BULLETIN 13 (2), 1970

ON AN INTERPOLATION THEOREM OF ZYGMUND AND KOIZUMI

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1. **Introduction.** Let $(X, \mathscr{M}, \mu)$ and $(Y, \mathscr{N}, \nu)$ be two σ-finite measure spaces. An operator T, defined by $h=Tf$, which maps functions on X into functions on Y is called quasilinear if $T(f+g)$ is uniquely defined whenever Tf and Tg are defined, and if

$$(1.1) \qquad T(f+g) \leq \mathscr{K}\,(|Tf|+|Tg|),$$

where $\mathscr{K} \geq 1$ is independent of f and g. If $\mathscr{K}=1$ the operator T is called sublinear.

We recall that a (complex valued) function f belongs to $L_{p,\mu}(X)$ $1 \leq p \leq \infty$, if f is μ-measurable and its norm

$$(1.2) \qquad \begin{aligned} \|f\|_{p,\mu} &= \left(\int_X |f|^p \, d\mu\right)^{1/p} \quad 1 \leq p < \infty \\ \|f\|_{\infty,\mu} &= \text{ess sup } |f| \end{aligned}$$

is finite. The operator T is said to be of strong type (r, s) $1 \leq r, s \leq \infty$ if there exists a constant A independent of f, such that

$$(1.3) \qquad \|Tf\|_{s,\nu} \leq A\,\|f\|_{r,\mu}.$$

Observe that if T is initially defined for simple functions only, and if $1 \leq r < \infty$ then there is a unique extension of T to all f in $L_{r,\mu}$, preserving (1.3).

Let f be defined on X and $E_f(t)=\{x\colon |f(x)|>t\}$, then $D_f^\mu = D_f$, the distribution unction of f, is defined by

$$D_f^\mu(t) = \mu(E_f(t)).$$

A quasilinear operator T which satisfies

$$D_{Tf}^\nu(t) \leq \left(\frac{A\,\|f\|_{r,\mu}}{t}\right)^s \quad 1 \leq r, s \leq \infty, s < \infty$$

is said to be of weak type (r, s).

The following theorem is an extension of a result of A. Zygmund [3, Ch. XII, Theorem 4.22] to the case of totally σ-finite measure spaces:

THEOREM 1 (S. Koizumi, [1], [2]). *Suppose $\mu(X)$ and $\nu(Y)$ are both infinite and T is a quasilinear operator of weak type (a, a) and (b, b), $1 \leq a < b < \infty$. Let ϕ be a*

Received by the editors December 30, 1969.

https://doi.org/10.4153/CMB-1970-045-3 Published online by Cambridge University Press

continuous increasing function defined on the nonnegative real line and vanishing at the origin. If

$$\phi(2u) = O(\phi(u)) \qquad u \to \infty, u \to 0 \tag{1.4}$$

$$\int_u^\infty \frac{\phi(t)}{t^{b+1}}\,dt = O\left(\frac{\phi(u)}{u^b}\right) \quad u \to \infty \tag{1.5}$$

$$\int_1^u \frac{\phi(t)}{t^{a+1}}\,dt = O\left(\frac{O(u)}{u^a}\right), \quad u \to \infty \tag{1.6}$$

$$\int_u^1 \frac{\phi(t)}{t^{b+1}}\,dt = O\left(\frac{\phi(u)}{u^b}\right), \quad u \to 0 \tag{1.7}$$

$$\int_0^u \frac{\phi(t)}{t^{a+1}}\,dt = O\left(\frac{\phi(u)}{u^a}\right), \quad u \to 0, \tag{1.8}$$

then $h=Tf$ is uniquely defined for all f for which $\phi(|f|)$ is μ integrable and

$$\int_Y \phi(|h|)\,d\nu \le A \int_X \phi(|f|)\,d\mu,$$

where A is independent of f.

Suppose $\phi(t)$ is "close" to t^a, $a \ge 1$. Then it may happen that (1.6) fails to hold. For example $\phi(t)=t^a \log t$, $a \ge 1$, does not satisfy (1.6). In [4, Theorem 2], A. Zygmund gave a modification of [3, Ch. XII, Theorem 4.22] for finite measure spaces which rectifies this deficiency. (See also [3, Ch. XII, Theorem 4.34] where the case $a=1$ is treated.)

In this note we extend the result of Zygmund [4, Theorem 2] to σ-finite measure spaces. The proof proceeds along the lines established by Zygmund and Koizumi.

The final section contains some applications.

Throughout, A denotes a constant independent of f not necessarily the same at each occurrence. As usual, R and R^+ denote the real, respectively, positive real line. Furthermore, we introduce the notation $\phi_s(i)=\phi(2^i)2^{-is}$, $s \ne 0$, and $\phi_0(i)=\phi(i)$, $i=0, \pm 1, \pm 2, \ldots$, similarly for ψ.

2. Interpolation theorem.

THEOREM 2. *Let $h=Tf$ be a quasilinear operator defined for all simple functions on $(X, \mathscr{M}, \mu)$ with values on $(Y, \mathscr{N}, \nu)$. Suppose T is of weak type (a, a) and (b, b), $1 \le a < b < \infty$, and ψ is a nonnegative, continuous, nondecreasing function on R^+ vanishing in a right-hand neighbourhood of zero. If ψ satisfies*

$$\psi(2u) = O(\psi(u)), \quad (u \to \infty) \tag{2.1}$$

and if

$$\phi(u) = u^a \int_0^u t^{-a-1}\psi(t)\,dt \tag{2.2}$$

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satisfies (1.5) *and* (1.7), *then*

$$\int_Y \psi(|h|)\,d\nu \le A\int_X \phi(|f|)\,d\mu. \tag{2.3}$$

In particular, T can be uniquely extended to the space of all $\phi(|f|)-\mu$ integrable functions preserving (2.3).

Proof. Observe that (2.1) implies $\phi(2u)=O(\phi(u))$, for by (2.2)

$$\phi(u) \ge u^a \int_{u/2}^{u} t^{-a-1}\psi(t)\,dt \ge u^a\psi(u/2)\int_{u/2}^{u} t^{-a-1}\,dt = \psi(u/2)\,\frac{2^a-1}{a},$$

so that

$$\psi(u) = O(\phi(u)),\ u\to\infty.$$

Moreover, by (2.2), $\phi(u)/u$ is bounded away from zero as $u\to\infty$, so that $|f|$ is integrable whenever $\phi(|f|)$ is.

Let f be a simple function and $\mathscr{K}=1$. For $\mathscr{K}>1$ the argument of the proof follows in the same way.

By hypotheses

$$\int_Y \psi(|h|)\,d\nu = \int_{R^+} D_h^\nu(y)d\psi(y) = \sum_j \int_{3\cdot 2^j}^{3\cdot 2^{j+1}} D_h^\nu(y)\,d\psi(y)$$

$$\le A\left(\sum_{j\ge 0} D_h^\nu(3\cdot 2^j)\psi(j)+\sum_{j<0} D_h^\nu(3\cdot 2^j)\psi(j)\right) \equiv A(S_1+S_2).$$

For fixed positive j, write $f=f_1+f_2+f_3$, where

$$f_1 = \begin{Bmatrix} f & \text{if } 1\le |f| < 2^j \\ 0 & \text{otherwise} \end{Bmatrix},$$

$$f_2 = \begin{Bmatrix} f & \text{if } 2^j\le |f| \\ 0 & \text{otherwise} \end{Bmatrix},$$

$$f_3 = \begin{Bmatrix} f & \text{if } 0\le |f| < 1 \\ 0 & \text{otherwise} \end{Bmatrix},$$

and $h_p=Tf_p$, $p=1,2,3$. Since $E_h(3\cdot 2^j)\le\bigcup_{p=1}^3 E_{h_p}(2^j)$, one obtains

$$D_h(3\cdot 2^j) \le \sum_{p=1}^{3} D_{h_p}(2^j). \tag{2.4}$$

Also, since T is of weak type (a,a) and (b,b)

$$D_{h_p}(2^j) \le A\left(2^{-jb}\int_X |f_p|^b\,d\mu\right),\quad p = 1,3$$

$$D_{h_2}(2^j) \le A\left(2^{-ja}\int_X |f_2|^a\,d\mu\right),$$

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so that by (2.5)

$$S_1 \leq A \sum_{j\geqslant 0} \psi(j)\left\{2^{-jb}\int_X |f_1|^b\,d\mu+2^{-ja}\int_X |f_2|^a\,d\mu+2^{-jb}\int_X |f_3|^b\,d\mu\right\}$$

$$\equiv A(S_1^1+S_1^2+S_1^3),$$

respectively. If $\eta_i=\{x\in X: 2^{i-1}\leq |f|<2^i, i=0, \pm 1, \pm 2, \pm 3,\ldots\}$ and $\varepsilon_i=\mu(\eta_i)$ then by (2.4) and (1.5)

$$S_1^1 \leq \sum_{j\geq 0}\psi_b(j)\sum_{0\leq i\leq j} 2^{ib}\varepsilon_i = A\sum_{i\geq 0} 2^{ib}\varepsilon_i \sum_{j\geq i}\psi_b{}^{(j)}$$

$$\leq A\sum_{i\geq 0} 2^{ib}\varepsilon_i\sum_{j\geq 1}\phi_b(j) \leq A\sum_{i\geq 0}\varepsilon_i 2^{ib}\sum_{j\leq 1}\int_{2^j}^{2^{j+1}}\frac{\phi(t)}{t^{b+1}}\,dt$$

$$= A\sum_{i\geq 0}\varepsilon_i 2^{ib}\int_{2^i}^{\infty}\frac{\phi(t)}{t^{b+1}}\,dt = A\sum_{i\geq 0}\phi(i)\varepsilon_i$$

$$\leq A\sum_{i\geq 0}\int_{\eta_i}\phi(|f|)\,d\mu \leq A\int_{X_1}\phi(|f|)\,d\mu,$$

where $X_1=\{x\in X: |f(x)|\geq 1\}$.

By (2.2)

$$S_1^2 \leq A\sum_{j\geq 0}\psi_a(j)\sum_{i\geqslant j}\int_{2^i}^{2^{i+1}} |f_2|^a\,d\mu \leq A\sum_{j\geq 0}\psi_a(j)\sum_{i\geq j} 2^{a(i+1)}\varepsilon_{i+1}$$

$$= A\sum_{i\geq 0} 2^{a(i+1)}\varepsilon_{i+1}\sum_{0\leq j\leq i}\psi_a(j) \leq A\sum_{i\geq 0} 2^{a(i+1)}\varepsilon_{i+1}\sum_{0\leq j\leq i}\int_{2^j}^{2^{j+1}} t^{-a-1}\psi(t)\,dt$$

$$= A\sum_{i\geq 0} 2^{a(i+1)}\varepsilon_{i+1}\int_1^{2^{i+1}} t^{-a-1}\psi(t)\,dt \leq A\sum_{i\geq 0}\phi(i+1)\varepsilon_{i+1} \leq A\sum_{i\geq 0}\int_{\eta_i}\phi(|f|)\,d\mu$$

$$= A\int_{X_1}\phi(|f|)\,d\mu.$$

To estimate S_1^3 we observe that

$$S_1^3 \leq A\sum_{j\geq 0}\psi_b(j)\int_{X_2}|f|^b d\mu,$$

where $X_2 = \{x: |f(x)|<1, x\in X\}$. Hence by (1.5) and (1.7)

$$S_1^3 \leq A\int_{X_2}|f|^b\,d\mu\sum_{j\geq 0}\phi_{b+1}(j)\int_{2^j}^{2^{j+1}} dt \leq A\int_{X_2}|f|^b\,d\mu\sum_{j\geq 0}\int_{2^j}^{2^{j+1}}\frac{\phi(t)}{t^{b+1}}\,dt$$

$$\leq A\int_{X_2}|f|^b\,d\mu\int_1^{\infty}\frac{\phi(t)}{t^{b+1}}\,dt \leq A\int_{X_2}\phi(|f|)\,d\mu.$$

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Next, for fixed negative j, let $f=f_4+f_5+f_6$, where

$$f_4 = \begin{cases} f & \text{if } 2^j \le |f| < 1 \\ 0 & \text{otherwise} \end{cases},$$

$$f_5 = \begin{cases} f & \text{if } 0 \le |f| < 2^j \\ 0 & \text{otherwise} \end{cases},$$

$$f_6 = \begin{cases} f & \text{if } 1 \le |f| \\ 0 & \text{otherwise} \end{cases},$$

and $h_p=Tf_p$, $p=4, 5, 6$. Then as before

$$D_h(3\cdot 2^j) \le \sum_{p=4}^{6} D_{h_p}(2^j) \le A\left\{2^{-aj}\int_X |f_4|^a\,d\mu+2^{-jb}\int_X |f_5|^b\,d\mu+2^{-ja}\int_X |f_6|^a\,d\mu\right\}$$

and

$$\begin{aligned} S_2 &\le A\sum_{j<0}\psi(j)\left\{2^{-ja}\int_X |f_4|^a\,d\mu+2^{-jb}\int_X |f_5|^b\,d\mu+2^{-ja}\int_X |f_6|\,d\mu\right\} \\ &\equiv A(S_2^1+S_2^2+S_2^3), \end{aligned}$$

respectively. The estimations of S_2^1, S_2^2, and S_2^3 are similar to those obtained for S_1^1, S_1^2 and S_1^3 and are therefore omitted.

The extension of T to $\phi(|f|)-\mu$ integrable functions follows now from [2, Lemmas A_1 and A_2].

3. **Applications.** Let $X=Y=R$, μ the ordinary Lebesgue measure and ν defined by

$$\nu(E) = \int_E y^{-2}\,dy, \quad y \ne 0,\ E \subset R.$$

Define T by $(Tf)(y)=y\hat{f}(y)$, where $\hat{f}$ is the Fourier transform of f. It is well known that T is of weak type (1, 1) and Plancherel's theorem shows that T is of type (2, 2). Thus with $a=1$ and $b=2$, Theorem 2 yields:

THEOREM 3. *If f is measurable and $\phi(|f|)$ integrable, then $\hat{f}$ the Fourier transform of f is defined and*

$$\int_R \psi(|y\hat{f}(y)|)\frac{dy}{y^2} \le A\int_R \phi(|f(x)|)\,dx. \tag{3.1}$$

COROLLARY 1. *If $\psi(t)=t$, $t>1$ and zero otherwise, then*

$$\int_{X_1} |y^{-1}\hat{f}(y)|\,dy \le A\int_R |f(x)|\ln^+|f(x)|\,dx,$$

where $X_1=\{xR: |y\hat{f}(y)|>1\}$, and $\ln^+|f|=\ln|f|$ if $|f|>1$ and zero otherwise.

COROLLARY 2. *If* $\psi(t)=t(\ln^+ t)^s$, $s>0$ *then* (3.1) *yields*

$$\int_R |y^{-1}\hat{f}(y)|\,(\ln^+ |y\hat{f}(y)|)^s\,dy \le A\int_R |f(x)|\,(\ln^+ |f(x)|)^{s+1}\,dx.$$

Another example involving the Hilbert transform is the following:

Let f be a complex valued measurable function over R. $\tilde{f}$ the Hilbert transform of f is defined by

$$\tilde{f}(x) = \lim_{\varepsilon\to 0+}\int_{|x-t|\ge\varepsilon}\frac{f(t)}{t-x}\,dt$$

provided the limit exists. Let μ and ν be defined by

$$\nu(E) = \mu(E) = \int_E \frac{dx}{1+|x|^\alpha},\quad 0\le\alpha<1, E\subset R$$

then [2, Theorems 3 and 4] show that $Tf=\tilde{f}$ is of type (p,p), $p>1$, and of weak type (1, 1). Applying Theorem 2 with $a=1$ and $b=p>1$ we obtain:

THEOREM 4. *If f is measurable* $\phi(|f|)-\mu$ *integrable, then* $\tilde{f}$ *exists and*

$$\int_R \psi(|\tilde{f}|)\,d\mu \le A\int_R \phi(|f|)\,d\mu.$$

In particular with $\psi(t)=t(\ln^+ t)^s$, $s>0$ *this yields*

$$\int_R |\tilde{f}(x)|\,(\ln^+ |\tilde{f}(x)|)^s\frac{dx}{1+|x|^\alpha} \le A\int_R |f(x)|\,(\ln^+ |f(x)|)^{s+1}\frac{dx}{1+|x|^\alpha},\quad 0\le\alpha<1.$$

ACKNOWLEDGEMENT. The author wishes to thank Professor T. M. K. Davison for some helpful discussions on the subject and the referee for suggesting the improvement of Theorem 2, to all a, such that $1\le a<b$.

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