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ON SOME TWO-DIMENSIONAL CRACKS IN LINEAR ISOTROPIC ELASTICITY

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1. Introduction

In this paper the method of conformal transformation is used to consider a class of crack problem in linear isotropic elasticity. The conformal transformation used is the Joukowski transformation so that in general the crack profile is an aerofoil. This is a useful profile to consider since as well as providing information about the general aerofoil crack it includes as special cases the circular arc crack, the straight line crack and, as will be shown, can also be used to obtain similar results to those obtained by Bowie (1956) for the crack originating at the boundary of a circular hole. Thus by varying the parameters in the one elastic solution it is possible to obtain the elastic solution for a number of crack profiles. An examination of the stresses at the crack tip is made since this gives useful information about the direction of crack extension and also gives information necessary for the application of the fracture criterion which is derived in section 6. Some particular crack profiles are considered in detail in section 7.

2. The elastic problem

Consider the region, in the complex σ-plane, $\sigma = u + iv$, outside of the circle $|\sigma| = 1$. This region may be mapped, by the conformal transformation

$$z(\sigma) = x(\sigma) + iy(\sigma) = A\sigma + B + \frac{C^2}{A\sigma + B} \tag{1}$$

onto the exterior of a hole with boundary Σ in the z-plane. In (1) A, B and C denote complex constants. We wish to determine the stress distribution, in plane strain, due to hydrostatic pressure p inside the hole whose boundary in the z-plane is Σ. The method is the standard one of conformal transformation (see for example Green and Zerna (1954)).

Set $\sigma = e$, $\zeta = \xi + i\eta$ and regard ξ, η as orthogonal curvilinear coordinates in the z-plane. Then it is required to find functions $f(\sigma)$, $g(\sigma)$, regular in $|\sigma| > 1$ and $O(\sigma^{-1})$ as $|\sigma| \to \infty$, such that

$$\sigma_{\xi\xi}(0,\eta) + i\sigma_{\xi\eta}(0,\eta) = -p \tag{2}$$

where the components of stress $\sigma_{\xi\xi}$, $\sigma_{\eta\eta}$ and $\sigma_{\xi\eta}$ are given by

$$\sigma_{\xi\xi} + \sigma_{\eta\eta} = 4\left[\frac{f'(\sigma)}{z'(\sigma)} + \frac{\bar{f}'(\bar{\sigma})}{\bar{z}'(\bar{\sigma})}\right] \tag{3}$$

$$\sigma_{\xi\xi} - \sigma_{\eta\eta} + 2i\sigma_{\xi\eta} = -4\frac{\bar{\sigma}}{\sigma}\frac{d\sigma}{dz}\frac{d}{d\bar{\sigma}}\left[z\frac{\bar{f}'(\bar{\sigma})}{\bar{z}'(\bar{\sigma})} + \frac{\bar{g}'(\bar{\sigma})}{\bar{z}'(\bar{\sigma})}\right]. \tag{4}$$

The bar denotes the complex conjugate and the prime differentiation with respect to the argument of the function in question. The set of elastic equations is completed by the equation for the displacement components u_ξ, u_η which is

$$\mu(u_\xi + iu_\eta) = \left[\frac{\bar{\sigma}\bar{z}'(\bar{\sigma})}{\sigma z'(\sigma)}\right]^{\frac{1}{2}}\left[\kappa f(\sigma) - z\frac{\bar{f}'(\bar{\sigma})}{\bar{z}'(\bar{\sigma})} - \frac{\bar{g}'(\bar{\sigma})}{\bar{z}'(\bar{\sigma})}\right] \tag{5}$$

and the equation for traction on the surface

$$\tfrac{1}{2}(\sigma_{\xi\xi} + i\sigma_{\xi\eta}) = \frac{f'(\sigma)}{z'(\sigma)} + \frac{\bar{f}'(\bar{\sigma})}{\bar{z}'(\bar{\sigma})} - \frac{\bar{\sigma}}{\sigma z'(\sigma)}\left[z(\sigma)\frac{d}{d\bar{\sigma}}\frac{\bar{f}'(\bar{\sigma})}{\bar{z}'(\bar{\sigma})} + \frac{d}{d\bar{\sigma}}\frac{\bar{g}'(\bar{\sigma})}{\bar{z}'(\bar{\sigma})}\right], \tag{6}$$

where μ is the shear modulus and $\kappa = 3 - 4\nu$ where ν is Poisson's ratio. In particular, on the boundary $\xi = 0$, where $\sigma\bar{\sigma} = 1$, the condition (2) gives

$$-p/2 = \frac{f'(\sigma)}{z'(\sigma)} + \frac{\bar{f}'(\bar{\sigma})}{\bar{z}'(\bar{\sigma})} - \frac{\bar{\sigma}}{\sigma z'(\sigma)}\left[z(\sigma)\frac{d}{d\bar{\sigma}}\frac{\bar{f}'(\bar{\sigma})}{\bar{z}'(\bar{\sigma})} + \frac{d}{d\bar{\sigma}}\frac{\bar{g}'(\bar{\sigma})}{\bar{z}'(\bar{\sigma})}\right]. \tag{7}$$

Now from (1)

$$z'(\sigma) = A - \frac{C^2 A}{(A\sigma + B)^2}, \tag{8}$$

which has zeros at $A\sigma + B = \pm C$ and $\bar{z}'(\bar{\sigma})$ has zeros at $\bar{A}\bar{\sigma} + \bar{B} = \pm\bar{C}$. Now we require that

$$\left|\frac{-B \pm C}{A}\right| < 1, \tag{9}$$

so that the zeros of $z'(\sigma)$ and $z'(\bar{\sigma})$ all lie within the unit circle $|\sigma| = 1$. Also $z(\sigma)$, $z'(\sigma)$ have poles at $\sigma = -B/A$. By imposing the condition that

$$|-B/A| < 1. \tag{10}$$

we ensure that the poles have modulus less than unity. Equation (7) may be solved for $f(\sigma)$, $g(\sigma)$ by using Cauchy's theorem in the following way.

Choose a complex number s, $|s| > 1$, multiply (7) by $z'(\sigma)(\sigma - s)^{-1}/2\pi i$ and integrate with respect to σ round a contour Γ consisting of the unit circle $|\sigma| = 1$ traversed in the clockwise direction. As σ describes this contour $\bar{\sigma} = 1/\sigma$ describes a contour $\bar{\Gamma}$ consisting of the circle traversed in the anti-clockwise direction. Then

$$(11)\qquad \begin{aligned}(-p/2)\frac{1}{2\pi i}\int_\Gamma \frac{z'(\sigma)d\sigma}{\sigma - s} &= \frac{1}{2\pi i}\int_\Gamma \frac{f'(\sigma)\,d\sigma}{\sigma - s}\\ &+ \frac{1}{2\pi i}\int_\Gamma\left[\frac{\bar{f}'(\bar{\sigma})z'(\sigma)}{\bar{z}'(\bar{\sigma})} - \frac{\bar{\sigma}}{\sigma}(z(\sigma)\bar{F}'(\bar{\sigma}) + \bar{G}'(\bar{\sigma}))\right]\frac{d\sigma}{\sigma - s},\end{aligned}$$

where

$$(12)\qquad \bar{\sigma} = 1/\sigma, \bar{F}'(\bar{\sigma}) = \frac{d}{d\bar{\sigma}}\left[\frac{\bar{f}'(\bar{\sigma})}{\bar{z}'(\bar{\sigma})}\right], \; \bar{G}'(\bar{\sigma}) = \frac{d}{d\bar{\sigma}}\left[\frac{\bar{g}'(\bar{\sigma})}{\bar{z}'(\bar{\sigma})}\right].$$

Now let Δ be the circle $|\sigma| = R > |s|$ in the σ-plane, described clockwise. Then by Cauchy's theorem for the annulus $1 < |\sigma| < R$ in which $z'(\sigma)$ is regular

$$\frac{1}{2\pi i}\int_{\Gamma+\Delta}\frac{z'(\sigma)\,d\sigma}{\sigma - s} = z'(s).$$

Also as $R \to \infty$

$$\frac{1}{2\pi i}\int_\Delta \frac{z'(\sigma)\,d\sigma}{\sigma - s} \to A.$$

Hence

$$(13)\qquad (-p/2)\frac{1}{2\pi i}\int_\Gamma \frac{z'(\sigma)\,d\sigma}{\sigma - s} = -\tfrac{1}{2}p[z'(s) - A].$$

Apply a similar method to the first integral on the right of (11), remembering

$$\int_\Delta f'(\sigma)(\sigma - s)^{-1}\,d\sigma \to 0,$$

as $R \to \infty$ from the condition on f as $\sigma \to \infty$. Then

$$(14)\qquad \frac{1}{2\pi i}\int_\Gamma \frac{f'(\sigma)\,d\sigma}{\sigma - s} = f'(s).$$

Call the third term of (11) I, and set $\sigma = 1/\bar{\sigma}$. Then

$$I = \frac{1}{2\pi i}\int_{\bar{\Gamma}}\left[\frac{-\bar{f}'(\bar{\sigma})z'(1/\bar{\sigma})}{\bar{\sigma}^2\bar{z}'(\bar{\sigma})} + z(1/\bar{\sigma})\;\bar{F}'(\bar{\sigma}) + \bar{G}'(\bar{\sigma})\right]\frac{\bar{\sigma}\,d\bar{\sigma}}{1 - s\bar{\sigma}}.$$

Integrate the term containing $\bar{F}'(\bar{\sigma})$ by parts. Then since all the functions are single valued, and $\bar{F}(\bar{\sigma}) = \bar{f}'(\bar{\sigma})/\bar{z}'(\bar{\sigma})$

$$I = \frac{1}{2\pi i}\int_{\bar{\Gamma}}\left[\frac{-z(1/\bar{\sigma})\bar{f}'(\bar{\sigma})}{(1-s\bar{\sigma})^2\,\bar{z}'(\bar{\sigma})} + \frac{\bar{\sigma}\,\bar{G}'(\bar{\sigma})}{1-s\bar{\sigma}}\right]d\bar{\sigma}.$$

Again apply Cauchy's theorem to the annulus $1 < |\bar{\sigma}| < R$ and let $R \to \infty$. The only function in the bracket which is not regular in the annulus is $\bar{z}(1/\bar{\sigma})$, which has a simple pole at $\bar{\sigma} = -A/B$ with residue $-AC^2/B^2$ so

(15) $$I = -\left[\frac{AC^2\bar{f}'(-A/B)}{(B+As)^2\bar{A}[1-\bar{C}^2B^2(A\bar{A}-BP)^{-2}]}\right].$$

Use of (11), (13) (14) and (15) with some rearrangement gives

(16) $$f'(s) = \tfrac{1}{2}p\,\frac{AC^2}{(As+B)^2} + \frac{AC^2\bar{f}'(-A/B)}{(As+B)^2K},$$

where

(17) $$K = \bar{A}[1-\bar{C}^2B^2/H],\ H = (A\bar{A}-B\bar{B})^2$$

From (16)

(18) $$f'(-\bar{A}/\bar{B}) = A\bar{B}^2C^2H^{-1}[\tfrac{1}{2}p + K^{-1}\bar{f}'(-A/B)].$$

Hence

(19) $$\bar{f}'(-A/B) = \bar{A}B^2\bar{C}^2H^{-1}[\tfrac{1}{2}p + \bar{K}^{-1}f'(-\bar{A}/\bar{B})].$$

Substituting for $f(-\bar{A}/\bar{B})$ from (18) in (19) we obtain

(20) $$\bar{f}'(-A/B) = \frac{\tfrac{1}{2}p\,\bar{A}B^2\bar{C}^2H^{-1}\,[1 + A\bar{B}^2C^2H^{-1}\bar{K}^{-1}]}{[1 - A\bar{A}B^2\bar{B}^2C^2\bar{C}^2H^{-2}K^{-1}\bar{K}^{-1}]}$$

By substituting (20) into (16) we obtain the final expression for $f'(s)$.

A similar application of Cauchy's theorem yields an expression for $\bar{g}'(\sigma)$ in terms of $f(\sigma)$, $z(\sigma)$ and their derivatives. The expression is

(21) $$\bar{G}(\bar{\sigma}) = -\tfrac{1}{2}p\,z(1/\bar{\sigma}) - f(1/\bar{\sigma}) - z(1/\bar{\sigma})\,\bar{F}(\bar{\sigma}).$$

This completes the elastic solution.

3. The condition for a crack tip

The analysis of the previous section may be used to determine the stress distribution in an infinite medium containing a hole with boundary Σ. The exact shape of the hole is determined from (1) once the values of A, B and C are known. For example, if we put

$$A = (a+b)/2,\ B = 0,\ C^2 = (a^2-b^2)/4$$

where a and b are real then the boundary Σ is the ellipse $x^2/a^2 + y^2/b^2 = 1$. However the profiles of greatest interest are those which include a crack tip. For

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such profiles

(22) $$|A| = |B + C| \text{ or } |A| = |B - C|,$$

the crack having two tips if $|A| = |B - C|$ and $|A| = |B + C|$. If either (or both) of the equations (22) are satisfied then it is apparent that condition (9) is not satisfied and the conditions under which the solution of the previous section was obtained are not strictly satisfied. This difficulty is best resolved by considering that any crack profile is obtained by shrinking suitable profiles given by (1) and for which (9) and (10) are satisfied onto the crack. This procedure is analogous to the way in which the elliptical hole was allowed to degenerate into the straight line crack to obtain the first solution to that particular problem (Griffith (1921)).

4. Stress at the crack tip

The stress at the crack tip is of considerable importance in fracture mechanics and in this section we show we may obtain a complete picture of the stress round the crack tip by using the results of section 2. From (3)

$$\sigma_{xx} + \sigma_{yy} = \sigma_{\xi\xi} + \sigma_{\eta\eta} = 8\mathscr{R}\{f'(\sigma)/z'(\sigma)\}$$

where $\mathscr{R}$ denotes the real part of a complex number. Using (16) we obtain

(23) $$\sigma_{xx} + \sigma_{yy} = 8\mathscr{R}\left\{\frac{C^2[\frac{1}{2}p + \bar{f}'(-A/B)/K]}{[(A\sigma + B)^2 - C^2]}\right\}.$$

Also from (1)

(24) $$\sigma = \frac{-(2B - z) + (z^2 - 4C^2)^{\frac{1}{2}}}{2A}$$

so that substituting in (23)

(25) $$\sigma_{xx} + \sigma_{yy} = 8\mathscr{R}\left\{\frac{2C^2[\frac{1}{2}p + \bar{f}'(-A/B)/K}{(z^2 - 4C^2)^{\frac{1}{2}}\,[z + (z^2 - 4C^2)^{\frac{1}{2}}]}\right\}$$

Suppose there is a crack tip at $z = 2C$ (so $|A| = |B - C|$) then letting $z = 2C + re^{i\theta}$ we obtain

(26) $$\sigma_{rr} + \sigma_{\theta\theta} = 2\mathscr{R}\left\{\frac{C^{\frac{1}{2}}[p + 2\,\bar{f}'(-A/B)/K]}{r^{\frac{1}{2}}\,e^{i\theta/2}}\right\}, \text{ as } r \to 0.$$

In plane extension, the stress distribution in the vicinity of the crack tip has been shown by Irwin (1958) to always take the form

(27) $$\sigma_{rr} = \sigma_{\beta\beta} = k_1\,(2/r)^{\frac{1}{2}}\cos(\tfrac{1}{2}\beta) - k_2\,(2/r)^{\frac{1}{2}}\sin(\tfrac{1}{2}\beta), \quad \text{as } r \to 0$$

(28) $$\sigma_{\beta\beta} = (2r)^{-\frac{1}{2}}\cos(\tfrac{1}{2}\beta)\;[k_1\cos^2(\tfrac{1}{2}\beta) - \tfrac{3}{2}k_2\sin\beta], \quad \text{as } r \to 0,$$

(29) $$\sigma_{r\beta} = (8r)^{-\frac{1}{2}}\cos(\tfrac{1}{2}\beta)\;[k_1\sin\beta + k_2(3\cos\beta - 1)], \quad \text{as } r \to 0,$$

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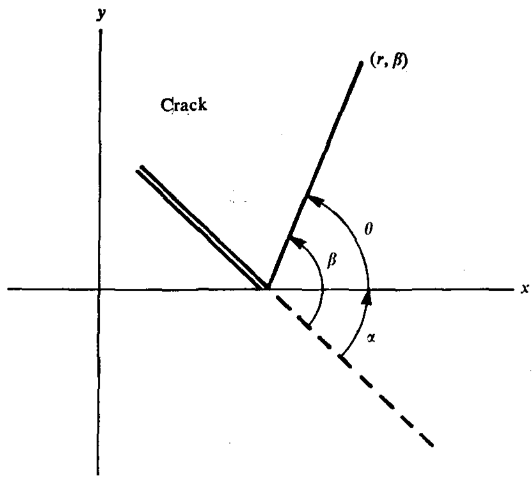

Figure 1

where k_1 and k_2 are stress intensity factors and (r, β) are polar coordinates measured from a crack tip and from the line of extension of the crack respectively (see Figure 1). Now from (26) it follows that

$$\sigma_{rr} + \sigma_{\beta\beta} = \sigma_{rr} + \sigma_{\theta\theta} = 2\mathscr{R}\{Mr^{-\frac{1}{2}}e^{-i\theta/2}\} \tag{30}$$
$$= 2\mathscr{R}\{Mr^{-\frac{1}{2}}e^{i(\alpha-\beta)/2}\} \text{ as } r \to 0,$$

where

$$M = C^{\frac{1}{2}}[p + 2\bar{f}'(-A/B)/K] \tag{31}$$
$$= m_1 + im_2, \; m_1, \; m_2 \text{ real.}$$

Hence

$$\sigma_{rr} + \sigma_{\beta\beta} = 2^{\frac{1}{2}}[m_1 \cos(\tfrac{1}{2}\alpha) - m_2 \sin(\tfrac{1}{2}\alpha)]\,(2/r)^{\frac{1}{2}} \cos(\tfrac{1}{2}\beta) \tag{32}$$
$$+ \; 2^{\frac{1}{2}}[m_2 \cos(\tfrac{1}{2}\alpha) + m_1 \sin(\tfrac{1}{2}\alpha)]\,(2/r)^{\frac{1}{2}} \sin(\tfrac{1}{2}\beta) \text{ as } r \to 0,$$

so that, by comparison with (27), the stress intensity factors are

$$\begin{cases} k_1 = 2^{\frac{1}{2}}[m_1 \cos(\tfrac{1}{2}\alpha) - m_2 \sin(\tfrac{1}{2}\alpha)], \\ k_2 = -2^{\frac{1}{2}}\,[m_2 \cos(\tfrac{1}{2}\alpha) + m_1 \sin(\tfrac{1}{2}\alpha)]. \end{cases} \tag{33}$$

Once the transformation (1) is known m_1 and m_2 may be determined through (31), (21), (19) and (17). In order to find the angle α we note that the crack profile is given by the equation

$$z = Ae^{i\theta} + B + \frac{C^2}{Ae^{i\theta} + B}.$$

Elementary differentiation yields

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$$\frac{dy}{dx} = \frac{\mathscr{I}[iAe^{i\theta} - iC^2Ae^{i\theta}(Ae^{i\theta}+B)^{-2}]}{\mathscr{R}[iAe^{i\theta} - iC^2Ae^{i\theta}(Ae^{i\theta}+B)^{-2}]}.$$

At the crack tip $e^{i\theta} = (-B + C/A)$ so an application of l'Hôpital's rule gives

$$\frac{dy}{dx} = \frac{\mathscr{I}[(B-C)/C^2]^2}{\mathscr{R}[(B-C)/C^2]^2}. \tag{34}$$

Equation (34) may be used to determine α, since at the crack tip

$$\frac{dy}{dx} = \tan(\pi - \alpha). \tag{35}$$

By using (35) and the formula for M the stress intensity factors may be determined through (33) once the transformation (1) and the load p are known. The complete stress pattern at the crack tip is then given through equations (27), (28) and (29).

5. The direction of crack extension

The commonly recognized hypotheses for the extension of cracks in brittle materials under slowly applied plane loads are (Sih and Erdogan (1963))

a) The crack extension starts at the tip in the radial direction

b) The crack extension starts in the plane perpendicular to the direction of greatest tension.

These hypotheses imply that the crack will start to grow from the crack tip in the direction along which the tangential stress $\sigma_{\beta\beta}$ is maximum and the shear stress $\sigma_{r\beta}$ is zero. Now from (28)

$$\begin{aligned}\frac{\partial\sigma_{\beta\beta}}{\partial\beta} = (2r)^{\frac{1}{2}}\{&-\tfrac{1}{2}\sin(\tfrac{1}{2}\beta)\,[k_1\cos^2(\tfrac{1}{2}\beta) - \tfrac{3}{2}k_2\sin\beta] \\ &+ \cos(\tfrac{1}{2}\beta)\,[-k_1\cos(\tfrac{1}{2}\beta)\sin(\tfrac{1}{2}\beta) - \tfrac{3}{2}k_2\cos\beta]\} \text{ as } r\to 0,\end{aligned} \tag{36}$$

and, in particular, when $\beta = 0$

$$\frac{\partial\sigma_{\beta\beta}}{\partial\beta} = (2r)^{-\frac{1}{2}}\{-3k_2/2\},\ \sigma_{r\beta} = (8r)^{-\frac{1}{2}}k_2, \tag{37}$$

which shows that the crack will only extend along its tangent at the tip if $\underline{k}_2 = 0$. Now $\underline{k}_1$ and $\underline{k}_2$ are the symmetric and skew-symmetric stress intensity factors respectively so it is apparent that the crack will only extend along its tangent if, at the crack tip, there is a stress distribution which is symmetrical about the line $\beta = 0$.

6. A fracture criterion

A possible criterion for the extension of the cracks discussed in this paper is derived in this section. The criterion is similar to one used by Sih and Erdogan

(1963) for predicting the critical stress for the extension of a straight crack in a plate under plane loading and transverse shear.

For a straight crack extending in its own direction (Figure 2) it is well-known (see Irwin (1958)) that the normal stress and displacement near the crack tip take the form

$$\sigma_{yy} = pN/r^{\frac{1}{2}}, \ \sigma_{xy} = 0, \tag{38}$$

$$u_y = \mp \frac{4(1-\nu^2)}{E} Npr^{\frac{1}{2}} \ \text{as} \ r \to 0, \tag{39}$$

where N is a constant, p is the (constant) applied normal traction over the faces of the crack and E is Young's modulus. Now consider an extension δl to the crack in its own direction (Figure 2). The normal displacement on the new crack and

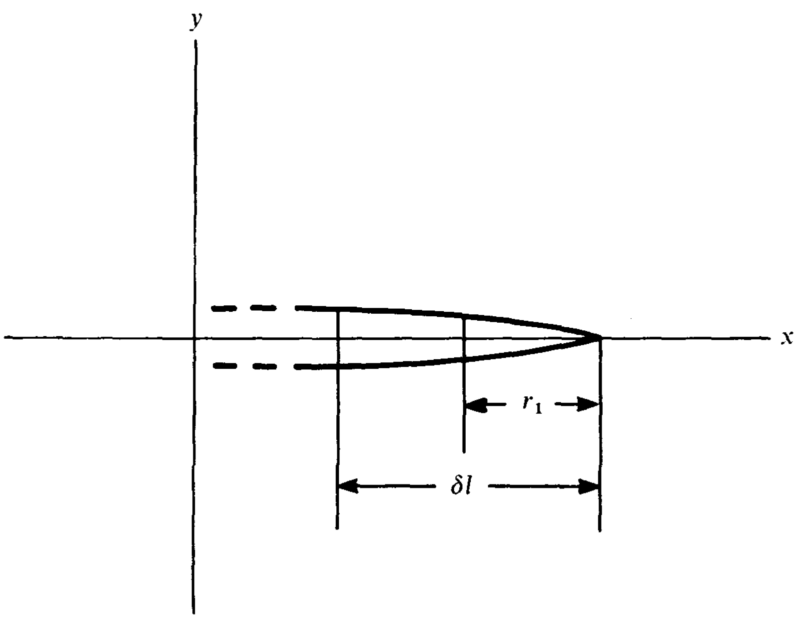

Figure 2

distribution of normal stress at these points prior to the formation of the new crack are, according to (38) and (39) given to within small quantities by

$$u_y = 4(1-\nu^2)\, E^{-1} Npr_1^{\frac{1}{2}}, \ \sigma_{yy} = pN(\delta l - r_1)^{-\frac{1}{2}}. \tag{40}$$

Hence the energy released in the formation of the new crack surface is given by

$$\delta W = \frac{4(1-\nu^2)\, N^2 p^2}{E} \int_0^{\delta l} \frac{r_1}{(\delta l - r_1)^{\frac{1}{2}}} dr_1$$

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$$= 2(1 - \nu^2)\, E^{-1} N^2 p^2 \pi \delta l. \tag{41}$$

The crack extension also has a surface energy $2T\delta l$ where T is the surface energy of the material per unit area of surface. Hence Griffith's criterion for extension of the straight crack is that p exceeds a critical pressure p_c given by

$$\frac{(1-\nu^2)\pi}{TE}\, p_c^2 = \frac{1}{N^2}. \tag{42}$$

According to our hypotheses for crack extension the crack will start to grow in a straight line in a direction along which the tangential stress $\sigma_{\theta\theta}$ is maximum and the shear stress $\sigma_{r\theta}$ is zero. Hence along the line of crack extension conditions (38) are satisfied with a suitable choice of N. Since the crack extension is straight it is reasonable to assume that displacement over the extension is of the same form as the displacement for the extension to a straight line crack. Hence the displacement is given by (39) and thus (42) may be regarded as a fracture criterion for the extension of the cracks discussed in this paper.

7. Some particular crack profiles

For a crack tip at $z = 2C$ we require that

$$|A| = |B - C| \tag{43}$$

and for a tip at $z = -2C$ we require

$$|A| = |B + C| \tag{44}$$

Now if we require the crack tip or tips to be on the real axis then the imaginary part of C will be zero and we write $C = c$ where c is real. Also since the circle $|\sigma| = 1$ in the σ-plane is unaffected by a rotation of the u, v axes it follows that we may choose $A = a$ (a real) without loss of generality. We also write $B = b + ib'$ where b and b' are real so that the transformation (1) may be written

$$z = a\sigma + b + ib' + \frac{c^2}{a\sigma + b + ib'} \tag{45}$$

Now from (34) the angle the tangent at the crack tip makes with the x-axis is given by

$$\frac{dy}{dx} = \tan(\pi - \alpha) = \frac{\mathscr{I}[b + ib' - c]^2}{\mathscr{R}[b + ib' - c]^2}$$

$$= \frac{2b'(b-c)}{(b-c)^2 - b'^2}. \tag{46}$$

If we wish to generate crack profiles which make a constant angle with the x-axis then we require that

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$$\frac{2b'(b-c)}{(b-c)^2 - b'^2} = K \text{ (constant)}. \tag{47}$$

Examples of two profiles given by (45) (with $\sigma = e^{i\theta}$) with a, b, b' and c satisfying (43) and (47) (with $K = -4/3$) are shown in Figure 3. For the profile marked

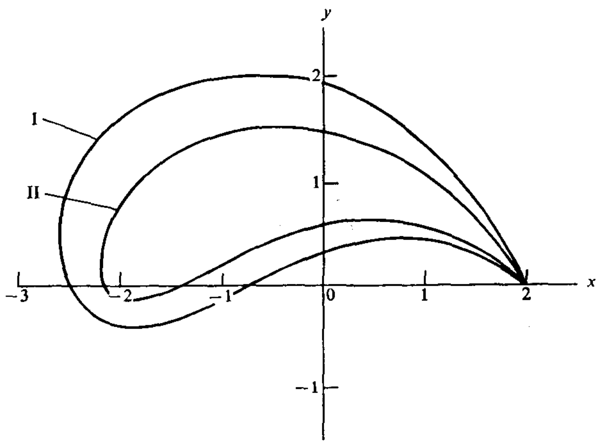

Figure 3

I $a = 0.83$, $b = 0.25$, $b' = 0.375$ and $c = 1$ while for the profile marked *II* $a = 1.397$, $b = -0.25$, $b' = 0.625$ and $c = 1$. When $b' = 0$ the transformation (45) carries the unit circle $|\sigma| = 1$ in the σ-plane into a symmetric aerofoil in the z-plane (Figure 4). When $b = 0$ the unit circle in the σ-plane transforms into an arc of a circle in the z-plane (Figure 6). Finally when $b = b' = 0$ the crack profile in the z-plane takes the form of a straight line crack lying along the x-axis between $z = -2c$ and $z = 2c$. We shall only consider in detail the symmetric aerofoil crack and the circular arc crack since this will be sufficient to illustrate the application of the analysis of the previous sections.

The symmetric aerofoil crack. In this case $b' = 0$ so that, from (34) and (35), the tangent to the crack at the crack tip lies along the x-axis and $\alpha = 0$. Also, from (31), after some algebra

$$M = m_1 + im_2 = pc^{\frac{1}{2}} \left[\frac{c^2 - 4bc + 3b^2}{c^2 - 4bc + 2b^2}\right], \tag{49}$$

so that from (33)

$$k_1 = \sqrt{2}\, pc^{\frac{1}{2}} \left[\frac{c^2 - 4bc + 3b^2}{c^2 - 4bc + 2b^2}\right], \; k_2 = 0. \tag{50}$$

It has been shown in section 5 that when $k_2 = 0$ the crack extends along its

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tangent at the crack tip and hence the symmetric aerofoil crack will extend along its tangent at the crack tip. When $b = 0$ so that the aerofoil degenerates into a straight line crack between $z = -2c$ and $z = 2c$ then $k_1 = \sqrt{2}\, pc^{\frac{1}{2}}$. We may

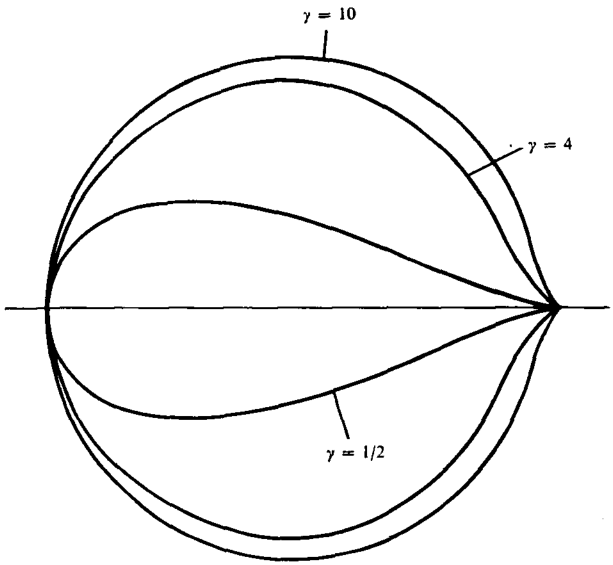

Figure 4

apply the fracture criterion (42) to the symmetric aerofoil crack. The appropriate value of N obtained by comparing (28) (with $\beta = 0$), (38) and (50) is

(51) $$N = c^{\frac{1}{2}} \left[\frac{c^2 - 4bc + 3b^2}{c^2 - 4bc + 2b^2}\right]$$

and hence the criterion is

(52) $$\frac{(1 - \nu^2)\pi\, p_c^2}{TE} = \frac{1}{c}\left[\frac{c^2 - 4bc + 2b^2}{c^2 - 4bc + 3b^2}\right]^2.$$

When $b = 0$ this reduces to the well-known Griffith critical stress for a straight line crack. If we put

(53) $$\begin{aligned} a &= (d/4)(1 + 2\gamma)(1 + \gamma)^{-1} \\ c &= (d/4)(1 + 2\gamma)(1 + \gamma)^{-2},\ b = -c\gamma, \end{aligned}$$

then by varying the parameter γ we generate a set of profiles of constant length d. Particular profiles for the cases $\gamma = \frac{1}{2}$, 4, 10 are shown in Figure 4. Using (53) the equation (52) may be written

$$\left[\frac{(1-\nu^2)\pi}{TE}\right]^{\frac{1}{2}} p_c = c^{-\frac{1}{2}}\left[\frac{1+4\gamma+2\gamma^2}{1+4\gamma+3\gamma^2}\right] \tag{54}$$

In Figure 5 values of the critical stress p_c for $d = 8$ are plotted against $1/\gamma$. It is apparent from the graph that for values of γ less than one the effect of the aerofoil shape is negligible and for all practical purposes the critical stress for a straight

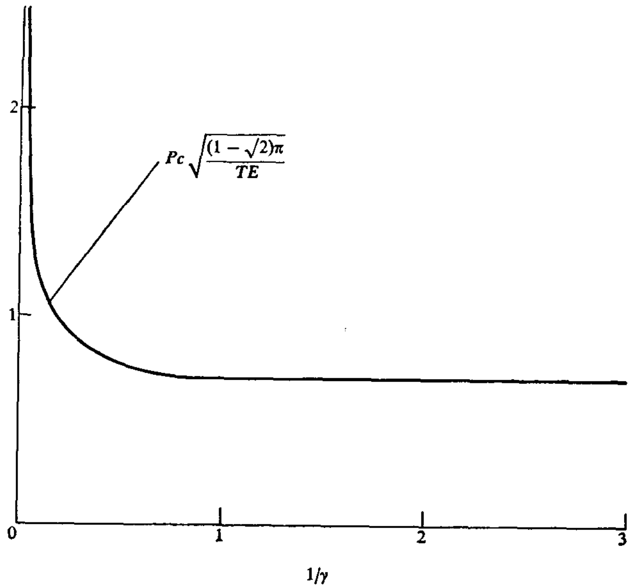

Figure 5

crack of length d could be used. However for the cases when γ is much greater than one the stress field is dominated by the local stress field of the circular hole which the aerofoil shape approximates for large γ and a large stress is required to initiate fracture. These results are similar to those obtained by Bowie (1956) for the crack originating at a circular hole. This similarity between the results of this paper and the results of Bowie is not altogether surprising since Bowie only uses an approximate mapping function which has the effect of rounding off the corners at the intersection between the crack and the hole.

The circular arc crack. In this case we put $b = 0$ and $a^2 = c^2 + b'^2$ so that the arc crack passes through the point $2ib'$, has tips at $z = \pm 2c$ and the radius of the circle of which the crack is an arc is a^2/b (Figure 6). The hoop stress $\sigma_{\beta\beta}$ and the shear stress $\sigma_{r\beta}$ have been determined numerically for $c = 1$, $b' = 0$ (the straight line crack), $b' = 0.6$ and $b' = 1$ and the results appears in graph form in Figure 7. The maximum values of $\sigma_{\beta\beta} r^{\frac{1}{2}}/p$ are (by comparison with (38)) the values of N occurring in equation (42) for the critical stress. In Table 1 the

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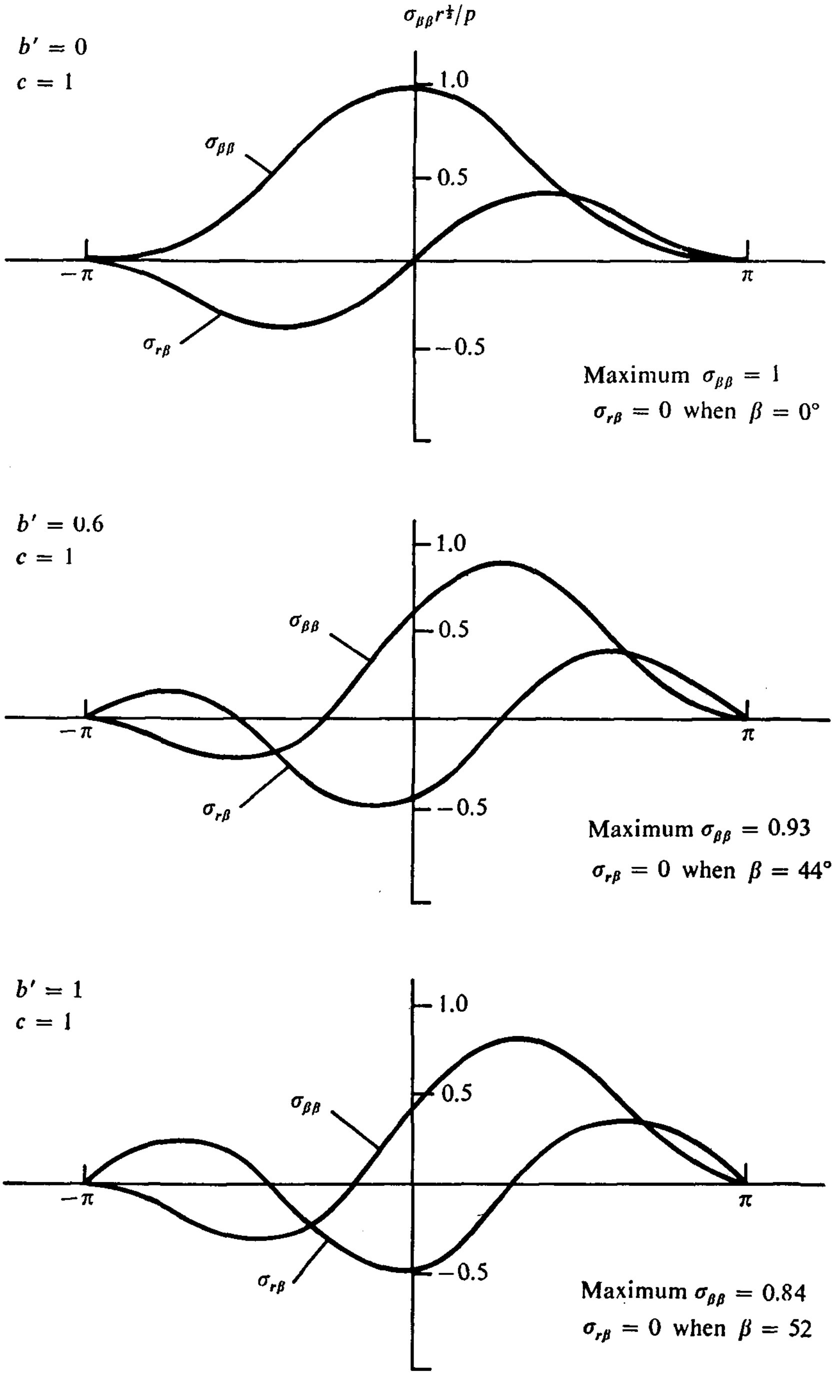

Figure 7

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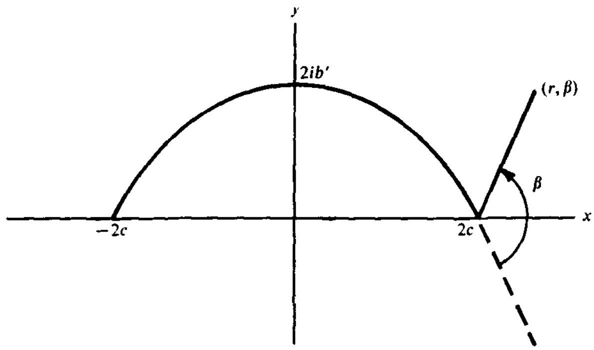

Figure 6

calculated values of $1/N^2$ are given for $c = 1$ and various values of b'. Once the constants v, T and E are known equation (42) can be used in conjunction with the values in Table 1 to give the critical stress p_c.

Table 1

b'	0	0.2	0.4	0.6	0.8	1.0
$1/N^2$	1.000	1.006	1.050	1.149	1.275	1.406

It is apparent that the arc crack is more stable than a straight crack of the same chord, in the sense that the arc crack requires a larger load to extend it.

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