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Determinantal rational surface singularities

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Abstract. In this paper we give explicit equations for determinantal rational surface singularities and prove dimension formulas for the T^1 and T^2 for those singularities.

Key words: rational singularity, determinantal variety.

1. Introduction

Let (X, x) be a germ of a normal surface singularity of embedding dimension e. Then the local ring $\mathcal{O}_X$ of X can be given as $\mathcal{O}_X = P/I$, where P is a power series ring in e indeterminates. One says that X is *determinantal* if the ideal I can be generated by the $t \times t$ minors of an $r \times s$ matrix with entries in P, with the condition that the codimension $e - 2$ is equal to the 'expected' codimension $(r - t + 1)(s - t + 1)$.

We consider *rational surface singularities*. For those we know that the multiplicity m is equal to $e - 1$ [1]. Wahl proved [13] that a *rational* surface singularity of embedding dimension e can be given by $m(m - 1)/2$ equations with *linear independent quadratic terms*. Using this, it is not hard to show

PROPOSITION [13] (3.2). *Let X be a rational determinantal surface singularity of multiplicity $m \geqslant 3$. Then equations for X can be given by the 2×2 minors of a $2 \times m$ matrix.*

Wahl also remarked that few rational surface singularities are determinantal.

THEOREM [13] (3.4). *Let (X, x) be a determinantal rational surface of multiplicity $m \geqslant 3$, and $(\widetilde{X}, E) \to (X, x)$ be the minimal resolution. Then E consists of one $(-m)$ curve and (possibly) some (-2) curves.*

The $(-m)$ curve we call the *central curve* from now on. The proof Wahl gives is not difficult. Let (X, x) be a determinantal rational surface singularity, given by the 2×2 minors of a matrix

$$\begin{pmatrix} f_1 & \cdots & f_m \\ g_1 & \cdots & g_m \end{pmatrix}.$$

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One has a rational map $(g_i\colon f_i)\colon X \to \mathbb{P}^1$. (This is independent of i.) One can define a modification $(\overline{X}, E_0) \to (X, x)$ (called the *Tjurina modification* by Van Straten [11]) by taking $\overline{X}$ the closure of the graph of this rational map. This $\overline{X}$ is then given by the following equations: ($(s\colon t)$ are homogeneous coordinates)

$$sf_1 = tg_1, \ldots, sf_m = tg_m.$$

There is an exceptional $\mathbb{P}^1$ in $\overline{X}$, given by the ideal generated by the f's and the g's. Wahl shows that $\overline{X}$ can only have rational double point singularities, and that the central curve has coefficient one in the fundamental cycle, from which he is able to deduce the Theorem.

Wahl also expected that the converse of this Theorem is true, and wrote down determinantal equations for some determinantal rational surface singularities with reduced fundamental cycle. (The proof of [13] 3.6 is incomplete, however.) Also Van Straten [11] wrote down equations for some so-called A_q^k singularities, which are almost the same as ours. The converse of Wahl's Theorem was shown by Röhr [10], as a special case of a much more general Theorem on *formats*. The purpose of this paper is to give eplicit equations for determinantal rational surface singularities, thereby also showing the converse of Wahl's Theorem. Wahl's Theorem restricts very much the shape of the resolution graph of a determinantal rational surface singularity: One has one $(-m)$ curve, with rational double point configurations (RDP-configurations) attached to it. Applying a rationality criterium (using a computation sequence for the fundamental cycle) one gets a list of how which RDP-configurations can be attached to the central curve. This is all well-known (and easy) and the list is written down in the first section.

Given a resolution graph of a rational determinantal singularity Γ one can try to write down (determinantal) equations, which define a singularity with resolution graph Γ. If one has those equations, it is relatively easy to check that the resolution graph is indeed Γ, using the Tjurina modification (remember the easy equations above for the Tjurina modification). This is done in section two.

The problem is that surface singularities in general are *not* determined by the analytic type of the resolution graph. (Laufer [8] wrote down all for which they do determine the singularity.) So, we do not know whether *all* rational surface singularities of multiplicity m and with one $(-m)$ curve in the minimal resolution have equations as given in section two (although this turns out to be the case). In section three we will resolve this problem. We will construct divisors on the minimal resolution of our singularity. Then we invoke Artin's Theorem, saying that if one has a divisor on the minimal resolution of a rational surface singularity, which intersects every exceptional curve trivial, then this divisor is principal, so of the form (f). (Given a divisor, the choice of f is determined up to a unit.) By constructing enough divisors, we get plenty of functions on X. Using then *additive* relations between the divisors, one gets *multiplicative* relations between the corresponding functions by choosing the functions, given their divisors, smart enough. So, then one has still to check whether there are additive relations between the functions. We will show that the relations between those functions generate the equations for the singularities. This will all be done in the third section.

Certainly our result is not the best possible, in the sense that some terms in the equations can be disposed of after coordinate transformations. To have this sorted out however, seems to require much more work.

Using the equations of determinantal rational surface singularities we are able to get dimension formulae for the T^1 and T^2 of a rational surface singularity, which are similar to the formulae for these modules for rational surface singularities with reduced fundamental cycle [6].

This will be done in the fourth section. The formula for T^2 says

$$\dim(T^2_X) = (m-1)(m-3) + \sum_{p \in \hat{X}} \dim(T^2_{\hat{X},p}),$$

where $\hat{X} \to X$ is the blow-up of X in its singular point. There is a similar formula for T^1. So, the dimensions of those vector spaces are more or less calculable from the resolution. Hopefully this result will be a beginning of an understanding of the deformation theory of determinantal rational surface singularities.

It is possible to write down equations for the more general class of so-called *quasi-determinantal* rational surface singularities. These singularities were also characterized (in terms of their resolutions graph) by Röhr. We will report on that in a future paper. At the moment we are not able to get a similar result for the T^1 and T^2 of a quasi-determinantal rational surface singularity. This problem seems to be much harder than the corresponding question for the determinantal ones.

1. Rational double point configurations

Let (X, x) be a normal surface singularity with minimal resolution $(\tilde{X}, E) \to (X, x)$. Let $E = \cup E_i$ be the irreducible decomposition of E. The fundamental cycle Z by definition is the *minimal* positive cycle with support on E subject to the condition that $Z \cdot E_i \leqslant 0$ for all exceptional divisors E_i. The fundamental cycle can be computed by means of a computation sequence [7] 4.1, as follows.

Let $Z_0 := E$. Given Z_k, if there is an exceptional curve F with $Z_k \cdot F > 0$ then define $Z_{k+1} := Z_k + F$. If on the other hand $Z_k \cdot F \leqslant 0$ for all exceptional divisors F then put $Z = Z_k$.

This process stops. Computation sequences are useful not only for computing Z, but also because of the following

RATIONALITY CRITERIUM (1.1). (X, x) *is rational if and only if the following two conditions hold*

- *Every exceptional curve is a* $\mathbb{P}^1$.
- *If* Z_k *appears in a computation sequence for* Z *and if* $Z_k \cdot F > 0$ *then* $Z_k \cdot F = 1$.

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For a rational surface singularity the fundamental cycle also gives information about the multiplicity m and embedding dimension e: one has $-Z^2 = m = e - 1$, see [1].

From now on we will assume that (X, x) is rational of multiplicity m and that there is one exceptional curve, say E_0, on the minimal resolution which has self-intersection $(-m)$. For convenience, we call such a singularity determinantal rational (although we have not proved yet that such a singularity is determinantal). The curve E_0 we call the *central* curve. Although the following proposition is well-known, we include a proof.

PROPOSITION (1.2). *Let (X, x) be a rational surface singularity of multiplicity m with one $(-m)$ curve E_0 on the minimal resolution. Then*

- *All other exceptional curves have self-intersection -2.*
- *The coefficient of the fundamental cycle Z at the central curve E_0 is one.*

Proof. Let K be the canonical divisor on the minimal resolution. Then one has the adjunction formulas

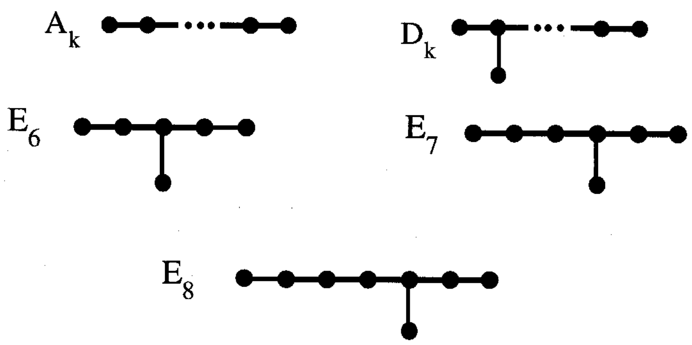
- $E_i \cdot K = -2 - E_i^2$ for all i. Note that this number is always nonnegative, because we work on the minimal resolution. In particular we have $E_0 \cdot K = m - 2$.
- $Z \cdot K = -2 - Z^2 = m - 2$.

Now write $Z = \sum a_i E_i$ with $a_i > 0$ and compute

$$(m-2) = ZK = a_0 E_0 K + \sum_{i \neq 0} a_i E_i K = a_0(m-2) + \sum_{i \neq 0} a_i(-2 - E_i^2).$$

Because $a_i > 0$ for all i it follows that $a_0 = 1$ and $E_i^2 = -2$ for all $i \neq 0$. $\square$

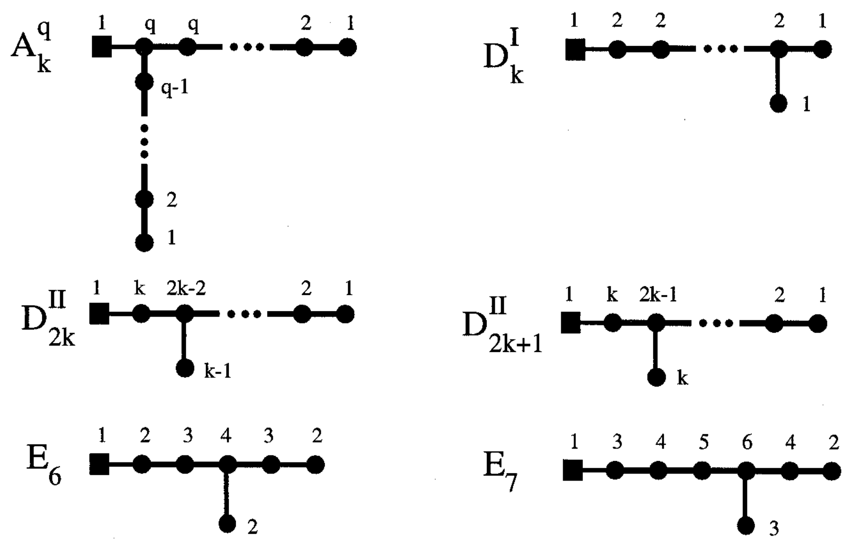
As any sub-configuration of the minimal resolution of a rational surface singularity contracts itself to a rational surface singularity, the structure of the resolution graph of a determinantal rational surface singularity is quite simple: one has a central $(-m)$ curve and rational double point configurations (RDP-configurations) intersecting this central curve in different points. The list of rational double points of course is very well known, the famous A, D, E list



Of course, a dot denotes a (-2) curve.

Because of the rationality condition however, the central curve cannot intersect an arbitrary curve of a RDP-configuration. Below we list the possibilities of intersections of the central curves with the different RDP-configurations.

PROPOSITION/DEFINITION (1.3). *Rational double point configurations can intersect the central curve only as in one of the following cases*



The box denotes the central curve. All other curves are (-2) curves. The number of them is $k+q-1$ in case A_k^q, otherwise it is the suffix. The number written at each vertex is the coefficient of the corresponding curve in the fundamental cycle. For each rational double point configuration R_a we define the *multiplicity* $m(a)$ as the coefficient of the fundamental cycle at the unique curve of the rational double point configuration intersecting the central curve. So, we assumed implicitly that the self-intersection of the central curve is at most minus the coefficient of the fundamental cycle of the curve adjacent to it. In case $Z \cdot E_0 < 0$ we will say that there are $-Z \cdot E_0$ A_0^1 rational double point configurations. The multiplicity of such an A_0^1 configuration we define to be *one*. This done formally, the number of rational double point configurations is exactly the number of irreducible components of a generic hyperplane section of the surface singularity. In fact, sometimes we will identify an A_0^1 with a smooth non-compact curve, which intersects the central curve transversally on the minimal resolution.

Sketch of proof. We try to attach the central curve to one of the rational double point configurations. From the rationality criterium it follows that there cannot be two vertices of valence three in the resolution graph. So except for the case A_k^q one has to attach the central curve to an endpoint of the D, E configuration. Using the

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rationality criterium it is tedious to check that one is left with the possibilities as written down in the list. □

2. Equations for determinantal rational surface singularities

We consider arbitrary rational double point configurations which we denote by $R_a, 0 \leqslant a \leqslant t$. (Recall our convention on A_0^1 rational double point configurations.) The multiplicity of R_a we denote by $m(a)$.

We will write down equations for all determinantal rational surface singularities with these given rational double point configurations. This will be done in the following two definitions.

DEFINITION (2.1). *Let x be an independent variable, and for each rational double point configuration consider variables $y_{ia}, 0 \leqslant i \leqslant m(a)-1$.*

For each rational double point configuration R_a consider matrices M_a (For simplicity we will not write the suffix a in the variables y_{ia})

- A_0^1:

$$M_a = \begin{pmatrix} y_0 \\ x \end{pmatrix}.$$

- A_k^q : *Define numbers r and p by*

$$k = qr - p; 0 \leqslant p \leqslant q-1,$$

$$M_a = \begin{pmatrix} y_0 & \cdots & y_{p-1} & w & y_{p+1} & \cdots & y_{q-2} & y_{q-1} \\ y_1 & \cdots & y_p & y_{p+1} & y_{p+2} & \cdots & y_{q-1} & xy_0 \end{pmatrix}.$$

$$w = y_p + x^r + \text{Rest};$$

$$\text{Rest } \in (xy_0, \ldots, xy_{p-1}, y_{p+1}, \ldots, y_{q-1}).$$

- D_k^I:

$$M_a = \begin{pmatrix} y_0 & y_1 \\ y_1 & w \end{pmatrix}.$$

$$w = x^2 + y_0^{k-1} + \lambda x y_0^q \quad \textit{for some function } \lambda$$

and q is the integral part of $(k+1)/2$.

- D_{2k}^{II}:

$$M_a = \begin{pmatrix} y_0 & y_1 & \cdots & y_{k-2} & w \\ y_1 & y_2 & \cdots & y_{k-1} & x^2 \end{pmatrix},$$

$$w = y_{k-1} + y_0^2 + \text{Rest};$$

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Rest $\in (y_0y_1, \ldots, y_0y_{k-2}, xy_0, \ldots, xy_{k-2})$.

- D^{II}_{2k+1}:

$$M_a = \begin{pmatrix} y_0 & y_1 & \ldots & y_{k-2} & w \\ y_1 & y_2 & \ldots & y_{k-1} & y_0^2 \end{pmatrix},$$

$$w = y_{k-1} + x^2 + \text{Rest};$$

$$\text{Rest} \in (xy_0, \ldots, xy_{k-2}, y_0^2, \ldots, y_0y_{k-2}).$$

- E_6:

$$M_a = \begin{pmatrix} y_0 & y_1 \\ y_1 & w \end{pmatrix},$$

$$w = y_0^2 + x^3 + \lambda x^2 y_0 \quad \textit{for some function } \lambda.$$

- E_7:

$$M_a = \begin{pmatrix} y_0 & y_1 & y_2 \\ y_1 & y_2 & w \end{pmatrix},$$

$$w = y_0^3 + x^2 + \text{Rest};$$

$$\text{Rest} \in (xy_1, y_1^2, xy_0^2, y_0^2y_1, xy_2).$$

DEFINITION (2.2). *Fix a double point configuration, say R_0. For all other rational double point configuration R_a, $1 \leqslant a \leqslant t$ consider units u_a and v_a in $\mathbb{C}\{x, y_{ia}\}$. Suppose that for $a \neq b$ the constants $u_a(0)$ and $u_b(0)$ are not equal. Consider the matrix*

$$N_a = \begin{pmatrix} 1 & 0 \\ u_a & v_a \end{pmatrix} M_a.$$

So to get N_a from M_a we multiply the second row of M_a by the unit v_a, and then we add u_a times the first row to the second row. Moreover we put $N_0 = M_0$. We then put

$$N = (N_0 N_1 \ldots N_t)$$

THEOREM (2.3). *Fix rational double point configurations, $R_0 \ldots R_t$, and let N be a matrix defined as above. For every choice for w_a, u_a and v_a (with the restrictions* *as above) the* 2×2 *minors of the matrix* N *define a rational surface singularity* X *of multiplicity* $m = \sum_{a=0}^{t} m(a)$*, having rational double point configurations* $R_0, \ldots, R_t$*. Moreover on the minimal resolution of* X *there is a* $(-m)$ *curve.*

Conversely any rational surface singularity X *with a* $(-m)$ *curve on the minimal resolution, and rational double point configurations* $R_0 \ldots R_t$ *can be defined by the* 2×2 *of a matrix* N *as defined above, for suitable choices of* w_a, u_a *and* v_a*.*

Proof. Here we only prove the first statement. The proof of the converse will take the whole of the next section. We write

$$N = \begin{pmatrix} f_1 & \ldots & f_m \\ g_1 & \ldots & g_m \end{pmatrix}$$

and we consider the Tjurina modification

$$p\colon (\overline{X}, E_0) \to (X, x)$$

defined by the equations

$$sf_1 = tg_1; \ldots; sf_m = tg_m.$$

This map is well-defined, precisely because X is defined by the 2×2 minors of N. The $(s\colon t)$ are homogeneous coordinates on E_0, which is a $\mathbb{P}^1$. The curve E_0 is mapped by p to the singular point of X. Let c_a be the constant term of u_a for all rational double point configurations. Then

CLAIM. In the equations given above of the Tjurina modification $\overline{X}$ one can, away from the point $(c_a\colon 1)$, eliminate the variables $y_{ib}; b \neq a$. (i.e. locally they occur with independent linear terms.)

We will look away from the point $(1\colon 0)$. The investigation locally at the point $(1\colon 0)$ is left to the reader. In the first row of M_a there is always a linear part of type

$$(y_{0a} \ldots y_{m(a)a}).$$

We denote the second row of M_a by

$$(h_{0a} \ldots h_{m(a)a}).$$

Then one notes (case by case check) that h_{ia} does not contain the terms y_{ja} for $j \leqslant i$, and also not linear terms of type y_{ib} for $b \neq a$. Also the term x never occurs

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in the second row. We have the following equation for $\overline{X}$ in the chart $t = 1$

$$sy_{ia} - (u_a y_{ia} + v_a h_{ia}) = 0.$$

Because v_a is a unit and h_{ia} does not contain the linear terms mentioned above, we can successively eliminate $y_{0a} \dots y_{m(a)a}$ away from $s = u_a(0) = c_a$. This proves the claim, and more. It also shows that away from the points

$$(0:1), (c_1:1), \dots, (c_t:1)$$

on E_0 the Tjurina modification is smooth. In fact, away from those points one can eliminate all y_{ia}, leaving us with the variables s, x. The Tjurina modification is given by m equations, and we have $m+2$ variables locally. We conclude that $\overline{X}$ is smooth with parameters s, x away from the points $(0\colon 1), (c_1\colon 1), \dots, (c_t\colon 1)$ on E_0. Moreover it follows that the (lift of the) function x vanishes with order one on the curve E_0.

We now investigate the singularities at the points $(c_{ia}\colon 1)$ for all rational double point configurations R_a. As mentioned above, all other y_{ib} for $b \neq a$ can be eliminated. So we are left then with the equations for the Tjurina modification coming from the part N_a. But by doing the coordinate transformation: $s \mapsto s + u_a$, and after that, multiplying s by the unit v_a, we just might consider the matrix M_a. Therefore, we have to investigate the Tjurina modification for every matrix M_a at the point $(0\colon 1) \in E_0$. We claim that for each M_a we have the rational double point configuration R_a. This is routine case by case check which we will do in two cases. The other cases are left to the reader. We omit the suffix a in doing this check.

(1) D_k^I: We write $y_0 = y$ and $y_1 = z$. The equations for the Tjurina modification are $sy = z; sz = w$. We eliminate z and get

$$s^2 y = x^2 + y^{k-1} + \lambda x y^q.$$

This indeed is a D_k singularity. To see where the central curve E_0, which is given by $x = y = 0$, intersects the D_k configuration, we blow-up. We look at the s-chart. So replace (x, y, s) by (sx, sy, s). The strict transform has equation

$$sy + x^2 + s^{k-3} y^{k-1} + \lambda s^{q-1} x y^q$$

and the exceptional locus is given by $s = x^2 = 0$. So the strict transform has an A_1 singularity at $(0,0,0)$, and the strict transform of E_0, which still is given by $x = y = 0$ goes through it. Now it is well known, and easy to check that the utmost left curve in the D_k configuration

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is obtained by resolving the A_1 singularity of the strict transform, and indeed the central curve intersects it.

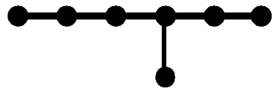
(2) E_7: In the Tjurina modification we eliminate the variables y_1 and y_2. After writing $y_0 = y$ the singularity on the Tjurina modification has the equation

$$s^3 y = y^3 + x^2 + \text{Rest}; \text{Rest} \in (sxy, s^2y^2, sy^3).$$

This indeed is an E_7 singularity. The central curve is given by $x = y = 0$. We blow up and look in the s-chart. The strict transform is given by:

$$s^2 y = sy^3 + x^2 + \text{Rest}; \text{Rest} \in (sxy, s^2y^2, sy^3)$$

and the exceptional locus is the $\mathbb{P}^1$ given by $s = x^2 = 0$. In the E_7 configuration



it is the utmost right curve. The strict transform has a singularity of type D_6 in $(0,0,0)$ and the strict transform of E_0 goes through it. Now the proof goes on as in the D_k^I case, and we conclude that the curve E_0 goes through the utmost left curve of the D_6 configuration, which together with the curve $x^2 = 0$ gives the E_7 configuration.

Let us recapitulate what we proved by now. On the minimal resolution of our singularity we have the central curve E_0 and rational double point configurations $R_0, \ldots, R_t$. All exceptional curves are $\mathbb{P}^1$'s, and only the central curve might not be a (-2) curve. What we are left to show is that the central curve has self-intersection $-m = -\sum m(a)$. This can be done directly, by calculating the vanishing order of the function x on every exceptional curve. But we can also argue as follows. The vanishing order of x along the exceptional curve of R_a intersecting the central curve must be at least $m(a)$. This is because the maximal ideal cycle is at least the fundamental cycle Z. As the vanishing order of x along the central curve is one, we deduce that $E_0^2 \leqslant -m$. Using the rationality criterium, one sees that X is rational of multiplicity $-E_0^2$. But our singularity is given by $m(m-1)/2$ equations with linear independent quadratic part (a tedious check). Therefore, by Wahl's structure Theorem for equations of rational surface singularities, we deduce $m = -E_0^2$. $\square$

3. Divisors on the minimal resolution

We consider a rational surface singularity (X, x) of multiplicity m, with a $(-m)$ curve on the minimal resolution $(\tilde{X}, E)$. Rather we consider good representatives for those. We will embed X in complex space. For this, we need functions on X, which generate the maximal ideal of the local ring $\mathcal{O}_X$. To obtain equations, one has to determine the relations between these functions. The fundamental tool in constructing functions on rational surface singularities is given in the following Theorem of Artin.

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THEOREM (3.1) [1] (proof of Theorem 4). *Let* $\pi\colon(\tilde{X},E)\to(X,x)$ *be the* (*minimal*) *resolution of a rational surface singularity* X. *Let* Y *be a Weil-divisor on* $\tilde{X}$ *with the condition that* $Y\cdot E_i=0$ *for all irreducible components* E_i *of* E. *Then* Y *is a* principal *divisor, i.e.* $Y=(y\pi)$ *for some* $y\in\mathcal{O}_X$.

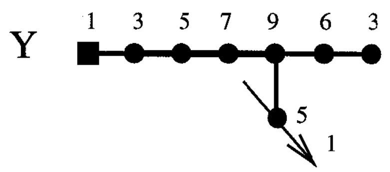
Of course, a function y as in the Theorem is determined up to a unit in $\mathcal{O}_X$ by the divisor Y.

Moreover we will need the following Theorem of Artin, which he did not formulate either. A proof is contained in loc. cit.

THEOREM (3.2) [1] (proof of Theorem 4). *Let* X *be a rational surface singularity,* $\pi\colon(\tilde{X},E)\to(X,x)$ *be the minimal resolution. Write the fundamental cycle* Z *as* $Z=\sum r_iE_i$. *Let* H *be a divisor on* $\tilde{X}$ *with* $d_i:=H\cdot E_i\leqslant 0$ *for all* i. *Let* $\mathcal{O}(-H)=\{f\in\mathcal{O}_X\colon(f\pi)\geqslant H\}$. *Then the number of generators of the ideal* $\mathcal{O}(-H)$ *is equal to* $1+\sum_i d_ir_i$.

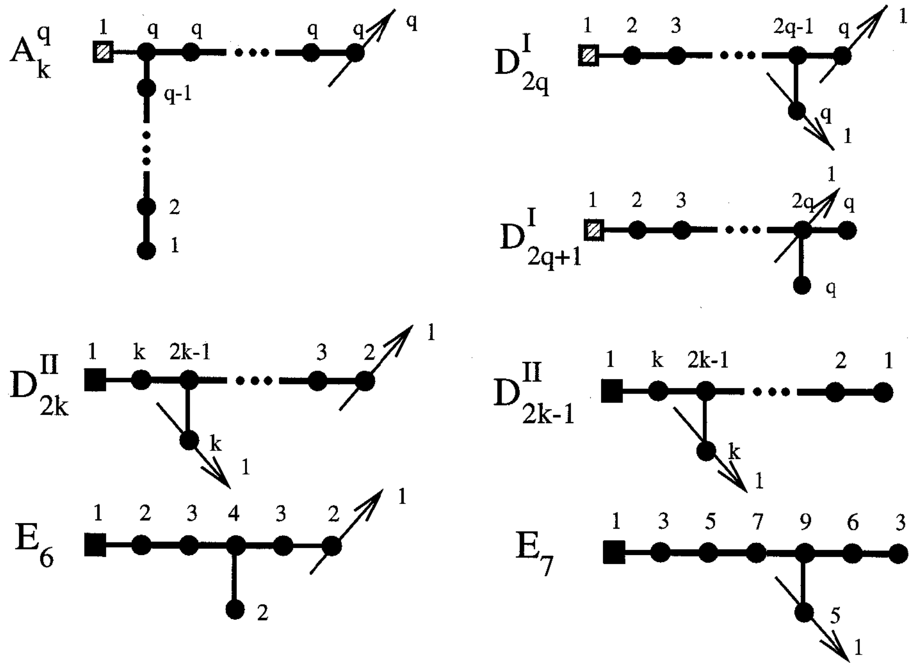
Our first job in this section is to write down divisors on the minimal resolution of a determinantal surface singularity. Such a divisor Y on $\tilde{X}$ can be decomposed as $Y=C+N$. Here C is a compact divisor, and therefore has support on the exceptional divisor E, and N is a non-compact divisor, i.e. a divisor whose support has a finite number of intersection points with the exceptional divisor. *In this paper we only consider divisors on* $\tilde{X}$, *for which each irreducible component of the support of the non-compact part* N *intersects exactly one exceptional divisor transversally.* So such a divisor therefore does not pass through an intersection point of two exceptional divisors.

For the compact part C of Y we use the dual graph notation; writing $C=\sum a_iE_i$ we put the number a_i at the vertex in the dual graph which corresponds to the exceptional curve E_i. For the non-compact part N, write $N=\sum b_jN_j$. Then for all j we draw an arrow through the unique vertex on the dual graph, which corresponds to the curve the N_j intersects. Moreover we will write the number b_j near this arrow. In the example



the non-compact part consist of a smooth branch on $\tilde{X}$ with multiplicity one. (Of course, its image on X is not smooth.) This divisor satisfies the condition of Artin's Theorem, i.e. intersects every exceptional divisor trivially. As it is usually a very easy exercise to check that the conditions of Artin's Theorem are satisfied, we immediately will write $Y=(y)$, indicating that the divisor is principal.

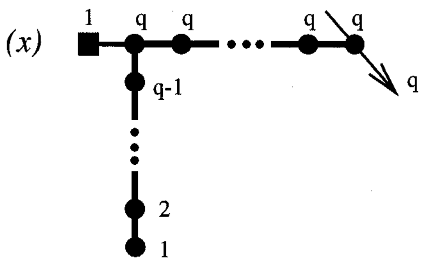
We begin with writing down the divisor (x) of a function x. We write down the restrictions to each RDP-configuration and the central curve. The divisor (x) contains all A_0^1 singularities (which by our convention are non-compact branches intersecting the central curve E_0) with multiplicity one. For the other RDP-configurations we define

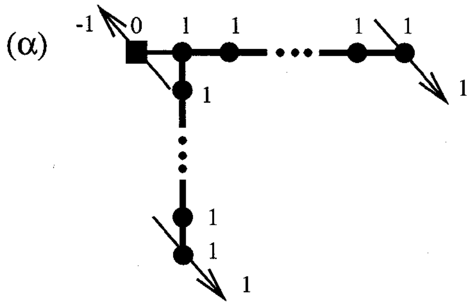


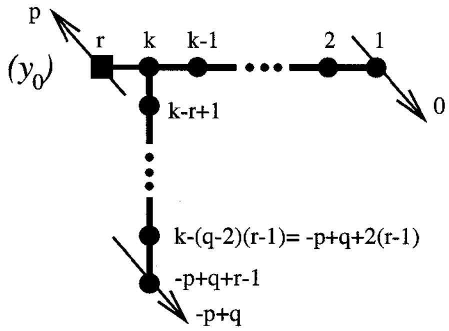
So Artin's Theorem gives us a function x. This function x is fixed *once and for all.* Remark that x is in the maximal ideal, but not in the square of the maximal ideal, because the divisor (x) is strictly less than $2Z$. For every rational double point configuration R_a we will now define certain divisors (y_{ia}) and (w_a) of functions on the minimal resolution. We will only write down the *restriction* to the rational double point configuration R_a, and the coefficient at the central curve E_0. Those restrictions are extended to divisors on the whole minimal resolution by putting on the *complement* of R_a: $(y_{ia}) = c_a \cdot (x)$ where c_a is the coefficient of (y_{ia}) at the central curve E_0. The non-compact divisor which is drawn through the central curve we call P (P for pole divisor). The divisor P is supposed not to intersect any RDP-configuration. For the moment we will suppress the suffix a for the divisors (y_{ia}) and (w_a). For completeness we rewrite the divisor (x). A remark in advance: If the number of y's is small, we will usually write $y_0 = y$ and $y_1 = z$.

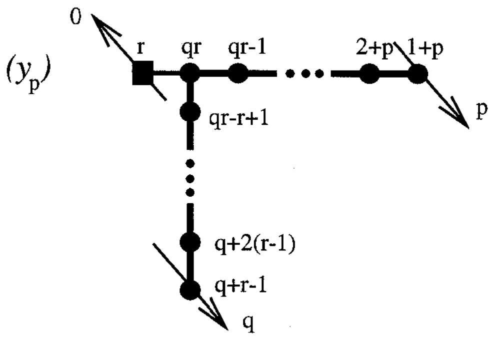
Case A_0^1. Let C be the non-compact branch of the A_0^1 configuration. Then we define $(y) = (x) + P - C$.

Case A_k^q.

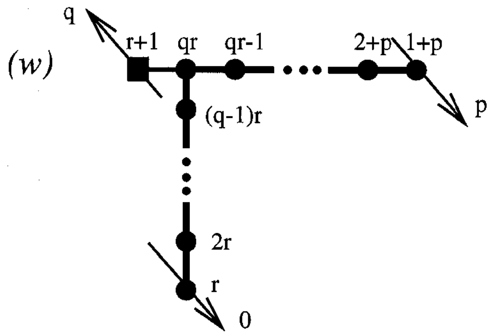


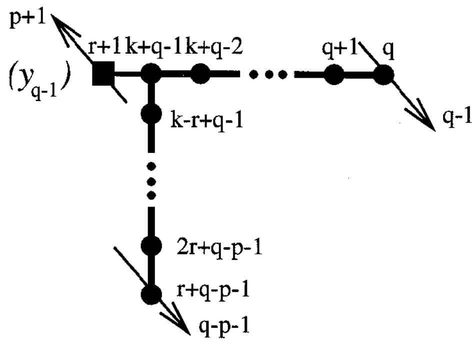






We moreover define divisors (y_i) for $1 \leqslant i \leqslant p-1$ by $(y_i) = (y_0) + i \cdot (\alpha)$.





We define the divisors (y_i) for $p+1 \leqslant i \leqslant q-1$ by $(y_i) = (w) + i \cdot (\alpha)$.

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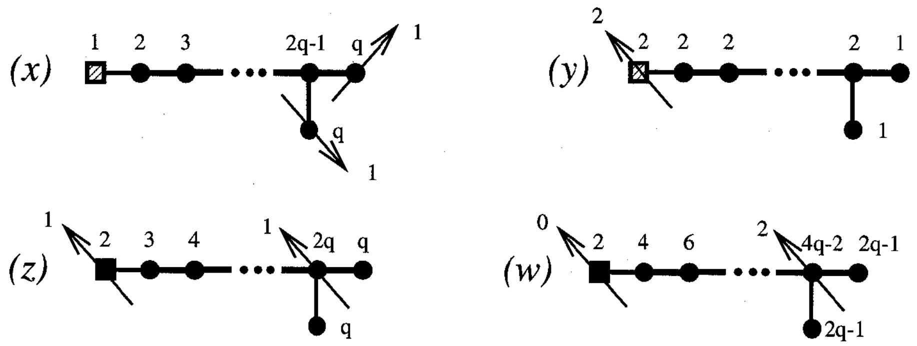
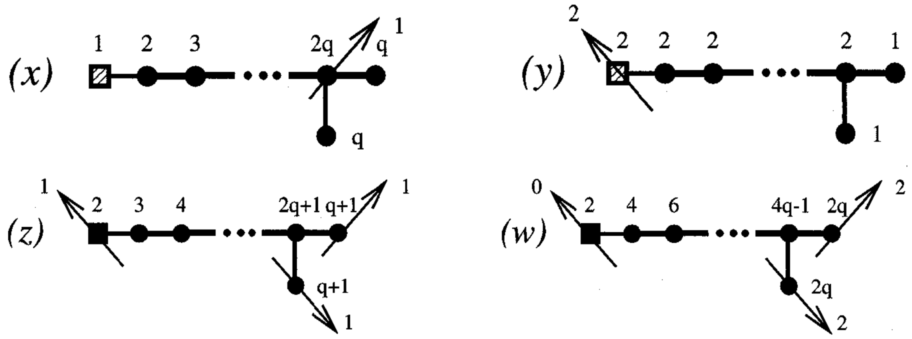
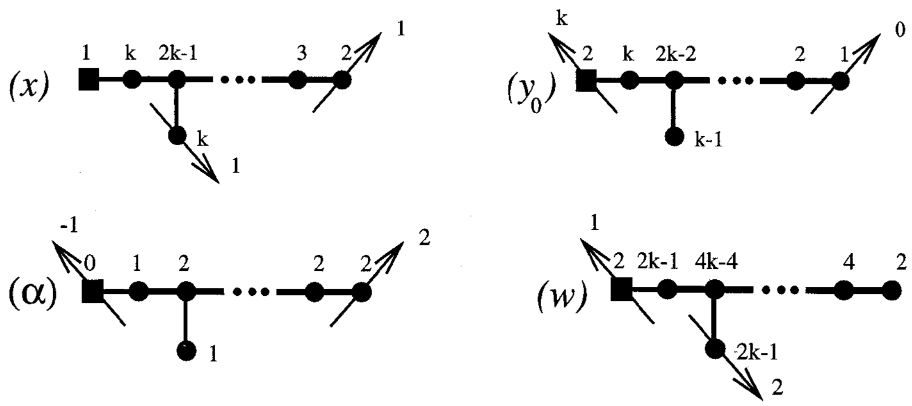
Case D^I_{2q}.



Case D^I_{2q+1}.

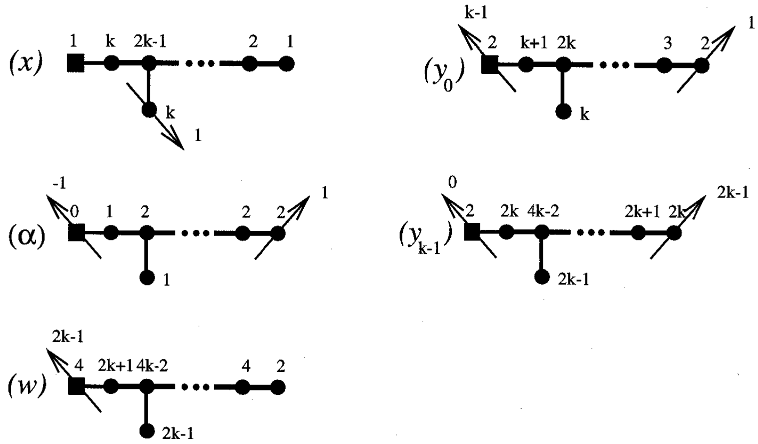


Case D^{II}_{2k}.



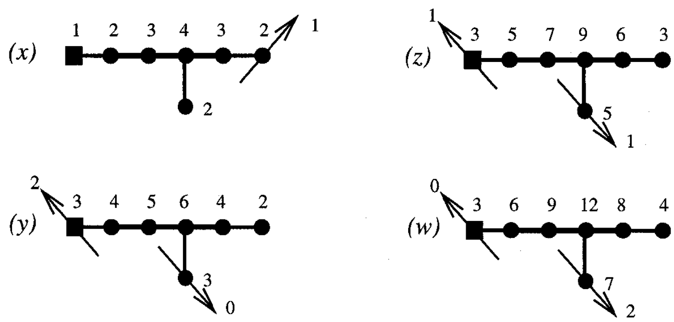
We moreover define divisors (y_i) by setting $(y_i) = (y_0) + i(\alpha)$ *for* $0 \leqslant i \leqslant k-1$.

Case D^{II}_{2k+1}.

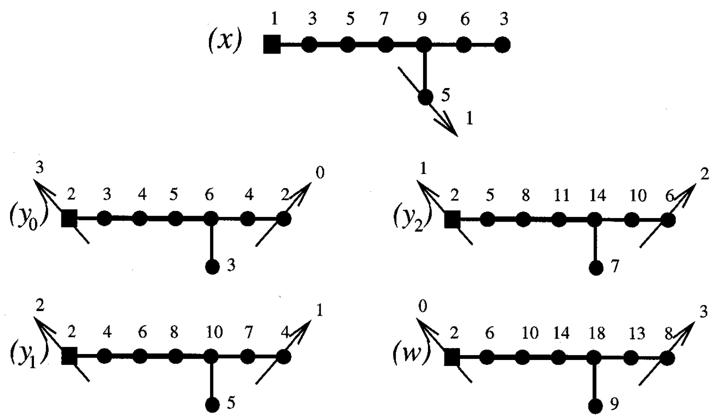


We moreover define divisors (y_i) by setting $(y_i) = (y_0) + i(\alpha)$ *for* $0 \leqslant i \leqslant k-2$.

Case E_6.



Case E_7.



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PROPOSITION (3.3). *The functions x and y_{ia} where R_a run over all rational double configurations generate the maximal ideal of $\mathcal{O}_X$.*

Proof. The number of y_{ia} is exactly the multiplicity m of the singularity. Suppose that

$$cx + \sum c_{ia} y_{ia} \in \mathbf{m}_X^2$$

for some $c, c_{ia} \in \mathbb{C}$. We will show that the coefficients c, c_{ia} are zero. As by Artin one knows that the number of generators of the maximal ideal is $m+1$, this suffices to prove the Proposition. Of course it suffices to prove that the c_{ia} are zero, as assuming this, $c = 0$ follows immediately from the fact that x is not in the square of the maximal ideal. We first consider the ideal $J := (x, y_{ia}\colon R_a$ is *not* an A_0^1 configuration). The strict transform of the zero set of J on the minimal resolution consist exactly of the non-compact part C of (x) passing through the central curve. The number of irreducible components of those is exactly the number of A_0^1 singularities. For every R_a, which is an A_0^1 singularity, the function y_{0a} vanishes identically on all but one of the irreducible components of C, and is a parameter on the irreducible component belonging to R_a. Now it follows immediately that $c_{ia} = 0$ for all A_0^1 configurations R_a.

So we may suppose that the above sum is only over all non-A_0^1 configurations. We now look at a fixed rational double point configuration R_a which is not an A_0^1 configuration. There is exactly one irreducible component C_a of (x) passing through an exceptional curve of R_a. (This is by construction of the divisor (x).) We put $x' = x + \varepsilon \cdot y_{0a}$, which is a small perturbation of our function x. For ε general enough, a case by case check shows that the unique irreducible component C_a' of $(x'\pi)$ passing through an exceptional curve E_a of R_a is smooth and *reduced*. The minimal vanishing order of a function in **m** along E_a is $m(a)$, the coefficient of the fundamental cycle at the curve E_a. From the definition of the functions y_{ib}, for $b \neq a$, it follows that they vanish with order at least $2m(a)$ along E_a. Here we use that R_b is *not* an A_0^1 configuration. Also all functions in the square of the maximal ideal vanish with order at least $2m(a)$ along E_a. As by construction the function y_{ia} vanishes with order $m(a)+i$ along E_a, it follows that the classes of y_{ia} $(i \leqslant m(a)-1)$ in the local ring of C_a' generate its maximal ideal. Thus, we conclude that $c_{0a} - c \cdot \varepsilon, c_{1a}, \ldots$ are zero. As this is true for all small ε, also $c_{0a} = 0$. □

Note that in all cases, (except in case A_0^1) we also wrote down the divisor of a function w. From the proposition it follows that w must be expressible in the y_{ia} and the function x. We will make this somewhat more explicit in the following proposition.

PROPOSITION (3.4). *One can choose w and the y_{ia} such that in $\mathcal{O}_X$ the 2×2 minors of the matrix M_a of Definition* (2.1) *are identically zero. In particular, one can express the function $w_a \in \mathcal{O}_X$ as done in Definition* (2.1).

Proof. We suppress the suffix a in this proof. For each RDP-configuration (except A_0^1) we consider an ideal I in the local ring of the singularity.

- A_k^q: $(y_p, x^r, xy_0, \ldots xy_{p-1}, y_{p+1}, \ldots, y_{q-1})$,
- D_k^I: $(x^2, y^{k-1}, xy^q, zx, zy^{q-1})$,
- D_{2k}^{II}: $(y_{k-1}, y_0^2, y_0y_1, \ldots, y_0y_{k-2}, xy_0, \ldots, xy_{k-2})$,
- D_{2k+1}^{II}: $(y_{k-1}, x^2, xy_0, \ldots, xy_{k-2}, y_0^2, \ldots, y_0y_{k-2})$,
- E_6: (y_0^2, x^3, x^2y_0),
- E_7: $(y_0^3, x^2, xy_1, y_1^2, xy_0^2, y_0^2y_1, xy_2)$.

In each case we define the divisor H as the infimum of the divisors of functions appearing in the definition of the above ideal I. A case by case check shows that $w \in \mathcal{O}(-H)$. We now claim that $I = \mathcal{O}(-H)$. First of all, a case by case check, using Artin's Theorem (3.2), shows that the number of generators of $\mathcal{O}(-H)$ is exactly the number of generators we used to define I. To prove the claim, one therefore has to show that the functions used to define I are linearly independent modulo $\mathbf{m}\mathcal{O}(-H)$. This is done by looking at vanishing orders of the functions along certain exceptional divisors.

To give an example, look at the case D_{2k}^{II}. Here the divisor H is given by the left-hand side of the following picture. On the right-hand side, we rewrite the coefficient of the fundamental cycle on this rational double point configuration (which is also the maximal ideal cycle, as we have a rational surface singularity).

We give some of the exceptional curves names, as indicated by the above picture. Now suppose

$$ay_{k-1} + \sum a_i y_0 y_i + \sum b_i x y_i \in \mathbf{m}\mathcal{O}(-H)$$

for constants a, a_i, b_i. We have to show that they are all zero. The vanishing order of y_{k-1} along A is $2k-1$. All other functions in our list have higher vanishing order along this curve. As an element in $\mathbf{m}\mathcal{O}(-H)$ has vanishing order at least $3k-1$ along A it follows that $a = 0$. Elements in $\mathbf{m}\mathcal{O}(-H)$ have vanishing order at least $6k-4$ along the curve F. The vanishing orders of

$$y_0^2, \ldots, y_0 y_{k-2}, xy_1, \ldots, xy_{k-2}$$

along F are respectively

$$4k-2, 4k, \ldots, 6k-6, 4k-1, \ldots, 6k-5.$$

Every order o with $4k-2 \leqslant o \leqslant 6k-5$ occur exactly once. It follows that a_i and b_i are all zero. This shows indeed that in this case $I = \mathcal{O}(-H)$.

All other cases are treated in a similar way, by looking at vanishing order at certain exceptional divisors. We therefore leave the other cases to the reader.

Because, as already remarked, $w \in \mathcal{O}(-H)$, it follows that w can be written as a combination of the generators of the ideal I: writing $I = (g_1, \ldots, g_s)$ in the order as written above, we have $w = \sum_{i=1}^{s} a_i g_i$ for some a_i. We now claim that a_1 and a_2 are units. This again is done by looking at the vanishing order along certain exceptional divisors. We again take the above example. For a_1 we look at the vanishing order along A: the function w vanishes with order $2k-1$ there. But y_{k-1} is the only generator vanishing with order $2k-1$ there; the other vanish with higher order. Therefore a_1 must be a unit. For a_2, look at the non-compact curve intersecting the utmost right exceptional curve. The function y_0^2 is the only one generator not vanishing there. As w by construction does not vanish there either, a_2 must be a unit. Again all other cases are treated in a similar way.

After redefining some functions (if necessary), we may suppose that $a_1 = a_2 = 1$. We now define rational functions $\alpha = \alpha_a$ in each case. We moreover redefine some generators of the local ring in such a way, that the 2×2 minors of the matrix M_a vanish identically on $\mathcal{O}_X$. We treat some cases in more detail, leaving the remaining cases to the reader.

- A_k^q:

$$\alpha = \frac{y_{p+1}}{w},$$

$$y_i = y_p \alpha^{i-p}; i \leqslant p-1,$$

$$y_i = y_{p+1} \alpha^{i-p-1}; i \geqslant p+1.$$

- D_k^I: Choose a new w, such that $z^2 = yw$. By a coordinate change we can dispose of the terms zy^{q-1} and y^{k-1} in the expression for w. Now define $\alpha := w/z$.
- D_{2k}^{II}: We deduce that $x^{2k-2} y_0 / w^{k-1} y_{k-1} = v$ is a unit, because its divisor is empty. Let β be a unit with $\beta^{2k-1} = 1/v$. We know replace y_0 with βy_0, y_{k-1} with $\beta^2 y_{k-1}$ and w with $\beta^2 w$. With these new choices we define the rational function $\alpha = \alpha_a$ by $\alpha = x^2/w$. Finally define $y_i = \alpha^i y_0$ for $1 \leqslant i \leqslant k-1$. □

PROPOSITION (3.5). *Let R_a and R_b be two rational double point configurations, and α_a and α_b the corresponding rational functions as defined in the previous proof. Then there exist units $u, v \in \mathcal{O}_X$ such that*

$$\alpha_a - v\alpha_b = u.$$

Proof. The pole divisor of both α_a and α_b on the minimal resolution consist of the *same* branch P intersecting the central curve transversally. The image of P

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on X is *smooth*, as the generic hyperplane section vanishes with multiplicity one on the central curve, hence has vanishing order one on P. The image of P on X we also denote by P. Consider a function ϕ in $\mathcal{O}_X$ whose non-compact divisor on the minimal resolution is equal to $P+$ REST, where REST has no points in common with P, and is reduced. Using Artin's Theorem, such functions are easy to construct. Consider the functions $\phi, \alpha_a\phi, \alpha_b\phi$. Because the pole divisor of α_a is just P, and the rational function α_a has degree one (hence does not vanish) on the central curve, the vanishing order on P of the function $\alpha_a\phi$ is exactly the vanishing order of ϕ along the central curve. Moreover the function $\alpha_a\phi$ vanishes on REST. As the same statements hold for $\alpha_b\phi$, it follows that modulo ϕ one has an equality

$$\alpha_a\phi = v\alpha_b\phi \quad \text{for some unit } v \in \mathcal{O}_X.$$

We therefore have an equality

$$\alpha_a\phi - v\alpha_b\phi = u\phi$$

for some $u \in \mathcal{O}_X$. We divide by ϕ

$$\alpha_a - v\alpha_b = u.$$

Because the zero divisors of α_a and α_b are completely different, even if restricted to the central curve, if follows that u is a unit. □

Proof of the second statement of Theorem (2.3). We fix a rational double point configuration R_0. For every other rational double point configuration R_a we have, by Proposition (3.5) units u_a and v_a in $\mathcal{O}_X$ such that

$$\alpha_0 = u_a + v_a\alpha_a.$$

Therefore, the 2×2 minors of the matrix N of Definition (2.2) are identically zero as elements of $\mathcal{O}_X$.

By abuse of notation we consider the x, y_{ia} as variables, so are parameters for the embedding space of X. Take lifts u_a, v_a in $\mathbb{C}\{x, y_{ia}\}$ which are also units. Then the 2×2 minors of N are in the ideal defining our singularity X. We claim that they generate the ideal defining X. Suppose the contrary, i.e. there is a function f which vanishes identically on X but which is not in the ideal generated by the 2×2 minors of N. But in the previous section we saw that the space X' defined by the 2×2 minors of N define a rational surface singularity, in particular it is a normal surface singularity. But X is contained in the zero locus of f on X', which then is a (maybe non-reduced) curve singularity. But this is a contradiction, because we assumed that X is a rational *surface* singularity. □

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4. The T^1 and T^2 of a determinantal rational surface singularity

Let X be a determinantal rational surface singularity. In this section we give formulas for T^1_X and T^2_X. In obtaining the results of this section, experiments with the computer algebra system Singular [5] were helpful. Basic for us is the following result.

THEOREM (4.1) [2] (5.1.1). *Let X be a rational surface singularity of multiplicity m. Then the number of generators of T^2_X is $(m-1)(m-3)$.*

Behnke and Christophersen in their paper gave examples of rational surface singularities where the dimension of T^2 is exactly $(m-1)(m-3)$. Further investigations on the dimension of T^2 for rational surface singularities were carried out in [6]. Although formulated differently in loc.cit., their result can be stated as

THEOREM (4.2) [6] (3.16 B) and (1.10). *Let X be a rational surface singularity with reduced fundamental cycle, of multiplicity $m \geqslant 3$. Let $\hat{X}$ be obtained from X by blowing-up the singular point. Then*

$$\dim(T^2_X) = (m-1)(m-3) + \sum_{p\in\hat{X}} \dim(T^2_{\hat{X},p}).$$

The usefulness of this Theorem lies in the fact that the right-hand side can be computed by a inductive procedure. Indeed, one has the following result of Tjurina.

THEOREM (4.3) [12]. *Let $\hat{X} \to X$ be the blow-up of X at the singular point of a rational surface singularity. Let X' be the space obtained from the minimal resolution of X by contracting all exceptional curves which intersects the fundamental cycle trivially. Then X' is isomorphic to $\hat{X}$.*

For a general rational surface singularity, the inequality

$$\dim(T^2_X) \geqslant (m-1)(m-3) + \sum_{p\in\hat{X}} \dim(T^2_{\hat{X},p})$$

has been proved recently by Christophersen and Gustavsen [3]. One cannot expect equality in general however, a counterexample is given in [2].

In order to investigate T^2 for rational determinantal singularities we recall the following result of Behnke and Christophersen.

PROPOSITION (4.4) [2] (2.1.1). *Let $f_1, \ldots, f_n, g_1, \ldots, g_n$ be elements of the maximal ideal of $\mathbb{C}\{x_1, \ldots, x_e\}$. Let X be a* Cohen–Macaulay *singularity defined by the* 2×2-*minors of*

$$\begin{pmatrix} f_1 & f_2 & \cdots & f_n \\ g_1 & g_2 & \cdots & g_n \end{pmatrix}.$$

Then the $\mathcal{O}_X$-module T^2_X is annihilated by the ideal $(f_1, \ldots, f_n, g_1, \ldots, g_n)$.

Before applying this proposition, we do one small coordinate change in the equations for rational determinantal singularities; In case we have an A^q_k singularity for which $p = 0$ (i.e. an A^q_{rq} singularity), we do the coordinate change

$$y_0 \mapsto y_0 - x^r.$$

Apart from this coordinate change, we assume that rational determinantal singularities are given by the equations of Section 2. We immediately deduce from these equations of determinantal rational surface singularities and the proposition of Behnke and Christophersen the following

PROPOSITION/DEFINITION (4.5). *Let X be a rational determinantal surface singularity, given by the equations described above. Then the module T^2_X is annihilated by all y_{ia}. Moreover T^2_X is annihilated by x^ϕ, where $\phi = \phi(X)$ is given by the minimum over ϕ_a for all rational double point configurations R_a. These ϕ_a are given by the following*

$$A^0_1 \quad \phi_a = 1,$$

$$A^q_{rq} \quad \phi_a = r + 1,$$

$$A^q_k \quad \phi_a = r, \qquad k = qr - p, \quad 0 < p \leqslant q - 1.$$

For all other rational double point configurations, one has $\phi_a = 2$.

PROPOSITION (4.6). *Let X be a rational determinantal surface singularity of multiplicity m. Then there exists a one parameter deformation $X_T \to T$ of X with on the* general *fiber $\phi = \phi(X)$ rational surface singularities of multiplicity m. This deformation occurs on the Artin component. By openness of versality, one might even assume that these singularities are all cones over rational normal curves of degree m.*

Proof. Look at the equations of a determinantal rational surface singularity. (Note the coordinate change in case A^q_{rq} we did above). We are going to perturb the matrix which give the equations for the determinantal rational surface singularities. Then deform the singularity by taking the 2×2 minors of the perturbed matrix. This deformation occurs on the Artin component, by a result of Wahl [14], (3.2). In the sub-matrix belonging to a rational double point configuration R_a, there is a term x^{ϕ_a} occuring (with coefficient 1). We are going to perturb the matrix by just perturbing these terms. Fix pairwise different numbers $c_1, \ldots c_\phi$, which are all different from 0. Then perturb the term x^{ϕ_a} by $(x - tc_1) \cdots (x - tc_\phi)(x^{\phi_a - \phi})$. For $t \neq 0$ we are getting singularities at $y_{ia} = 0$ and $x = c_i t$, for $i = 1, \ldots, \phi$. A tedious check shows that at these points the singularity has multiplicity m.

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THEOREM (4.7). *Let X be a determinantal rational surface singularity of multiplicity $m \geqslant 3$. Then*

$$\dim(T^2_X) = (m-1)(m-3) + \sum_{p \in \hat{X}} \dim(T^2_{\hat{X},p}).$$

Proof. We first claim that

$$(m-1)(m-3)\phi = (m-1)(m-3) + \sum_{p \in \hat{X}} \dim(T^2_{\hat{X},p}).$$

This is just an investigation of the blow-up of a rational determinantal surface singularity, using the result of Tjurina. From Wahl's result on the structure of the resolution of a rational determinantal surface singularities, and Tjurina's result on the blow-up of we deduce that we have the following two possibilities.

(1) The fundamental cycle Z intersects the central curve strictly negative, i.e. we have an A_1^0 singularity. Then $\phi = 1$, and on the first blow up $\hat{X}$ we just have rational double points. So $\sum_{p \in \hat{X}} \dim(T^2_{\hat{X},p}) = 0$, which proves the theorem in this case.

(2) Z intersects the central curve trivially. Then on $\hat{X}$ we have, apart from rational double points, just one rational determinantal surface singularity, say X'. We claim that $\phi(X') = \phi(X) - 1$. This just a case by case check, using Tjurina's description and the computation sequence for the fundamental cycle Z. For instance suppose that one has a E_6, E_7, D_k^I or D_k^{II} configuration for X, then X' has an A_1^0 singularity, as the fundamental cycle for X' now will intersect the central curve negatively. So we just have to investigate the A_k^q case, which is easy, either using the resolution and Tjurina's result, or using the equations and the definition of ϕ immediately.

This proves the claim. As remarked before, the inequality $\geqslant$ in the statement of the Theorem is a general result by Christophersen and Gustavsen. But in our case it can also be deduced quite elementary: It is well-known that the dimension of T^2_X for a rational surface singularity of multiplicity m ($m \geqslant 3$) is at least $(m-1)(m-3)$. Use the above deformation into ϕ rational surface singularities of multiplicity m (the multiplicity of X) and semi-continuity of the dimension of T^2 to get the inequality $\geqslant$. For the other inequality we use again that the number of generators of T^2_X is $(m-1)(m-3)$. Furthermore we know that T^2_X is annihilated by the functions y_{ia} and x^ϕ, see (4.5). So we deduce that dim $(T^2_X) \leqslant (m-1)(m-3)\phi$.$\square$

As a corollary of the result of T^2_X, and the existence of the special one parameter deformation, one also gets a result on the dimension of T^1_X, and on the surjectivity of the obstruction map.

COROLLARY (4.8). *Let X be a determinantal rational surface singularity, of multiplicity m and let $\phi = \phi(X)$. Let $(\tilde{X}, E) \to (X, x)$ be the minimal resolution. Let $\Theta_{\tilde{X}}$ be the tangent sheaf of $\tilde{X}$. Then*

$$\dim(T^1_X) = (m-3)\phi + \dim(H^1(\Theta_{\tilde{X}})).$$

Proof. We look at the one parameter deformation X_T of X which has ϕ cones over the rational normal curve of degree m on the general fiber. Look at the associated long exact sequence of cotangent modules

$$\cdots \to T^1_{X_T/T} \xrightarrow{\cdot t} T^1_{X_T/T} \xrightarrow{\alpha} T^1_X \to T^2_{X_T/T}$$
$$\xrightarrow{\cdot t} T^2_{X_T/T} \xrightarrow{\beta} T^2_X \ldots$$

The dimension of T^2 for a cone over the rational normal curve of degree m is $(m-1)(m-3)$, so the $\mathbb{C}\{t\}$-module $T^2_{X_T/T}$ has rank at least $\phi(m-1)(m-3)$. Hence the image of β has at least dimension $\phi(m-1)(m-3)$, which we just proved to be the dimension of T^2_X. Therefore β is *surjective*, and it follows that there are no other singularities on a general fiber, apart maybe from rational double and triple points. As a finitely generated $\mathbb{C}\{t\}$-module, the rank of $T^2_{X_T/T}$ is dim(coker $(\cdot t))-$ dim(ker$(\cdot t)$). Therefore multiplication by t is injective on $T^2_{X_T/T}$. From the exact sequence it follows that α is *surjective* too. The proof now literally goes as in [6], proof of (3.16A), which we repeat here. One knows that $\dim(H^1(\Theta_{\tilde{X}}))$ is the dimension of the Artin component, which is well-known to be smooth. We denote by cod (X) the codimension of the Artin component in T^1_X. The statement of the Theorem simply is

$$\mathrm{cod}(X) = (m-3)\phi.$$

By Greuel and Looijenga [4] the dimension of the image of α (so in our case $\dim(T^1_X)$) is the dimension of the Zariski-tangent space at a general point of $j(T)$, where $j(T) \to$ the base space of a semi-universal deformation of X, is a map, inducing by base change the given one parameter deformation $X_T \to T$. Now $j(T)$ lies on the Artin component, which is smooth. Openness of versality gives that the codimension of the Artin component is additive. The codimension of the Artin component of the cone of the rational normal curve of degree m is $m-3$ [9], and as one has ϕ of those on the general fiber, the result follows. □

COROLLARY (4.9). *The 'obstruction map' for a determinantal rational surface singularity is surjective, i.e. the minimal number of equations to describe the base space of a semi-universal deformation of a determinantal rational surface singularity X is the dimension of T^2_X.*

Proof. Just repeat the argument of [6] (4.2).

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