21) is Levi-umbilical.*

A special solution of (30) is given by

$$r = \text{constant}, \qquad \alpha \equiv \pi/2 \mod \pi.$$

In the case $\alpha = \pi/2$, we have $\mu = 2\cot 2r$, $\nu = 0$ and $\dot{\alpha} + \sin\alpha \cot r = \lambda = \cot r$. Hence $M_\gamma$ is a geodesic sphere of radius $r$ $(0 < r < \pi/2)$ with $k = 2\cot r$.

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https://doi.org/10.1017/nmj.2016.63 Published online by Cambridge University Press