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RESEARCH ARTICLE

Nonemptiness of severi varieties on enriques surfaces

Ciro Ciliberto[1], Thomas Dedieu[2], Concettina Galati[3] and Andreas Leopold Knutsen[4]

[1]Dipartimento di Matematica, Università di Roma Tor Vergata, Via della Ricerca Scientifica, Roma, 00173, Italia; e-mail: cilibert@mat.uniroma2.it.
[2]Institut de Mathématiques de Toulouse–UMR5219, Université de Toulouse–CNRS, UPS IMT, F-31062 Toulouse Cedex 9, France; e-mail: thomas.dedieu@math.univ-toulouse.fr.
[3]Dipartimento di Matematica ed Informatica, Università della Calabria, Via P. Bucci, cubo 31B, Arcavacata di Rende, CS, 87036, Italia; e-mail: concettina.galati@unical.it.
[4]Department of Mathematics, University of Bergen, Postboks 7800, 5020, Bergen, Norway; e-mail: andreas.knutsen@math.uib.no.

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Abstract
Let (S, L) be a general polarised Enriques surface, with L not numerically 2-divisible. We prove the existence of regular components of all Severi varieties of irreducible nodal curves in the linear system $|L|$, that is, for any number of nodes $\delta = 0, \dots, p_a(L) - 1$. This solves a classical open problem and gives a positive answer to a recent conjecture of Pandharipande–Schmitt, under the additional condition of non-2-divisibility.

Contents

1. Introduction

Let S be a smooth, projective complex surface and L a line bundle on S. Let $p_a(L) = \frac{1}{2}L \cdot (L + K_S) + 1$ denote the arithmetic (or sectional) genus of L. For any integer δ satisfying $0 \leq \delta \leq p_a(L)$, we denote by $V_{|L|,\delta}(S)$ the *Severi variety* parametrising irreducible δ-nodal curves in $|L|$. A heuristic count shows that the *expected dimension* of $V_{|L|,\delta}(S)$ is $\dim(|L|) - \delta$. Severi varieties were introduced by Severi in [30, Anhang F], where he proved that all Severi varieties of irreducible δ-nodal curves of degree d in $\mathbb{P}^2$ are nonempty and smooth of the expected dimension. Severi also claimed irreducibility of such varieties, but his proof contains a gap. The irreducibility was proved by Harris in [21].

Severi varieties on other surfaces have received much attention in recent years, especially in connection with enumerative formulas computing their degrees (see [1, 2, 3, 14, 19, 22, 26, 36, 37]). Nonemptiness, smoothness, dimension, and irreducibility for Severi varieties have been widely investigated on various rational surfaces (see, e.g., [20, 31, 33, 34, 35]), as well as K3 and abelian surfaces (see, e.g., [5, 24, 25, 26, 27, 32, 39]). Extremely little is known on other surfaces. In particular, Severi varieties may have unexpected behaviour: Examples are given in [7] of surfaces of general type with reducible Severi varieties, and also with components of dimension different from the expected one.

In this paper, we consider the case of Enriques surfaces. If S is an Enriques surface, it is known (cf. [9, Proposition 1]) that $V_{|L|,\delta}(S)$, *if nonempty*, is smooth and every irreducible component has dimension either $p_a(L) - \delta - 1$ or $p_a(L) - \delta$. Moreover, if S is general in moduli, the latter case can only occur if L is 2-divisible in $\mathrm{Pic}(S)$. Any component of dimension $p_a(L) - \delta - 1$ is called *regular*, and these components can only be nonempty for $\delta \leq p_a(L) - 1$, that is, they parametrize nodal curves of genus at least one. The nonemptiness problem has remained open until now.

For any integer $g \geq 2$, let $\mathcal{E}_g$ denote the moduli space of complex polarised Enriques surfaces (S, L) of *(sectional) genus* g, that is, S is an Enriques surface and L is an ample line bundle on S, such that $L^2 = 2g - 2$. Thus, g is the arithmetic genus of all curves in the linear system $|L|$. The spaces $\mathcal{E}_g$ have many irreducible components. A way to determine this has recently been given in [23], after partial results were obtained in [10] (cf. Theorem 5.7 below).

Denote by $\mathcal{E}_g[2]$ the locus in $\mathcal{E}_g$ parametrising pairs (S, L), such that L is 2-divisible in $\mathrm{Num}(S)$. The main result of this paper settles the existence of regular components of all Severi varieties on general polarised Enriques surfaces outside $\mathcal{E}_g[2]$:

Theorem 1.1. *Let (S, L) be a general element of any irreducible component of $\mathcal{E}_g \setminus \mathcal{E}_g[2]$. Then $V_{|L|,\delta}(S)$ is nonempty and has a regular component, of dimension $g - 1 - \delta$, for all $0 \leq \delta < g$.*

By [9, Corollary 1], the theorem follows as soon as one proves the case of maximal δ, that is, $\delta = g-1$, in which case, the parametrised curves are elliptic.

We note that Theorem 1.1 implies a conjecture due to Pandharipande and Schmitt regarding smooth curves of genus $g \geq 2$ on Enriques surfaces (see [28, Conjecture 5.1]). Our result implies this conjecture for curves whose classes are not 2-divisible (see [28, Proposition 2.2 and text after Conjecture 5.1]).

We shall prove Theorem 1.1 by degenerating a general Enriques surface to the union of two surfaces R and P, birational to the symmetric square of a general elliptic curve and the projective plane, respectively, and glued along a smooth elliptic curve T numerically anticanonical on each surface. We need the assumption that L is not 2-divisible to ensure that the degenerations of the curves we are interested in do not contain the curve $T = R \cap S$ (see Lemma 3.6), which is well known to be a major issue in the general context of degenerations.

We introduce the degenerations we need in Section 2. On such a semistable limit, we identify suitable curves that deform to elliptic nodal curves on the general Enriques surface and with the prescribed linear

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equivalence class that are rigid, that is, they do not move in a positive dimensional family. As remarked above, this suffices to prove the theorem. The aforementioned suitable curves consist, apart from some (-1)-curves as components, of an irreducible nodal elliptic curve C_R on R and an irreducible nodal rational curve C_P on P intersecting at one single point on T, where both C_R and C_P are smooth and have a contact of high order. Such curves are members of so-called logarithmic Severi varieties on the surfaces on which they lie. We develop all necessary tools and results on such varieties on the two types of surfaces in question in Section 3.

The analysis of the conditions under which the limit curves actually deform to rigid nodal elliptic curves on the general Enriques surface is performed in the crucial Section 4. This includes the most delicate part of our proof (Propositions 4.2 and 4.3), which consists in showing that the abovementioned curves C_R and C_P can be put together nicely. We end up eventually with numerical conditions to be verified by the line bundles determined on each component of the limit surfaces. An important ingredient next is the description of all components of moduli spaces of polarised Enriques surfaces in terms of decompositions of the polarising line bundles into effective isotropic divisors as developed recently in [10, 23], which we review in Section 5. The corresponding identification of suitable isotropic Cartier divisors on the limit surfaces is done in Section 6. Finally, Section 7 is devoted to exhibiting, for each component of the moduli spaces of polarised Enriques surfaces, a suitable isotropic decomposition of the limit polarising line bundle, such that its restriction on each component verifies the conditions necessary to deform the curves mentioned above.

2. Flat limits of Enriques surfaces

In this section, we will introduce the semistable degenerations of general Enriques surfaces that we will use in our proof of Theorem 1.1.

Let E be a smooth elliptic curve. Denote by $\oplus$ (and $\ominus$) the group operation on E and by e_0 the neutral element. Let $R := \mathrm{Sym}^2(E)$ and $\pi : R \longrightarrow E$ be the (Albanese) projection map sending $x + y$ to $x \oplus y$. We denote the fibre of π over a point $e \in E$ by

$$\mathfrak{f}_e := \pi^{-1}(e) = \{x + y \in \mathrm{Sym}^2(E) \mid x \oplus y = e \text{ (equivalently, } x + y \sim e + e_0)\},$$

which is the $\mathbb{P}^1$ defined by the linear system $|e + e_0|$ (here, and throughout the paper, $\sim$ denotes linear equivalence of divisors). We denote the algebraic equivalence class of the fibres by $\mathfrak{f}$. Symmetric products of elliptic curves have been studied in detail in [4], to which we will frequently refer in this paragraph.

For each $e \in E$, we define the curve $\mathfrak{s}_e$ (called D_e in [4]) as the image of the section $E \longrightarrow R$ mapping x to $e + (x \ominus e)$. We let $\mathfrak{s}$ denote the algebraic equivalence class of these sections, which are the ones with minimal self-intersection, namely, 1 (cf. [4]). We note that $\mathrm{Sym}^2(E)$ is the $\mathbb{P}^1$-bundle on E with invariant -1. We observe for later use that for $x \neq y$, we have

$$\mathfrak{s}_x \cap \mathfrak{s}_y = \{x + y\}. \tag{1}$$

We also note that

$$K_R \sim -2\mathfrak{s}_{e_0} + \mathfrak{f}_{e_0}.$$

Let η be any of the three nonzero 2-torsion points of E. The map $E \longrightarrow R$ defined by $e \mapsto e + (e \oplus \eta)$ realises E as an unramified double cover of its image curve $T := \{e + (e \oplus \eta) \mid e \in E\}$, which is a smooth elliptic curve. We have

$$T \sim -K_R + \mathfrak{f}_\eta - \mathfrak{f}_{e_0} \sim 2\mathfrak{s}_{e_0} - 2\mathfrak{f}_{e_0} + \mathfrak{f}_\eta,$$

by [4, (2.10)]. In particular, $T \nsim -K_R$ and $2T \sim -2K_R$.

Embed T as a cubic in $P := \mathbb{P}^2$. Consider nine (possibly coinciding) points $y_1, \dots, y_9 \in T$. Divide the nine points into two subsets, say of i and $9-i$ points, with $0 \le i \le 9$. Let $\widetilde{R} \to R$ and $\widetilde{P} \to P$, respectively, denote the blowups at the schemes on T determined by these two subsets of i and $9-i$ points, respectively. Denote by $\mathfrak{e}_R$ and $\mathfrak{e}_P$ the total exceptional divisors on $\widetilde{R}$ and $\widetilde{P}$, respectively, and denote the strict transforms of $T, \mathfrak{s}, \mathfrak{f}$ with the same symbols. We have

$$T \sim 2\mathfrak{s}_{e_0} - 2\mathfrak{f}_{e_0} + \mathfrak{f}_\eta - \mathfrak{e}_R \nsim -K_{\widetilde{R}} \sim 2\mathfrak{s}_{e_0} - \mathfrak{f}_{e_0} - \mathfrak{e}_R \text{ on } \widetilde{R}, \tag{2}$$

$$2T \sim -2K_{\widetilde{R}} \text{ on } \widetilde{R}, \tag{3}$$

$$T \sim 3\ell - \mathfrak{e}_P \sim -K_{\widetilde{P}} \text{ on } \widetilde{P}, \tag{4}$$

where ℓ is the pullback on $\widetilde{P}$ of a general line in P. Define $X = \widetilde{R} \cup_T \widetilde{P}$ as the surface obtained by gluing $\widetilde{R}$ and $\widetilde{P}$ along T. Denote by $\mathcal{D}_{[i]}$ the family of such surfaces. It is easy to see that $\mathcal{D}_{[i]}$ is irreducible of dimension 10 (when one also allows E to vary in moduli). We define $\mathcal{D} := \cup_{i=0}^{9} \mathcal{D}_{[i]}$.

Let X be a member of $\mathcal{D}$. The *first cotangent sheaf* $T^1_X := \mathfrak{ext}^1_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X)$ of X (cf. [29, Corollary 1.1.11] or [16, Section 2]) satisfies

$$T^1_X \simeq \mathcal{N}_{T/\widetilde{R}} \otimes \mathcal{N}_{T/\widetilde{P}}$$

by [16, Proposition 2.3], which is trivial if and only if the nine points satisfy the condition

$$y_1 + \cdots + y_9 \in |\mathcal{N}_{T/R} \otimes \mathcal{N}_{T/P}|. \tag{5}$$

Thus, X is *semistable* if and only if (5) holds (cf. [16, Definition (1.13)] and [17, (0.4)]). We will denote by $\mathcal{D}^*_{[i]}$ the subfamily of $\mathcal{D}_{[i]}$ consisting of semistable surfaces. It is easy to see that $\mathcal{D}^*_{[i]}$ is irreducible of dimension 9. We define $\mathcal{D}^* := \cup_{i=0}^{9} \mathcal{D}^*_{[i]}$.

We recall that a Cartier divisor, or a line bundle, $\mathcal{L}$ in $\mathrm{Pic}(X)$, is a pair (L', L''), such that $[L'] \in \mathrm{Pic}(\widetilde{R})$, $[L''] \in \mathrm{Pic}(\widetilde{P})$ and $L'|_T \simeq L''|_T$. Since T is numerically equivalent to the anticanonical divisor on both $\widetilde{R}$ and $\widetilde{P}$, we have

$$\mathcal{L}^2 = (L')^2 + (L'')^2 = 2p_a(L') - 2 + 2p_a(L'') - 2 + 2d, \quad d := L' \cdot T = L'' \cdot T.$$

The canonical divisor K_X is represented by

$$K_X = (K_{\widetilde{R}} + T, 0) = (\mathfrak{f}_\eta - \mathfrak{f}_{e_0}, 0) \;\text{ in }\; \mathrm{Pic}(\widetilde{R}) \times \mathrm{Pic}(\widetilde{P}).$$

In particular, by (2)–(4), we have

$$K_X \nsim 0 \;\text{ and }\; 2K_X \sim 0. \tag{6}$$

By [23, Lemma 3.5], the Cartier divisor K_X is the only nonzero torsion element of $\mathrm{Pic}(X)$ (the proof is for $i = 2$ but carries over to the general case).

Remark 2.1. There are exactly two elements of $\mathrm{Pic}^0(\widetilde{R}) \simeq E$ that restrict trivially on T, namely, $\mathcal{O}_{\widetilde{R}}$ and $\mathcal{O}_{\widetilde{R}}(\mathfrak{f}_\eta - \mathfrak{f}_{e_0})$ (see [23, Lemma 3.3]). Accordingly, for any $[L'] \in \mathrm{Pic}(\widetilde{R})$ and $[L''] \in \mathrm{Pic}(\widetilde{P})$, such that $L' \cdot T = L'' \cdot T$, there are two line bundles $\overline{L}'$ on $\widetilde{R}$ numerically equivalent to L' such that $(\overline{L}', L'')$ is a line bundle on X. By (6), their difference is K_X. These line bundles are numerically equivalent, and we will denote by $[L', L'']$ their numerical equivalence class.

By (5), if X is semistable, it also carries the Cartier divisor ξ represented by the pair

$$\xi = (T, -T) \sim (2\mathfrak{s}_{e_0} - 2\mathfrak{f}_{e_0} + \mathfrak{f}_\eta - \mathfrak{e}_R, -3\ell + \mathfrak{e}_P) \tag{7}$$

in $\mathrm{Pic}(\widetilde{R}) \times \mathrm{Pic}(\widetilde{P})$ (see [17, (3.3)]).

The central result for our purposes is:

Theorem 2.2. *Let $y_1, \dots, y_9 \in T$ be general, such that $X = \widetilde{R} \cup_T \widetilde{P}$ is a member of $\mathcal{D}^*$.*

There is a flat family $\pi : \mathfrak{X} \to \mathbb{D}$ over the unit disc, such that $\mathfrak{X}$ is smooth and, setting $S_t := \pi^{-1}(t)$, we have that

- *$S_0 = X$, and*
- *S_t is a smooth general Enriques surface for $t \neq 0$.*

Furthermore, denoting by $\iota_t : S_t \subset \mathfrak{X}$ the inclusion, there is a short exact sequence

$$0 \longrightarrow \mathbb{Z} \cdot \xi \longrightarrow \mathrm{Pic}(X) \simeq H^2(\mathfrak{X}, \mathbb{Z}) \xrightarrow{\iota_t^*} H^2(S_t, \mathbb{Z}) \simeq \mathrm{Pic}(S_t) \longrightarrow 0.$$

Proof. This follows from [23, Proposition 3.7, Theorem 3.10 and Corollary 3.11] in the case where X lies in $\mathcal{D}^*_{[2]}$. Once we have the statement in this case, we can prove it in the other cases by making a birational transformation of $\mathfrak{X}$ to flop any of the exceptional curves between $\widetilde{P}$ and $\widetilde{R}$ (see, for example [12, Section 4.1], where the flop is called a *1-throw*). □

3. Logarithmic Severi varieties

Theorem 1.1 will be proved by degenerating a general Enriques surface to a surface $\widetilde{R} \cup_T \widetilde{P}$ in $\mathcal{D}^*$. It will be essential to construct curves on $\widetilde{R} \cup_T \widetilde{P}$ that will deform to nodal irreducible elliptic curves on the general Enriques surface. As we will see in Section 4, the good limit curves on $\widetilde{R}$ and $\widetilde{P}$ are nodal curves with high order tangency with T at the same point on each component. These are members of so-called *logarithmic Severi varieties*, parametrising nodal curves with given tangency conditions to a fixed curve. This will be the topic of this section. We start with some general definitions and results:

Definition 3.1. Let S be a smooth projective surface, $T \subset S$ a smooth, irreducible curve and L a line bundle or a divisor class on S. Let g be an integer satisfying $0 \leq g \leq p_a(L)$.

For any effective divisor $\mathfrak{d} = m_1 p_1 + \cdots + m_l p_l$ on T, where the p_i are pairwise distinct, we denote by $V_{g,\mathfrak{d}}(S, T, L)$ the locus of curves in S, such that

- C is irreducible of geometric genus g and algebraically equivalent to L,
- denoting by $\nu : \widetilde{C} \to S$ the normalisation of C composed with the inclusion $C \subset S$, there exists $q_i \in \nu^{-1}(p_i)$, such that $\nu^* T$ contains $m_i q_i$, for all $i \in \{1, \dots, l\}$.

For any integer m satisfying $0 < m \leq L \cdot T$, we let $V_{g,m}(S, T, L)$ denote the locus of curves contained in some $V_{g,mp}(S, T, L)$ for some (nonfixed) $p \in T$.

We denote by $V^*_{g,m}(S, T, L)$ the open sublocus of $V_{g,m}(S, T, L)$ parametrising curves that are *smooth* at the intersection points with T and otherwise *nodal*.

In the sequel, $\equiv$ will denote numerical equivalence of divisors. We will need:

Proposition 3.2. *Let $S, T, L, \mathfrak{d}, g$ and m be as in Definition 3.1. Assume that $T \equiv -K_S$.*

(i) *If $L \cdot T > \sum_{i=1}^{l} m_i$, then all irreducible components of $V_{g,\mathfrak{d}}(S, T, L)$ have dimension $g - 1 + L \cdot T - \sum_{i=1}^{l} m_i$.*
(ii) *All irreducible components of $V_{g,m}(S, T, L)$ have dimension $g + L \cdot T - m$.*
(iii) *If $m \leq L \cdot T - 2$, then the general member $[C]$ in any component of $V_{g,m}(S, T, L)$ is smooth at its intersection points with T; moreover, if we fix $G \subset S$ any curve not having T as an irreducible component, and $\Gamma \subset S$ any finite set, then for general $[C]$, the curve C is transverse to G and does not intersect Γ.*
(iv) *If $m \leq L \cdot T - 3$, then the general member in any component of $V_{g,m}(S, T, L)$ is nodal.*

Proof. The result follows from [3, Section 2], as outlined in [15, Theorem (1.4)]. □

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3.1. Families of blownup surfaces

We will also need to work in families in the following way. For $S = R$ or P containing T as above and for any nonnegative integer n, we consider the family $\mathcal{S}^{\langle n\rangle} \to T^n$ with fibre $\mathrm{Bl}_{y_1,\ldots,y_n}(S)$ over $(y_1,\ldots,y_n) \in T^n$, the blowup of S at $y_1,\ldots,y_n$ (when the points are coinciding, this has to be interpreted as blowing up curvilinear schemes on T). To be precise, the fibres are *marked*, in the sense that their (total) exceptional divisors are labelled with $1,\ldots,n$. Whenever we have a line bundle on a single surface $\mathrm{Bl}_{y_1,\ldots,y_n}(S)$, we can write it in terms of the generators of $\mathrm{Pic}(S)$ and of the exceptional divisors over each y_i, and thus we can extend it to a relative line bundle on the whole family $\mathcal{S}^{\langle n\rangle}$ in the obvious way. We will therefore mostly not distinguish notationally between a relative line bundle L and its restriction to any surface in the family.

Similarly, there is for all $i = 1,\ldots,n$ a relative (total) exceptional divisor $\mathfrak{e}_i$ on $\mathcal{S}^{\langle n\rangle}$, whose fibre over a point $(y_1,\ldots,y_n) \in T^n$ is the exceptional divisor on $\mathrm{Bl}_{y_1,\ldots,y_n}(S)$ over the point y_i, which we by abuse of notation still denote by $\mathfrak{e}_i$.

Definition 3.3. Let L be a relative line bundle on $\mathcal{S}^{\langle n\rangle}$. The *value of L on the ith exceptional divisor* is the number $L\cdot\mathfrak{e}_i$ on any fibre $\mathrm{Bl}_{y_1,\ldots,y_n}(S)$. We say that L is *positive on the ith exceptional divisor* if $L\cdot\mathfrak{e}_i > 0$.

We shall consider the relative Hilbert scheme

$$\mathcal{H}^{\langle n\rangle}_{S,L} \longrightarrow T^n$$

whose fibres are the Hilbert schemes of curves on $\mathrm{Bl}_{y_1,\ldots,y_n}(S)$ algebraically (or equivalently numerically) equivalent to L. We have a (possibly empty) scheme

$$\mathcal{V}^{\langle n\rangle}_{g,m}(S,T,L) \longrightarrow T^n$$

whose fibres are $V_{g,m}(\mathrm{Bl}_{y_1,\ldots,y_n}(S),T,L) \subset \mathcal{H}^{\langle n\rangle}_{S,L}$ (here, as usual, we denote by T, its strict transform on the blowup). Taking the closure in $\mathcal{H}^{\langle n\rangle}_{S,L}$, we obtain a (possibly empty) scheme with a morphism

$$\nu^{\langle n\rangle}_{g,m}(S,T,L) : \overline{\mathcal{V}^{\langle n\rangle}_{g,m}(S,T,L)} \longrightarrow T^n, \tag{8}$$

whose fibres we denote by

$$(\nu^{\langle n\rangle}_{g,m})^{-1}(y_1,\ldots,y_n) := \overline{V}^{\langle n\rangle}_{g,m}\big(\mathrm{Bl}_{y_1,\ldots,y_n}(S),T,L\big).$$

Note that for any $(y_1,\ldots,y_n) \in T^n$, one has

$$\overline{V_{g,m}(\mathrm{Bl}_{y_1,\ldots,y_n}(S),T,L)} \subseteq \overline{V}^{\langle n\rangle}_{g,m}\big(\mathrm{Bl}_{y_1,\ldots,y_n}(S),T,L\big).$$

3.2. Logarithmic Severi varieties on blowups of the symmetric square of an elliptic curve

Let $T \subset R = \mathrm{Sym}^2(E)$, as defined in Section 2. Let $y_1,\ldots,y_n \in T$, and let $\widetilde{R} := \mathrm{Bl}_{y_1,\ldots,y_n}(R)$ denote the blowup of R at $y_1,\ldots,y_n$, with (total) exceptional divisors $\mathfrak{e}_i$ over y_i. We denote the strict transforms of $\mathfrak{s}$, $\mathfrak{f}$ and T on $\widetilde{R}$ by the same names. We also still denote by $\pi : \widetilde{R} \to E$ the composition of the blowup $\widetilde{R} \to R$ with the Albanese morphism $R \to E$ (cf. beginning of Section 2). By (2)-(3), we have

$$T \equiv -K_{\widetilde{R}} \equiv 2\mathfrak{s} - \mathfrak{f} - \mathfrak{e}_1 - \cdots - \mathfrak{e}_n.$$

Definition 3.4. A line bundle or Cartier divisor L on $\widetilde{R}$ is *odd* if $L\cdot\mathfrak{f}$ is odd.

Notation 3.5. We denote by $\mathrm{Sym}^n(T)_m \subset \mathrm{Sym}^n(T)$ the subscheme consisting of divisors with a point of multiplicity $\geq m$.

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Lemma 3.6. *Let L be an odd line bundle or Cartier divisor on $\widetilde{R}$. Let m be any integer satisfying $1 \leq m \leq L \cdot T$. Then the following hold:*

(i) *No curve C in $\overline{V}_{1,m}(\widetilde{R}, T, L)$ contains T.*
(ii) *For any component $V \subset \overline{V}_{1,m}(\widetilde{R}, T, L)$, the restriction map*

$$V \longrightarrow \mathrm{Sym}^{L\cdot T}(T)_m$$
$$C \mapsto C \cap T$$

is well-defined, finite and surjective. In particular,

$$\dim(V) = L \cdot T - m + 1.$$

(iii) *For a general curve C in any component of $V^*_{1,m}(\widetilde{R}, T, L)$, let N be the reduced subscheme of $\widetilde{R}$ supported at the nodes of C, and Z any subscheme of $C \cap T$ of degree $C \cdot T - 1$. Then the linear system $|\mathcal{O}_{\widetilde{R}}(C) \otimes \mathcal{J}_{N \cup Z}|$ consists only of C.*

Proof. Assume that we have $C = hT + C'$ in $\overline{V}_{1,m}(\widetilde{R}, T, L)$ for some $h > 0$, with C' not containing T. We have $L \cdot \mathfrak{f} = C \cdot \mathfrak{f} = 2h + C' \cdot \mathfrak{f}$, whence $C' \cdot \mathfrak{f} > 0$ since L is assumed to be odd. Hence, C' has at least one component dominating E via $\pi : \widetilde{R} \to E$, and therefore C cannot be a limit of an elliptic curve. Thus, (i) follows.

It also follows that the restriction map in (ii) is everywhere defined. The fibre over a $Z \in \mathrm{Sym}^{L\cdot T}(T)$ consists of all curves C in V, such that $C \cap T = Z$. This must be finite, for otherwise, we would find a member of the fibre passing through an additional general point $p \in T$, a contradiction (using again that no curve in V contains T). Hence, the restriction morphism in (ii) is finite. We have $\dim(V) \geq L\cdot T + 1 - m$ by Proposition 3.2(ii) and semicontinuity, which equals $\dim\big(\mathrm{Sym}^{L\cdot T}(T)_m\big)$. The morphism is therefore surjective, and equality holds for the dimension. This proves (ii).

Let now C be a curve in $V^*_{1,m}(\widetilde{R}, T, L)$ and Z be any subscheme of $C \cap T$ of degree $C \cdot T - 1$. Let $\widehat{R} \to \widetilde{R}$ denote the blowup of $\widetilde{R}$ along Z, considered as a curvilinear subscheme of T, and let $\widehat{C}$ and $\widehat{T}$ denote the strict transforms of C and T, respectively, and $\widehat{L} := \mathcal{O}_{\widehat{R}}(\widehat{C})$. Then $\widehat{C}$ is a member of $V^*_{1,1}(\widehat{R}, \widehat{T}, \widehat{L})$. To prove (iii), we may reduce to proving that if X is a general member of a component of $V^*_{1,1}(\widehat{R}, \widehat{T}, \widehat{L})$, and N is the subscheme of its nodes, then the linear system $|\mathcal{O}_{\widehat{R}}(X) \otimes \mathcal{J}_N|$ consists only of X.

Let $\delta = p_a(C) - 1$. The variety $V^*_{1,1}(\widehat{R}, \widehat{T}, \widehat{L})$ is the open subset of the *Severi variety of δ-nodal curves, algebraically equivalent to* $\widehat{L}$ consisting of curves with nodes off $\widehat{T}$. All of its components have dimension $\widehat{L} \cdot \widehat{T} = 1$ by (ii) (or Proposition 3.2(ii)), and it is smooth by standard arguments (see, e.g. [11, Proposition 2.2]). Let W be any component of $V^*_{1,1}(\widehat{R}, \widehat{T}, \widehat{L})$. Then W is fibred over $\mathrm{Pic}^0(E) \simeq E$ in subvarieties $W_{\widehat{L}'}$ parametrising δ-nodal curves in $|\widehat{L}'|$, where $\widehat{L}'$ is any line bundle numerically equivalent to $\widehat{L}$. By (ii), the linear equivalence classes of the curves in W vary. Thus, $W_{\widehat{L}'}$ is nonempty for general $\widehat{L}'$, whence smooth and zero-dimensional. The tangent space to $W_{\widehat{L}'}$ at any point $[X]$ is isomorphic to $H^0(\widehat{L}' \otimes \mathcal{J}_N)/\mathbb{C}$, where N is the scheme of nodes of X (see, e.g. [8, Section 1]). In particular, for a general X in W, we have

$$\dim\big(|\mathcal{O}_{\widehat{R}}(X) \otimes \mathcal{J}_N|\big) = h^0(\mathcal{O}_{\widehat{R}}(X) \otimes \mathcal{J}_N) - 1 = \dim(W_{\widehat{L}'}) = 0,$$

whence $|\mathcal{O}_{\widehat{R}}(X) \otimes \mathcal{J}_N|$ consists only of X, as desired. This proves (iii). □

In view of part (iii) of the previous result, we introduce the following:

Notation 3.7. We let $V^{**}_{1,m}(\widetilde{R}, T, L)$ denote the open subvariety of $V^*_{1,m}(\widetilde{R}, T, L)$ parametrising curves C, such that, for N, its scheme of nodes and for every subscheme Z of $C \cap T$ of degree $C \cdot T - 1$, the linear system $|\mathcal{O}_{\widetilde{R}}(C) \otimes \mathcal{J}_{N \cup Z}|$ consists only of C.

The main existence result of this subsection is Proposition 3.9 right below. To state it, we need a definition:

Definition 3.8. A line bundle or Cartier divisor L on $\widetilde{R}$ is said to verify condition $(\star)$ if it is of the form $L \equiv \alpha\mathfrak{s} + \beta\mathfrak{f} - \sum_{i=1}^n \gamma_i \mathfrak{e}_i$, such that:

(i) $\alpha \geq 1$ and $\beta \geq 0$;
(ii) $\alpha \geq \gamma_i$ for $i = 1, \dots, n$;
(iii) $\alpha + \beta \geq \sum_{i=1}^n \gamma_i$;
(iv) $\alpha + 2\beta \geq \sum_{i=1}^n \gamma_i + 4$ (equivalently, $-L \cdot K_{\widetilde{R}} \geq 4$).

Proposition 3.9. *Let E and $y_1, \dots, y_n \in T$ be general. Assume that L is a line bundle on $\widetilde{R}$ that is odd (cf. Definition 3.4) and satisfies condition $(\star)$ (cf. Definition 3.8). Then, if $0 < m \leq L \cdot T - 3$, the variety $V^{**}_{1,m}(\widetilde{R}, T, L)$ (cf. Definitions 3.1 and 3.7) has pure dimension $L \cdot T - m + 1$. Moreover, for all curves $G \subset \widetilde{R}$ not having T as an irreducible component, the general member of $V^{**}_{1,m}(\widetilde{R}, T, L)$ intersects G transversely.*

Proof. By Proposition 3.2(ii)–(iv) and Lemma 3.6(iii), we only need to prove nonemptiness of $V_{1,m}(\widetilde{R}, T, L)$. Following an idea in the proof of [6, Theorem 3.10], we will prove this by induction on m. The base case $m = 1$ follows from [11, Proposition 2.3], which requires all of (i)–(iv) from condition $(\star)$.

Assume that we have proved nonemptiness of $V_{1,m}(\widetilde{R}, T, L)$ for some $1 \leq m \leq L \cdot T - 4$. By Lemma 3.6(ii), its general member C satisfies

$$C \cap T = mp_0 + p_1 + \cdots + p_l + p_{l+1}, \quad l = L \cdot T - m - 1 \geq 3,$$

where $p_0, \dots, p_{l+1}$ are pairwise distinct, general points on T. Set $\mathfrak{d} = mp_0 + p_1 + \cdots + p_l$. Then $V_{1,\mathfrak{d}}(\widetilde{R}, T, L) \neq \emptyset$ and all its components are one-dimensional, by Proposition 3.2(i). The general member in any component intersects T in $mp_0 + p_1 + \cdots + p_l + q$, where the point q varies in the family, by Proposition 3.2(iii). Pick a component $\overline{V}$ of its closure inside the component of the Hilbert scheme of $\widetilde{R}$ containing $|L|$. After a finite base change, we find a smooth projective curve B, a surjective morphism $B \to \overline{V}$ and a family

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"] \arrow[d, "g"'] & \widetilde{R} \\ B & \end{tikzcd}$$

of stable maps of genus one, such that, setting $\mathcal{C}_b := g^*b$ for any $b \in B$, the curve $f_*\mathcal{C}_b$ is a member of $\overline{V}$, and such that

$$f^*T = mP_0 + P_1 + \cdots + P_l + Q + W,$$

where

(I) P_i and Q are sections of g, for $i \in \{0, \dots, l\}$,
(II) $f(P_i) = p_i$, for $i \in \{0, \dots, l\}$,
(III) $f(Q) = T$,
(IV) $g_*W = 0$,
(V) $f_*W = 0$;

the latter property follows from the fact that no member of the family contains T, by Lemma 3.6(i). Property (III) implies that $f^{-1}(T)$ is connected as follows. Consider the Stein factorization $\mathcal{C} \xrightarrow{f'} R' \xrightarrow{h} \tilde{R}$ of f. Then $h^{-1}(T)$ is of pure dimension 1. Since all irreducible components of f^*T except Q are contracted by f, it follows that $h^{-1}(T) = f'(Q)$, in particular, it is irreducible. Eventually, since f' has connected fibres, $f^{-1}(T) = (f')^{-1}(h^{-1}(T))$ is connected.

In particular, P_0 and Q are connected by an effective (possibly zero) divisor $W' \subset f^{-1}(p_0) \cap \mathcal{C}_{b_0} \subset W$ for some $b_0 \in B$. Thus,

$$f_*\mathcal{C}_{b_0} \cap T = (m+1)p_0 + p_1 + \cdots + p_l, \quad l \geq 3. \tag{9}$$

By the generality of the points $p_0, \ldots, p_l$, they cannot be contained in any (-1)-curve on $\widetilde{R}$, nor can any two of them lie in a fibre of $\pi : \widetilde{R} \to E$. Consequently, $f_*\mathcal{C}_{b_0}$ cannot contain any rational component. Moreover, $f_*\mathcal{C}_{b_0}$ must be a reduced curve by (9). Therefore, $f_*\mathcal{C}_{b_0} = C$ is an irreducible curve of geometric genus one, hence, $\mathcal{C}_{b_0}$ consists of one smooth elliptic curve $\widetilde{C}$, such that $f(\widetilde{C}) = C$ and otherwise chains of rational curves contracted by f and attached to $\widetilde{C}$ at one single point each. Therefore, $f^{-1}(p_0) \cap \widetilde{C}$ is a single (smooth) point of $\widetilde{C}$, hence, $[C] \in V_{1,(m+1)p_0+p_1+\cdots+p_l}(\widetilde{R}, T, L)$ by (9), which implies $[C] \in V_{1,m+1}(\widetilde{R}, T, L)$. □

3.3. Logarithmic Severi varieties on blownup planes

Fix a smooth cubic curve $T \subset P = \mathbb{P}^2$. Let $y_1, \ldots, y_n \in T$, for $n \geq 0$, and consider the blowup $\widetilde{P} := \mathrm{Bl}_{y_1,\ldots,y_n}(P) \to P$ at $y_1, \ldots, y_n$. We denote the strict transforms of the general line on P by ℓ and by $\mathfrak{e}_i$ the (total) exceptional divisor over y_i. We denote still by T the strict transform of T. Note that $T \sim -K_{\widetilde{P}} \sim 3\ell - \mathfrak{e}_1 - \cdots - \mathfrak{e}_n$.

The next result parallels Lemma 3.6.

Lemma 3.10. *Let L be a line bundle or Cartier divisor on $\widetilde{P}$. Let m be any integer satisfying $1 \leq m \leq L \cdot T$. Then the following hold:*

(i) *No curve C in $\overline{V}_{0,m}(\widetilde{P}, T, L)$ contains T.*
(ii) *For any component $V \subset \overline{V}_{0,m}(\widetilde{P}, T, L)$, the restriction map*

$$\begin{aligned} V &\longrightarrow \mathrm{Sym}^{L\cdot T}(T)_m \\ C &\mapsto C \cap T \end{aligned}$$

is well-defined and finite, with image $|L \otimes \mathcal{O}_T| \cap \mathrm{Sym}^{L\cdot T}(T)_m$, which has codimension one. In particular,

$$\dim(V) = L \cdot T - m.$$

Proof. Since the members of $\overline{V}_{0,m}(\widetilde{P}, T, L)$ are limits of rational curves, none of them can contain T as a component, which proves (i). As in the proof of Lemma 3.6(ii), the restriction map is everywhere defined and finite. Its image lies in $|L|_T| \cap \mathrm{Sym}^{L\cdot T}(T)_m$. Since, by Proposition 3.2(ii) and semicontinuity, $\dim(V) \geq L \cdot T - m$, which equals $\dim\big(|L|_T| \cap \mathrm{Sym}^{L\cdot T}(T)_m\big)$, the latter is in fact the image. This proves (ii). □

The next result is about the relative version $\nu^{\langle n \rangle}_{g,m} : \overline{\mathcal{V}^{\langle n \rangle}_{g,m}(P, T, L)} \longrightarrow T^n$ of the logarithmic Severi variety $\overline{V}_{0,m}(\widetilde{P}, T, L)$ considered in Lemma 3.10 above (see Section 3.1).

Lemma 3.11. *(i) Assume that $n > 0$ and L is a relative line bundle that is positive on the i-th exceptional divisor. Fix a point $(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n) \in T^{n-1}$. Let $\mathcal{V}$ be any component of*

$$\{(\nu^{\langle n \rangle}_{g,m})^{-1}(y_1, \ldots, y_{i-1}, p, y_{i+1}, \ldots, y_n),\ p \in T\}.$$

Then the restriction map

$$\mathcal{V} \longrightarrow \mathrm{Sym}^{L\cdot T}(T)_m$$

is finite and surjective.

(ii) Assume, furthermore, that $n \geq 2$ and L is positive, with two different values, on the i-th and j-th exceptional divisor, $i < j$. Fix any linear series $\mathfrak{g}$ of type g^1_2 on T and any point $(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_{j-1}, y_{j+1}, \ldots, y_n) \in T^{n-2}$. Let $\mathcal{V}$ be any component of the subset

$$\{(\nu^{\langle n\rangle}_{g,m})^{-1}(y_1, \ldots, y_{i-1}, p, y_{i+1}, \ldots, y_{j-1}, q, y_{j+1}, \ldots, y_n) \mid p + q \in \mathfrak{g}\}.$$

Then the restriction map

$$\mathcal{V} \longrightarrow \mathrm{Sym}^{L\cdot T}(T)_m$$

is finite and surjective.

Proof. Assume $L\cdot \mathfrak{e}_i > 0$ for some i. Varying p, we obtain a one-dimensional nontrivial family of surfaces $\mathrm{Bl}_{y_1,\ldots,y_{i-1},p,y_{i+1},\ldots,y_n}(P)$ and a one-dimensional nonconstant family of line bundles whose restrictions to T yield a one-dimensional nonconstant family of line bundles. This together with Lemma 3.10(ii) yields (i).

Finally, assume $L \cdot \mathfrak{e}_i = a_i > 0$ and $L \cdot \mathfrak{e}_j = a_j > 0$, with $a_i \neq a_j$. Varying $p + q \in \mathfrak{g}$, we get a one-dimensional nontrivial family of surfaces $\mathrm{Bl}_{y_1,\ldots,y_{i-1},p,y_{i+1},\ldots,y_{j-1},q,y_{j+1},\ldots,y_n}(P)$ as above and a one-dimensional family of line bundles, all of the form $L' - a_i\mathfrak{e}_i - a_j\mathfrak{e}_j$, with L' fixed (on $\mathrm{Bl}_{y_1,\ldots,y_{i-1},y_{i+1},\ldots,y_{j-1},y_{j+1},\ldots,y_n}(P)$) and $\mathfrak{e}_i$ and $\mathfrak{e}_j$ varying with p and q. To prove (ii), we will prove that the family of restrictions

$$\left\{L'|_T - a_i p - a_j q\right\}_{p+q\in\mathfrak{g}}$$

to T is nonconstant. Assume that

$$L'|_T - a_i p - a_j q \sim L'|_T - a_i p' - a_j q' \quad \text{for} \quad p + q \neq p' + q' \in \mathfrak{g}. \tag{10}$$

This yields $(a_i - a_j)p \sim (a_i - a_j)p'$ and $(a_i - a_j)q \sim (a_i - a_j)q'$. For fixed $x \in T$, there are only finitely many points $y \in T$, such that $(a_i - a_j)x \sim (a_i - a_j)y$. For general $p + q, p' + q' \in \mathfrak{g}$, condition (10) is therefore not fulfilled. This finishes the proof of (ii). □

The main existence result of this subsection is the following:

Proposition 3.12. *Let $y_1, \ldots, y_n \in T$ be general, $n \leq 8$ and L be big and nef on $\widetilde{P}$. If $0 < m \leq L \cdot T - 3$, the variety $V^*_{0,m}(\widetilde{P}, T, L)$ is nonempty of dimension $T \cdot L - m$. Moreover, its general member intersects any fixed curve on $\widetilde{P}$ different from T transversely.*

Proof. This is an application of [6, Corollary 3.11]; there are some conditions to check, so we give a proof for completeness.

The statements about dimension and transversal intersection follow from Proposition 3.2(ii) and (iii), respectively, once nonemptiness is proved. By Proposition 3.2(iii)–(iv), we have that $V^*_{0,m}(\widetilde{P}, L) \neq \emptyset$ as soon as $V_{0,m}(\widetilde{P}, T, L) \neq \emptyset$, because of the condition $m \leq L \cdot T - 3$. We therefore have left to prove nonemptiness of $V_{0,m}(\widetilde{P}, T, L)$. We will prove this by induction on m, as in the proof of Proposition 3.9, again following an idea in the proof of [6, Theorem 3.10].

Since $y_1 \ldots, y_n \in T$ are general and $n \leq 8$, we may take $y_1, \ldots, y_n$ to be general points of $\mathbb{P}^2$ and T a general plane cubic containing them. Hence, $\widetilde{P}$ is a Del Pezzo surface, so that T is ample on it. It is then well-known, by [20, Theorems 3–4], that $V_{0,1}(\widetilde{P}, T, L) \neq \emptyset$.

Assume now that we have proved nonemptiness of $V_{0,m}(\widetilde{P}, T, L)$ for some $1 \leq m \leq L \cdot T - 4$. By Lemma 3.10(ii), its general member C satisfies

$$C \cap T = mp_0 + p_1 + \cdots + p_l + p_{l+1} + p_{l+2}, \quad l = L \cdot T - m - 2 \geq 2,$$

where $p_0, \ldots, p_{l+2}$ are distinct, and we may take $p_0, \ldots, p_{l+1}$ general on T.

For later purposes, we observe that, since there are only finitely many divisor classes D, such that $C - D > 0$ (by which we mean that $C - D$ is effective and nonzero), hence, $C \cdot T > D \cdot T$, and for each such class, the image of the restriction morphism $|D| \to \mathrm{Sym}^{D\cdot T}(T)$ has codimension one, the generality of the points implies that

$$\begin{aligned} &\text{there is no effective divisor } D \neq T \text{ such that} \\ &C - D > 0 \text{ and } D \cap T \subset \{p_0, p_1, \dots, p_l\}. \end{aligned} \tag{11}$$

Set $\mathfrak{d} = mp_0 + p_1 + \cdots + p_l$. Then $V_{0,\mathfrak{d}}(\widetilde{P}, T, L) \neq \emptyset$, and all its components are one-dimensional, by Proposition 3.2(i). The general member in any component intersects T in $mp_0 + p_1 + \cdots + p_l + q_1 + q_2$, where the points q_1, q_2 vary in the family, by Proposition 3.2(iii). Pick a component $\overline{V}$ of its closure inside the component of the Hilbert scheme of $\widetilde{P}$ containing $|L|$. After a finite base change, we find a smooth projective curve B, a surjection $B \to \overline{V}$ and a family

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"] \arrow[d, "g"'] & \widetilde{P} \\ B & \end{tikzcd}$$

of stable maps of genus zero, such that, setting $\mathcal{C}_b := g^*b$ for any $b \in B$, the curve $f_*\mathcal{C}_b$ is a member of $\overline{V}$, and such that

$$f^*T = mP_0 + P_1 + \cdots + P_l + Q_1 + Q_2 + W,$$

where

(I) P_i and Q_j are sections of g, for $i \in \{0, \dots, l\}$, $j \in \{1, 2\}$,
(II) $f(P_i) = p_i$, for $i \in \{0, \dots, l\}$,
(III) $f(Q_j) = T$, for $j \in \{1, 2\}$,
(IV) $g_*W = 0$,
(V) $f_*W = 0$;

the latter property follows from the fact that no member of $\overline{V}$ contains T, by Lemma 3.10(i). Since T is ample, f^*T is big and nef, hence, its support $f^{-1}(T)$ is connected as a consequence of Kawamata–Viehweg vanishing. Therefore, P_0 and Q_2 are connected by a chain $W' \subset W$, such that $W' \subset \mathcal{C}_{b_0}$ for some $b_0 \in B$, and $f(W') = p_0$. Thus,

$$f_*\mathcal{C}_{b_0} \cap T = (m+1)p_0 + p_1 + \cdots + p_l + q_1, \quad q_1 = f(Q_1), \quad l \geq 2. \tag{12}$$

Since T is ample, all components of $f_*\mathcal{C}_{b_0}$ intersect T. By (11) and (12), $f_*\mathcal{C}_{b_0}$ must be reduced and irreducible, say $f_*\mathcal{C}_{b_0} = C$, an irreducible rational curve. Since $\mathcal{C}_{b_0}$ has arithmetic genus 0, it consists of a tree of smooth rational curves, with one component $\widetilde{C}$, such that $f(\widetilde{C}) = C$, and the other components contracted by f. Therefore, $f^{-1}(p_0) \cap \widetilde{C} = (W' + P_0) \cap \widetilde{C}$ is a single (smooth) point of $\widetilde{C}$. It follows that $[C] \in V_{0,(m+1)p_0+p_1+\cdots+p_l}(\widetilde{P}, T, L)$, which implies $[C] \in V_{0,m+1}(\widetilde{P}, T, L)$. □

4. Deforming to rigid elliptic curves

As mentioned in the Introduction, to prove Theorem 1.1, it will suffice by [9, Corollary 1] to prove that $V_{|L|,g-1}(S)$ has a 0-dimensional component. We will call any element of such a 0-dimensional component a *rigid nodal elliptic curve*. We will prove the existence of such a curve on a general (S, L) in any component of $\mathcal{E}_g \setminus \mathcal{E}_g[2]$ by degeneration, using Theorem 2.2, constructing suitable curves on limit surfaces in $\mathcal{D}^*$ that will deform to rigid curves in $V_{|L|,g-1}(S)$. In this section, we will identify numerical conditions on limit line bundles under which deformations to such curves can be achieved. The general strategy of proof is given in the following:

Proposition 4.1. *Let $X = \widetilde{R} \cup_T \widetilde{P}$ be a general member of a component of $\mathcal{D}^*$ and $Y = C \cup_T D$ a curve on X, with $C \subset \widetilde{R}$ and $D \subset \widetilde{P}$, having the following properties: there are distinct points $x, p_1, \ldots, p_k, q_1, \ldots, q_l$ on T, for nonnegative integers k, l and a positive integer m, such that*

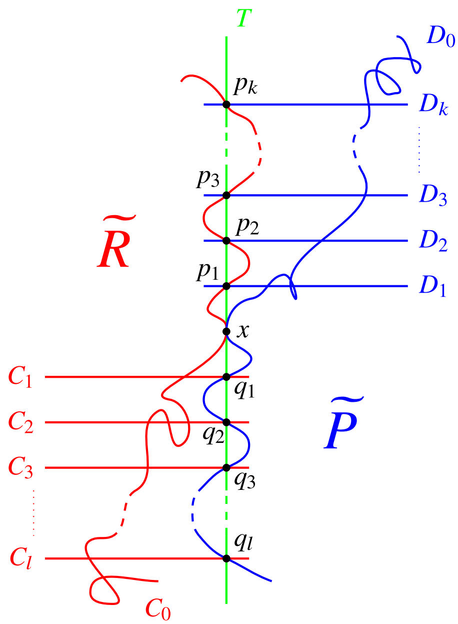
- *$C = C_0 + C_1 + \cdots + C_l$ is nodal, with $[C_0] \in V^{**}_{1,m}(\widetilde{R}, T, C_0)$ and C_i a (-1)-curve, $i \in \{1, \ldots, l\}$,*
- *$C_0 \cap T = mx + p_1 + \cdots + p_k$,*
- *C_0 is odd,*
- *$C_i \cap T = \{q_i\}$, $i \in \{1, \ldots, l\}$,*

and

- *$D = D_0 + D_1 + \cdots + D_k$ is nodal, with $[D_0] \in V^{*}_{0,m}(\widetilde{P}, T, D_0)$ and D_i a (-1)-curve, $i \in \{1, \ldots, k\}$,*
- *$D_0 \cap T = mx + q_1 + \cdots + q_l$,*
- *$D_i \cap T = \{p_i\}$, $i \in \{1, \ldots, k\}$.*

Then Y deforms to an irreducible rigid nodal elliptic curve on the general deformation S of X.

Proof. Here is a picture of what Y looks like:



We note that Y is Cartier, has an m-tacnode at x and is otherwise nodal. Moreover, an easy computation as in [23, p. 119] shows that

$$\dim(|Y|) = \frac{1}{2}Y^2 = p_a(Y) - 1. \tag{13}$$

We define

- N_{C_0} the scheme of nodes of C_0 and γ_0 its degree,
- N_C the scheme of intersection points between components of C and γ its degree,
- N_{D_0} the scheme of nodes of D_0 and δ_0 its degree,
- N_D the scheme of intersection points between components of D and δ its degree.

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Then $N_{C_0} \cup N_C \cup N_{D_0} \cup N_D$ is the set of nodes of Y off T. Since

$$\gamma_0 = p_a(C_0) - 1 = \frac{1}{2} C_0 \cdot (C_0 + K_{\widetilde{R}}) = \frac{1}{2}\left(C_0^2 - C_0 \cdot T\right) = \frac{1}{2}\left(C_0^2 - m - k\right)$$

and, similarly,

$$\delta_0 = p_a(D_0) = \frac{1}{2}\left(D_0^2 - m - l\right) + 1,$$

we compute

$$\begin{aligned} p_a(Y) &= \frac{1}{2}Y^2 + 1 = \frac{1}{2}\left(C^2 + D^2\right) + 1 = \frac{1}{2}\left(C_0^2 - l + 2\gamma + D_0^2 - k + 2\delta\right) + 1 \\ &= \frac{1}{2}\left(C_0^2 - m - k\right) + \frac{1}{2}\left(D_0^2 - m - l\right) + m + \gamma + \delta + 1 \\ &= \gamma_0 + (\delta_0 - 1) + m + \gamma + \delta + 1 = \gamma_0 + \gamma + \delta_0 + \delta + m. \end{aligned} \tag{14}$$

Let $\mathfrak{X} \to \mathbb{D}$ be the deformation of X to a general smooth Enriques surface S_t in Theorem 2.2. Then Y deforms to a Cartier divisor Y_t on S_t by the same theorem. By (13), we have $\dim(|Y|) = \dim(|Y_t|)$. Let $\mathfrak{D}$ be the sublinear system of $|Y|$ of curves with an $(m-1)$-tacnode at x and passing through $N_{C_0} \cup N_C \cup N_{D_0} \cup N_D$. We claim that

$$\mathfrak{D} \text{ consists only of } Y \text{ itself.}^{1} \tag{15}$$

Granting this for the moment, (13)-(15) yields that the codimension of $\mathfrak{D}$ is $m - 1 + \gamma_0 + \gamma + \delta_0 + \delta$. Thus, the hypotheses of [18, Theorem 3.3, Corollary 3.12 and proof of Theorem 1.1] are fulfilled,[2] and we conclude that, under the deformation of X to S_t, we may deform Y in such a way that the m-tacnode of Y at x deforms to $m-1$ nodes and the $\gamma_0 + \gamma + \delta_0 + \delta$ nodes of Y in the smooth locus of X are preserved, whereas the nodes of Y on T automatically smooth. Thus, Y deforms to a nodal curve $Y_t \subset S_t$ with a total of

$$\gamma_0 + \gamma + \delta_0 + \delta + m - 1 = p_a(Y) - 1 = p_a(Y_t) - 1$$

nodes (using (14)). Since one easily sees that no subcurve of Y is Cartier, Y_t is irreducible, whence Y_t is nodal and elliptic, as desired. It is rigid, as Y is rigid on X.

We have left to prove (15). To this end, let $A \cup_T B \in |Y|$ be a curve with an $(m-1)$-tacnode at x and passing through $N_{C_0} \cup N_C \cup N_{D_0} \cup N_D$, where $A \subset \widetilde{R}$ and $B \subset \widetilde{P}$. Then both A and B must intersect T in a scheme containing $(m-1)x$ and, moreover, $N_{C_0} \cup N_C \subset A$ and $N_{D_0} \cup N_D \subset B$.

The fact that $N_D \subset B$ implies that B must contain all (-1)-curves $D_1, \ldots, D_k$. Hence, $B = B_0 + D_1 + \cdots + D_k$, with $B_0 \sim D_0$. Similarly, the fact that $N_C \subset A$ implies that A must contain all (-1)-curves $C_1, \ldots, C_k$. Hence, $A = A_0 + C_1 + \cdots + C_k$, with $A_0 \sim C_0$. Since $A \cup_T B$ is Cartier, A_0 must pass through $p_1, \ldots, p_k$ and B_0 must pass through $q_1, \ldots, q_l$. Thus

$$A_0 \cap T \supset (m-1)x + p_1 + \cdots + p_k =: Z_C \quad \text{and} \quad B_0 \cap T \supset (m-1)x + q_1 + \cdots + q_l =: Z_D.$$

Hence, $A_0 \in |\mathcal{O}_{\widetilde{R}}(C_0) \otimes \mathcal{J}_{N_{C_0} \cup Z_C}|$. Since $\deg(Z_C) = C_0 \cdot T - 1$ and $[C_0] \in V^{**}_{1,m}(\widetilde{R}, T, C_0)$, this implies $A_0 = C_0$ (recall Definition 3.7), whence $A = C$.

[1] From a deformation-theoretic point of view, (15) implies that the *equisingular deformation locus of Y in $\mathfrak{X}$* is smooth and zero-dimensional (cf. [18, Lemma 3.4]), thus consisting only of the point $[Y]$.

[2] We remark that the hypothesis in [18] that both components of X are regular is not necessary; it suffices that $h^1(\mathcal{O}_X) = 0$, which is proved as in [23, Lemma 3.4].

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Similarly, $B_0 \in |\mathcal{O}_{\widetilde{P}}(D_0) \otimes \mathcal{J}_{N_{D_0} \cup Z_D}|$, whence, if $B_0 \neq D_0$, we would get

$$\begin{aligned} D_0^2 = B_0 \cdot D_0 &\geq \deg(Z_D) + 2\deg(N_{D_0}) = (D_0 \cdot T - 1) + 2p_a(D_0) \\ &= -D_0 \cdot K_{\widetilde{P}} - 1 + D_0^2 + D_0 \cdot K_{\widetilde{P}} + 2 > D_0^2, \end{aligned}$$

a contradiction. Thus, $B_0 = D_0$, whence $B = D$. This proves (15). □

The next two results form the basis for our proof of Theorem 1.1. We henceforth assume that E is a *general* elliptic curve.

Proposition 4.2. *Let R' (respectively, P') be a blowup of R (respectively, P) at $s \geq 0$ (respectively, $t \geq 1$) general points of T. Assume L' (respectively, L'') is a line bundle (or Cartier divisor) on R' (respectively, P') and k is an integer, such that the following conditions are satisfied:*

- (i) $s + t - 5 \leq k \leq \min\{3, s, t-1\}$,
- (ii) $L' \cdot T = L'' \cdot T$,
- (iii) $L' \equiv L_0' + C_1 + \cdots + C_k$, *where the C_i are disjoint (-1)-curves and L_0' satisfies condition $(\star)$ and is odd,*
- (iv) $L'' \sim L_0'' + D_1 + \cdots + D_k$, *where the D_i are disjoint (-1)-curves and L_0'' is big and nef,*
- (v) *there are $t - k$ additional (-1)-curves on P', mutually disjoint and disjoint from $D_1, \ldots, D_k$, such that L'' is positive on at least one of them.*

Then there are blowups $\widetilde{R} \to R'$ and $\widetilde{P} \to P'$ at distinct points of T, such that $\widetilde{R} \cup_T \widetilde{P}$ is general in a component of $\mathcal{D}^*$ and, denoting by $\widetilde{L}'$ and $\widetilde{L}''$ the pullbacks of L' and L'' to $\widetilde{R}$ and $\widetilde{P}$, respectively, there is a line bundle $\widetilde{L} \in [\widetilde{L}', \widetilde{L}'']$ (cf. Remark 2.1), such that $(\widetilde{R} \cup_T \widetilde{P}, \widetilde{L})$ deforms to a smooth polarised Enriques surface (S, L) with S containing an irreducible, rigid nodal elliptic curve in $|L|$.

Proof. Set $m := L_0' \cdot T - 3 = L_0'' \cdot T - 3$ (equality follows from assumptions (ii)–(iv)). By condition (iv) in $(\star)$, we have $m > 0$. The line bundle L_0' satisfies the conditions of Proposition 3.9. Hence, $V_{1,m}^{**}(R', T, L_0') \neq \emptyset$ and its general member intersects $C_1, \ldots, C_k$ transversely. Let C_0 be such a general member; we have $L' \equiv C_0 + C_1 + \cdots + C_k$ by assumption (iii).

The surface R' is a blowup of R at s general points $y_1, \ldots, y_s$ of T, where $k \leq s \leq 4$ by assumption (i). Denoting the exceptional divisor over y_i by $\mathfrak{e}_i$, the surface R' contains precisely s additional (-1)-curves $\mathfrak{e}_i'$, $i = 1, \ldots, s$, such that $\mathfrak{e}_i \cdot \mathfrak{e}_i' = 1$ and each $\mathfrak{e}_i + \mathfrak{e}_i'$ is a fibre of the projection $R' \to E$. Set $y_i' := \mathfrak{e}_i' \cap T$. Since $C_1, \ldots, C_k$ are disjoint (-1)-curves by assumption (iii), we have, after renumbering, that $C_i = \mathfrak{e}_i$ or $\mathfrak{e}_i'$. We will in the following, for simplicity, assume that $C_i = \mathfrak{e}_i$; the other cases can be treated in the same way by substituting y_i with y_i' at the appropriate places. Thus, we have

$$\begin{aligned} &C_0 \cap T = mp + p_1 + p_2 + p_3, \quad \text{for } \ p, p_1, p_2, p_3 \in T, \\ &C_i \cap T = y_i, \quad i \in \{1, \ldots, k\}. \end{aligned}$$

By Lemma 3.6(ii), the points p, p_1, p_2, p_3 are general on T (even for fixed $y_1, \ldots, y_k$). The points $y_1, \ldots, y_k$ are general on T by the assumption that R' is a blowup of R at *general* points on T.

The surface P' is a blowup of $\mathbb{P}^2$ at t general points on T, where $k < t \leq 5$ by assumption (i). The line bundle L_0'' is big and nef by assumption (iv), whence, by Proposition 3.12, we have $V_{0,m}^*(P', T, L_0'') \neq \emptyset$ and its general member intersects $D_1, \ldots, D_k$ transversely. Let D_0 be such a general member; we have $L'' \sim D_0 + D_1 + \cdots + D_k$ by assumption (iv).

By assumption (iv)–(v), $D_1, \ldots, D_k$ are disjoint (-1)-curves belonging to a set of t disjoint (-1)-curves, which we may therefore take as an exceptional set for a blowdown $P' \to \mathbb{P}^2$ centred at points that we denote by $x_1, \ldots, x_t \in T$. Furthermore, there is an exceptional curve, different from $D_1, \ldots, D_k$, call it D_{k+1}, such that $D_{k+1} \cdot L_0'' > 0$. We have

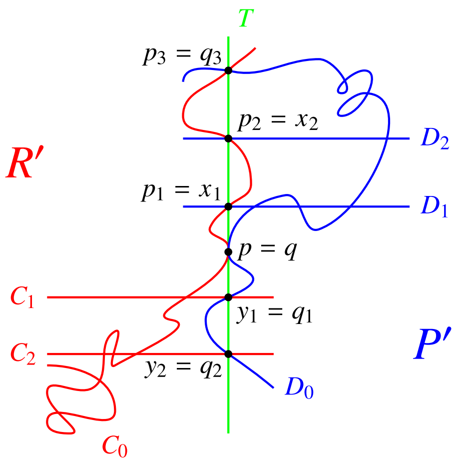
$$D_0 \cap T = mq + q_1 + q_2 + q_3, \quad \text{for} \quad q, q_1, q_2, q_3 \in T,$$
$$D_i \cap T = x_i, \quad i \in \{1, \ldots, k\}.$$

Since $D_{k+1} \cdot L_0'' > 0$, we can apply Lemma 3.11(i), which ensures that, even for fixed $x_1, \ldots, x_k$, by moving x_{k+1}, the intersection $D_0 \cap T$ is a general element in $\mathrm{Sym}^{L_0'' \cdot T}(T)_m$. Hence, the points q, q_1, q_2, q_3 are general on T. The points $x_1, \ldots, x_k$ are general on T by the assumption that P' is a blowup of P at *general* points on T.

Since the points $p, p_1, p_2, p_3, y_1, \ldots, y_k$ are general on T, and likewise the points $q, q_1, q_2, q_3, x_1, \ldots, x_k$, we may assume that our choices of C_0 and D_0 come with the identifications

$$p = q$$
$$x_i = p_i, \quad y_i = q_i \quad \text{for} \quad 1 \le i \le k \quad (\text{recall that } k \le 3),$$
$$p_i = q_i \quad \text{for} \quad k + 1 \le i \le 3 \quad (\text{if } k = 3, \text{ this condition is empty}),$$

so that we can glue R' and P' along T in such a way that $(C_0 + C_1 + \cdots + C_k, D_0 + D_1 + \cdots + D_k)$ is Cartier on $R' \cup_T P'$. The following picture shows the case $k = 2$:



By assumption (i), we have that $s + t \le k + 5 \le 8$. The set of points

$$y_1, \ldots, y_s, x_1, \ldots, x_t, (p_{k+1} = q_{k+1}), \ldots, (p_3 = q_3)$$

is therefore a set of $s + t + 3 - k \le 8$ general points on T. Pick now points $w_1, \ldots, w_{k+6-s-t}$ on T, general under the condition that

$$y_1 + \cdots + y_s + x_1 + \cdots + x_t + p_{k+1} + \cdots + p_3 + w_1 + \cdots + w_{k+6-s-t} \in |\mathcal{N}_{T/R} \otimes \mathcal{N}_{T/P}|.$$

Note that, in this way, we get a general divisor of $|\mathcal{N}_{T/R} \otimes \mathcal{N}_{T/P}|$. Then blowup either R' or P' at the points

$$(p_{k+1} = q_{k+1}), \ldots, (p_3 = q_3), w_1, \ldots, w_{k+6-s-t}$$

to obtain $\widetilde{R} \cup_T \widetilde{P}$, which is a general member of a component of $\mathcal{D}^*$, containing the inverse image Y of the curve $(C_0 + C_1 + \cdots + C_k) \cup_T (D_0 + D_1 + \cdots + D_k)$, which satisfies the conditions of Proposition 4.1.

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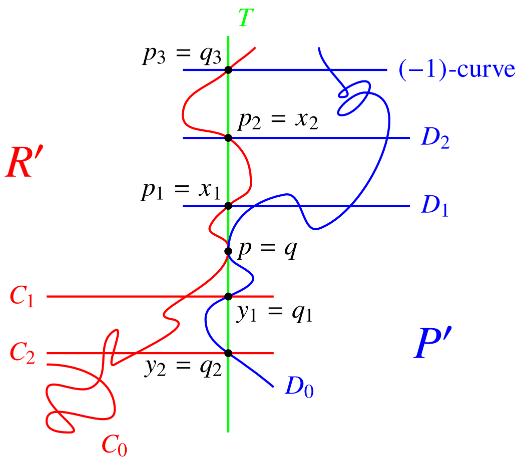
The result therefore follows by Proposition 4.1 (the following picture shows the inverse image of the curve in the previous image in the case $k = 2$: an additional (-1)-curve appears over $p_3 = q_3$).



□

Proposition 4.3. *Let $\widetilde{R}$ (respectively, $\widetilde{P}$) be a blowup of R (respectively, P) at 4 (respectively, 5) general points of T. Assume L' (respectively, L'') is a line bundle (or Cartier divisor) on $\widetilde{R}$ (respectively, $\widetilde{P}$), such that the following conditions are satisfied:*

(i) $L' \cdot T = L'' \cdot T$,
(ii) $L' \equiv L'_0 + C_1 + C_2 + C_3$, *where the C_i are disjoint (-1)-curves and L'_0 satisfies condition $(\star)$ and is odd,*
(iii) $L'' \sim L''_0 + D_1 + D_2 + D_3$, *where the D_i are disjoint (-1)-curves and L''_0 is big and nef,*
(iv) *there are two additional (-1)-curves D_4, D_5 on $\widetilde{P}$, mutually disjoint and disjoint from D_1, D_2, D_3, satisfying $L'' \cdot D_4 \neq L'' \cdot D_5$.*

Then there exist $\overline{R}$ (respectively, $\overline{P}$) a blowup of R (respectively, P) at 4 (respectively, 5) points of T and a line bundle $\overline{L}'$ on $\overline{R}$ (respectively, $\overline{L}''$ on $\overline{P}$), such that:

(a) *the pair $(\overline{R}, \overline{L}')$ (respectively, $(\overline{P}, \overline{L}'')$) is a deformation of $(\widetilde{R}, L')$ (respectively, $(\widetilde{P}, L'')$),*
(b) *the surface $\overline{R} \cup_T \overline{P}$ is a general member of $\mathcal{D}^*_{[4]}$ and $(\overline{L}', \overline{L}'')$ is a line bundle on it,*
(c) *$(\overline{R} \cup_T \overline{P}, (\overline{L}', \overline{L}''))$ deforms to a smooth polarised Enriques surface (S, L), such that $|L|$ contains an irreducible, rigid nodal elliptic curve.*

Proof. We argue as in the beginning of the previous proof with $k = 3$, $s = 4$ and $t = 5$, noting that L'' is positive on at least one of the two curves D_4, D_5. We find, as before, a Cartier divisor in a surface

$$Y := C \cup_T D \subset \widetilde{R} \cup_T \widetilde{P} =: X, \quad \widetilde{R} = \mathrm{Bl}_{y_1,y_2,y_3,y_4}(R), \quad \widetilde{P} = \mathrm{Bl}_{y_5,\dots,y_9}(P), \tag{16}$$

for general $y_i \in T$, $i \in \{1, \dots, 9\}$, such that, denoting by $\mathfrak{e}_i$ the exceptional divisor over y_i, one has

$$C = C_0 + \mathfrak{e}_1 + \mathfrak{e}_2 + \mathfrak{e}_3, \quad D = D_0 + \mathfrak{e}_5 + \mathfrak{e}_6 + \mathfrak{e}_7,$$

with $C \equiv L'$ and $D \sim L''$ both nodal, $C_0 \in V^{**}_{1,m}(\widetilde{R}, T, L'_0)$, $D_0 \in V^{*}_{0,m}(\widetilde{P}, T, L''_0)$, and we set $C_i = \mathfrak{e}_i$ for $i \in \{1, 2, 3\}$ and $D_i = \mathfrak{e}_{i+4}$ for $i \in \{1, 2, 3, 4, 5\}$. Moreover, setting $m := C_0 \cdot T - 3 = D_0 \cdot T - 3$, there is a point $x \in T$, such that

$$C_0 \cap T = mx + y_5 + y_6 + y_7, \quad D_0 \cap T = mx + y_1 + y_2 + y_3.$$

As in the previous proof, the points x, y_i are general on T.

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The problem now is that we cannot a priori guarantee that $\sum_{i=1}^{9} y_i \in |\mathcal{N}_{T/R} \otimes \mathcal{N}_{T/P}|$ to ensure that X is a member of $\mathcal{D}^*$ and therefore conclude, as in the previous proof; we only know that X is a member of $\mathcal{D}$. We will prove that we can create such a member of $\mathcal{D}^*$ without losing the 'nice' properties of Y.

Note first that condition (iv) says that $L'' \cdot \mathfrak{e}_8 \neq L'' \cdot \mathfrak{e}_9$. If $L'' \cdot \mathfrak{e}_i = 0$ for $i = 8$ or 9, we may contract $\mathfrak{e}_i$ and reduce to the case studied in the previous proposition. We will therefore assume that

$$L'' \cdot \mathfrak{e}_8 > 0, \quad L'' \cdot \mathfrak{e}_9 > 0 \quad \text{and} \quad L'' \cdot \mathfrak{e}_8 \neq L'' \cdot \mathfrak{e}_9. \tag{17}$$

Fix now, general $y_1, y_2, y_3, y_5, y_6, y_7 \in T$. Varying (x_4, x_8, x_9) in T^3, we obtain a 3-dimensional family of surfaces of the form

$$\mathrm{Bl}_{y_1,y_2,y_3,x_4}(R) \cup_T \mathrm{Bl}_{y_5,y_6,y_7,x_8,x_9}(P)$$

together with a family of line bundles (L', L''). There exists a relative Hilbert scheme $\mathcal{H}$ of effective Cartier divisors $C' \cup_T D'$ on these surfaces, such that on each surface the line bundle (C', D') is in the numerical equivalence class $[L', L'']$, cf. Remark 2.1 (considering L' and L'' as relative line bundles, see section 3.1). By our assumptions, we have a nonempty subscheme $\mathcal{W}$ of $\mathcal{H}$ with a dominating morphism $\mathcal{W} \to T^3$ whose fibre over a point (x_4, x_8, x_9) consists of pairs (X', Y'), such that

- ∘ $X' = R' \cup_T P'$ with $R' = \mathrm{Bl}_{y_1,y_2,y_3,x_4}(R)$ and $P' = \mathrm{Bl}_{y_5,y_6,y_7,x_8,x_9}(P)$,
- ∘ $Y' = C' \cup_T D'$, where

$$\begin{aligned} C' &= C'_0 + \mathfrak{e}_1 + \mathfrak{e}_2 + \mathfrak{e}_3 \equiv L' \\ D' &= D'_0 + \mathfrak{e}_5 + \mathfrak{e}_6 + \mathfrak{e}_7 \sim L'', \\ &\quad \text{with } C'_0 \in V^{**}_{1,m}(R', T, L'_0), D'_0 \in V^{*}_{0,m}(P', T, L''_0) \text{ intersecting all } \mathfrak{e}_i \text{ transversely,} \\ C'_0 \cap T &= mx' + y_5 + y_6 + y_7, \\ D'_0 \cap T &= mx' + y_1 + y_2 + y_3, \\ &\quad \text{for a point } x' \in T \setminus \{y_1, y_2, y_3, y_5, y_6, y_7\}. \end{aligned}$$

We once and for all substitute $\mathcal{W}$ with a dominating component containing $[(X, Y)]$ and will henceforth assume $\mathcal{W}$ is irreducible. Taking the closure in $\mathcal{H}$, we obtain a closed scheme with a surjective morphism

$$g : \overline{\mathcal{W}} \to T^3.$$

If $[(X', Y')] \in \overline{\mathcal{W}}$, then Y' looks like $C' \cup_T D'$ as above, except that $C'_0 \in \overline{V}_{1,m}(R', T, L'_0)$ and $D'_0 \in \overline{V}_{0,m}(P', T, L''_0)$, the intersection with the $\mathfrak{e}_i$s need not be transversal and x' may coincide with one of the points y_is (this follows from Lemmas 3.6(i) and 3.10(i)). In any event, (X', Y') comes equipped with a point $x' \in T$, which is the only point of intersection between the nonexceptional members of Y'. We will call this the *tacnodal point* of $[(X', Y')]$ (although it may be a worse singularity of Y' for special pairs). We therefore have a natural map

$$p : \overline{\mathcal{W}} \to T$$

sending a pair to its tacnodal point. For $x' \in T$, we set $\overline{\mathcal{W}}_{x'} := p^{-1}(x')$; this is the locus of pairs $[(X', Y')]$ of $\overline{\mathcal{W}}$ with tacnodal point x'.

Claim 4.4. The map g is finite and $\dim(\overline{\mathcal{W}}) = 3$.

Proof of claim. Since g is surjective, we only need to prove that g is finite. Fix any $X' = R' \cup_T P'$ as above, with $R' = \mathrm{Bl}_{y_1,y_2,y_3,x_4}(R)$ and $P' = \mathrm{Bl}_{y_5,y_6,y_7,x_8,x_9}(P)$, and assume $[(X', Y')] \in \overline{\mathcal{W}}$. Let $\overline{C_0} \subset R'$ and $\overline{D_0} \subset P'$ be the nonexceptional irreducible components of Y', elliptic and rational, respectively. They intersect only at the tacnodal point $x' \in T$, and otherwise, they intersect T in fixed points, as the (-1)-curves are fixed on each component of X'. Therefore, by Lemma 3.10(ii), there are finitely many possibilities for the point x' and the curve $\overline{D}_0$, and consequently, finitely many possibilities for the intersection $\overline{C_0} \cap T$. As $\overline{C_0}$ is odd by assumption (ii), there are, by Lemma 3.6(ii), only finitely many possibilities for $\overline{C_0}$ as well. This proves that g is finite. □

Let $(T^3)^*$ denote the two-dimensional subset of triples $(x_4, x_8, x_9) \in T^3$, such that

$$y_1 + y_2 + y_3 + x_4 + y_5 + y_6 + y_7 + x_8 + x_9 \in |\mathcal{N}_{T/R} \otimes \mathcal{N}_{T/P}|, \tag{18}$$

and let $\overline{\mathcal{W}}^* = g^{-1}((T^3)^*)$. This is the locus of pairs $[(X', Y')]$ of $\overline{\mathcal{W}}$, such that X' is semistable. For $x' \in T$, we set $\overline{\mathcal{W}}^*_{x'} := \overline{\mathcal{W}}^* \cap \overline{\mathcal{W}}_{x'}$.

We write $g_1 : \overline{\mathcal{W}} \to T$ and $g_2 : \overline{\mathcal{W}} \to T \times T$ for the composition of g with the projections onto the first factor and onto the product of the second and third factors, respectively. In other words, g_1 maps a pair $(X' = R' \cup_T P', Y')$ as above to x_4, whereas g_2 maps it to (x_8, x_9).

Claim 4.5. For all $x' \in T$, the following hold:

(i) $\overline{\mathcal{W}}^*_{x'} \neq \emptyset$ (hence, $\overline{\mathcal{W}}_{x'} \neq \emptyset$),
(ii) $(g_1)|_{\overline{\mathcal{W}}^*_{x'}}$ is surjective (hence, also $(g_1)|_{\overline{\mathcal{W}}_{x'}}$ is surjective).

Proof of claim. Fix any $x_4 \in T$ and consider $R' = \mathrm{Bl}_{y_1,y_2,y_3,x_4}(R)$. By Lemma 3.6(ii), there are finitely many curves $C_0' \in \overline{V}_{1,m}(R', T, L_0')$, such that $C_0' \cap T = mx' + y_5 + y_6 + y_7$. On the other hand, by Lemma 3.11(ii), taking property (17) into account, and considering the linear series

$$\mathfrak{g}(x_4) := |\mathcal{N}_{T/R} \otimes \mathcal{N}_{T/P}(-y_1 - y_2 - y_3 - x_4 - y_5 - y_6 - y_7)| \tag{19}$$

of type g^1_2 on T, there exist finitely many curves $D_0' \in \overline{V}_{0,m}(P', T, L_0'')$, with $P' = \mathrm{Bl}_{y_5,y_6,y_7,x_8,x_9}(P)$ for some $x_8 + x_9 \in \mathfrak{g}(x_4)$, such that $D_0'' \cap T = mx' + y_1 + y_2 + y_3$. Thus

$$\left(R' \cup_T P', (C_0' + \mathfrak{e}_1 + \mathfrak{e}_2 + \mathfrak{e}_3) \cup_T (D_0' + \mathfrak{e}_5 + \mathfrak{e}_6 + \mathfrak{e}_7)\right) \in \overline{\mathcal{W}}^*_{x'} \cap g_1^{-1}(x_4),$$

which proves (i) and (ii) (note, in particular, that the condition that $x_8 + x_9 \in \mathfrak{g}(x_4)$ is equivalent to (18)). □

Note that Claim 4.5 also implies that p is surjective, whence all fibres $\overline{\mathcal{W}}_{x'}$ are two-dimensional by Claim 4.4.

Now consider the map

$$\sigma : T \times T \longrightarrow \mathrm{Sym}^2(T),$$
$$(x, y) \mapsto x + y$$

and recall that there is a fibration $u : \mathrm{Sym}^2(T) \to T$ with fibres being the g^1_2s on T.

For all $x_4 \in T$, we consider the linear series $\mathfrak{g}(x_4) \subset \mathrm{Sym}^2(T)$ defined in (19) above; it is a g^1_2, hence, a fibre of u.

Claim 4.6. For all $x' \in T$, one has $u(\sigma(g_2(\overline{\mathcal{W}}_{x'}))) = T$, that is, $\sigma(g_2(\overline{\mathcal{W}}_{x'}))$ is not a union of fibres of u.

Proof of claim. Suppose to the contrary that $\sigma(g_2(\overline{\mathcal{W}}_{x'}))$ is a union of fibres of u. Then, for general $x_4 \in T$, we would have $\sigma(g_2(\overline{\mathcal{W}}_{x'})) \cap \mathfrak{g}(x_4) = \emptyset$, contradicting Claim 4.5(ii). □

Let now $x' \in T$ be general. Set $\mathcal{W}_{x'} = \overline{\mathcal{W}}_{x'} \cap \mathcal{W}$, which is nonempty, as $[(X, Y)] \in \mathcal{W}_x$. Since $\overline{\mathcal{W}}_{x'}$ is a general fibre of p and $\overline{\mathcal{W}}$ is irreducible, $\mathcal{W}_{x'}$ is dense in any component of $\overline{\mathcal{W}}_{x'}$. It follows that $g_2(\mathcal{W}_{x'})$ is dense in any component of $g_2(\overline{\mathcal{W}}_{x'})$. By the last claim, $\sigma(g_2(\mathcal{W}_{x'})) \cap \mathfrak{g}(x_4) \neq \emptyset$ for

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general $x_4 \in T$. Pick any $(x_8, x_9) \in g_2(\mathcal{W}_{x'}) \cap \sigma^{-1}(\mathfrak{g}(x_4))$. Then, by definition, there exists a pair $[(X^1, Y^1 = C^1 \cup_T D^1)] \in \mathcal{W}_{x'}$, such that

$$X^1 = \mathrm{Bl}_{y_1,y_2,y_3,x_4'}(R) \cup_T \mathrm{Bl}_{y_5,y_6,y_7,x_8,x_9}(P) \quad \text{for some} \quad x_4' \in T, \text{and}$$
$$y_1 + y_2 + y_3 + x_4 + y_5 + y_6 + y_7 + x_8 + x_9 \in |\mathcal{N}_{T/R} \otimes \mathcal{N}_{T/P}|.$$

In particular, $D^1 \subset \mathrm{Bl}_{y_5,y_6,y_7,x_8,x_9}(P)$ satisfies the conditions of Proposition 4.1 and

$$D^1 = D_0^1 + \mathfrak{e}_5 + \mathfrak{e}_6 + \mathfrak{e}_7, \quad D_0^1 \cap T = mx' + y_1 + y_2 + y_3. \tag{20}$$

As x_4 is general in T, one has $x_4 \in g_1(\mathcal{W}_{x'})$ (since $(g_1)|_{\overline{\mathcal{W}}_{x'}}$ is surjective, by Claim 4.5(ii)). Then, by definition, there exists a pair $[(X^2, Y^2 = C^2 \cup_T D^2)] \in \mathcal{W}_{x'}$, such that

$$X^2 = \mathrm{Bl}_{y_1,y_2,y_3,x_4}(R) \cup_T \mathrm{Bl}_{y_5,y_6,y_7,x_8',x_9'}(P) \quad \text{for some} \quad x_8', x_9' \in T.$$

In particular, $C^2 \subset \mathrm{Bl}_{y_1,y_2,y_3,x_4}(R)$ satisfies the conditions of Proposition 4.1 and

$$C^2 = C_0^2 + \mathfrak{e}_1 + \mathfrak{e}_2 + \mathfrak{e}_3, \quad C_0^2 \cap T = mx' + y_5 + y_6 + y_7. \tag{21}$$

Consider the pair

$$(\overline{X} = \mathrm{Bl}_{y_1,y_2,y_3,x_4}(R) \cup_T \mathrm{Bl}_{y_5,y_6,y_7,x_8,x_9}(P), \overline{Y} = C^2 \cup_T D^1).$$

Recall that $y_1, y_2, y_3, y_5, y_6, y_7$ were chosen general to start with, and x_4 and x' are also general by construction. Lemma 3.11(ii) implies that we may choose $x_8 + x_9$ general in $\mathfrak{g}(x_4)$. It follows that $\overline{X}$ is a general member of $\mathcal{D}^*_{[4]}$, that is, (b) holds. Properties (20) and (21) imply that the pair $(\overline{X}, \overline{Y})$ satisfies the conditions of Proposition 4.1 and therefore (c) holds. It is, moreover, clear that also (a) holds. □

5. Isotropic 10-sequences and simple isotropic decompositions

An important tool for identifying the various components of the moduli spaces of polarised Enriques surfaces is the decomposition of line bundles as sums of effective isotropic divisors. In this section, we will recall some notions and results from [10, 23].

Definition 5.1 [13, p. 122]. An *isotropic* 10-sequence on an Enriques surface S is a sequence of isotropic effective divisors $\{E_1, \dots, E_{10}\}$, such that $E_i \cdot E_j = 1$ for $i \neq j$.

It is well-known that any Enriques surface contains isotropic 10-sequences. Note that we, contrary to [13], require the divisors to be *effective*, which can always be arranged by changing signs. We also recall the following result:

Lemma 5.2 ([10, Lemma 3.4(a)], [13, Corollary 2.5.5]). *Let $\{E_1, \dots, E_{10}\}$ be an isotropic 10-sequence. Then there exists a divisor D on S, such that $D^2 = 10$ and $3D \sim E_1 + \cdots + E_{10}$. Furthermore, for any $i \neq j$, we have*

$$D \sim E_i + E_j + E_{i,j}, \quad \textit{with } E_{i,j} \textit{ effective isotropic}, \quad E_i \cdot E_{i,j} = E_j \cdot E_{i,j} = 2, \tag{22}$$

and $E_k \cdot E_{i,j} = 1$ *for* $k \neq i, j$. *Moreover,* $E_{i,j} \cdot E_{k,l} = \begin{cases} 1, & \textit{if } \{i,j\} \cap \{k,l\} \neq \emptyset, \\ 2, & \textit{if } \{i,j\} \cap \{k,l\} = \emptyset. \end{cases}$

The next result yields a ‘canonical’ way of decomposing line bundles:

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Proposition 5.3 [23, Theorem 5.7]. *Let L be an effective line bundle on an Enriques surface S, such that $L^2 > 0$. Then there are unique nonnegative integers $a_0, a_1, \dots, a_7, a_9, a_{10}$, depending on L, satisfying*

$$a_1 \geq \cdots \geq a_7, \quad and \tag{23}$$

$$a_9 + a_{10} \geq a_0 \geq a_9 \geq a_{10}, \tag{24}$$

such that L can be written as

$$L \sim a_1 E_1 + \cdots + a_7 E_7 + a_9 E_9 + a_{10} E_{10} + a_0 E_{9,10} + \varepsilon_L K_S, \tag{25}$$

for an isotropic 10-sequence $\{E_1, \dots, E_{10}\}$ (depending on L) and

$$\varepsilon_L = \begin{cases} 0, & \text{if } L + K_S \text{ is not 2-divisible in } \operatorname{Pic}(S), \\ 1, & \text{if } L + K_S \text{ is 2-divisible in } \operatorname{Pic}(S). \end{cases} \tag{26}$$

Remark 5.4. Although the coefficients a_i are unique, the isotropic 10-sequence in Proposition 5.3 is not unique, not even up to numerical equivalence or permutation, and nor is the presentation (25) (see [23, Remark 5.6]).

Definition 5.5 [23, Definitions 5.1, 5.8]. Let L be any effective line bundle on an Enriques surface S, such that $L^2 > 0$. A decomposition of the form (25) with coefficients satisfying (23), (24) and (26) is called a *fundamental presentation of L*. The coefficients $a_i = a_i(L)$, $i \in \{0, 1, \dots, 7, 9, 10\}$ and ε_L appearing in any fundamental presentation are called *fundamental coefficients of L* or *of* (S, L).

Remark 5.6. By [10, Lemma 4.8] (or [23, Theorem 1.3(f) and Proposition 5.5]), a line bundle L is 2-divisible in $\operatorname{Num}(S)$ if and only if all $a_i = a_i(L)$ are even, $i \in \{0, 1, \dots, 7, 9, 10\}$. In particular, by (26) or [10, Corollary 4.7], the number $\varepsilon = \varepsilon_L$ satisfies

$$\varepsilon = \begin{cases} 0, & \text{if some } a_i \text{ is odd}, \\ 0 \text{ or } 1, & \text{if all } a_i \text{ are even}. \end{cases} \tag{27}$$

This means that any 11-tuple $(a_0, a_1, \dots, a_7, a_9, a_{10}, \varepsilon)$ occurring as fundamental coefficients satisfies the conditions (23), (24) and (27). Conversely, for any such 11-tuple, we can choose any isotropic 10-sequence on any Enriques surface and write down the line bundle (25) having this 11-tuple as fundamental coefficients.

For any integer $g \geq 2$, let $\mathcal{E}_g$ denote the moduli space of complex polarised Enriques surfaces (S, L) of genus g, which is a quasiprojective variety by [38, Theorem 1.13]. Its irreducible components are determined by the fundamental coefficients, by the following:

Theorem 5.7 [23, Theorem 5.9]. *Given an irreducible component $\mathcal{E}$ of $\mathcal{E}_g$, all pairs (S, L) in $\mathcal{E}$ have the same fundamental coefficients. Different components correspond to different fundamental coefficients.*

The following technical result will be useful for our purposes:

Lemma 5.8. *Let (S, L) be an element of $\mathcal{E}_g \setminus \mathcal{E}_g[2]$. Set $a_i = a_i(L)$. Then one of the following holds:*

(i) *There are three distinct $k, l, m \in \{1, \dots, 7\}$, such that $a_i + a_k + a_l + a_m$ is odd for $i = 9$ or 10.*
(ii) *$a_0 > 0$ is odd, and all a_i for $i \neq 0$ are even.*
(iii) *$a_0 > 0$, and all a_i for $i \neq 0$ are odd.*

Proof. We first show that if $a_0 = 0$, then we end up in case (i). We have $a_9 = a_{10} = 0$ by condition (24). Moreover, by Remark 5.6, the set $I := \{i \in \{1, \dots, 7\} \mid a_i \text{ is odd}\}$ is nonempty. If $\sharp I \geq 3$, then for any distinct $k, l, m \in I$, we have that $a_i + a_k + a_l + a_m = a_k + a_l + a_m$ is odd for $i = 9$ and 10. If $\sharp I \leq 2$, we may pick $k \in I$ and two distinct $l, m \in \{1, \dots, 7\} \setminus I$; then again, $a_i + a_k + a_l + a_m = a_k + a_l + a_m$ is odd for $i = 9$ and 10.

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We may therefore assume that $a_0 > 0$.

Assume next that (i) does not hold; then we have

$$a_i + a_k + a_l + a_m \text{ is even for all } i \in \{9,10\} \text{ and distinct } k,l,m \in \{1,\dots,7\}. \tag{28}$$

This clearly implies that a_9 and a_{10} have the same parity.

Assume that a_9 and a_{10} are even. Then (28) implies that $a_k + a_l + a_m$ is even for all distinct $k,l,m \in \{1,\dots,7\}$. Hence, a_i is even for all $i \in \{1,\dots,7\}$. Then a_0 is odd by Remark 5.6, and we end up in case (ii).

Assume that a_9 and a_{10} are odd. Then (28) implies that $a_k + a_l + a_m$ is odd for all distinct $k,l,m \in \{1,\dots,7\}$. Hence, a_i is odd for all $i \in \{1,\dots,7\}$, yielding case (iii). □

6. Isotropic 10-sequences on members of $\mathcal{D}$

The notions of isotropic divisors and isotropic 10-sequences can be extended in the obvious way to all members of $\mathcal{D}$. Referring to [23, Section 3] for more details, we will in Example 6.1 below construct one such 10-sequence that we will use in the proof of Theorem 1.1 in the next section.

Recall that we have the points $y_1,\dots,y_9 \in T$, which are the blownup points on either R or P. We will now assume that $y_1,\dots,y_9$ are distinct, though the case of coinciding points can be treated similarly. Denote by $\mathfrak{e}_j$ the exceptional divisor over y_j, without fixing whether it lies on $\widetilde{R}$ or $\widetilde{P}$. View $y_j \in T \subset P$. The linear system of lines in P through y_j is a pencil inducing a g^1_2 on T, which has, by Riemann-Hurwitz, two members that are also two fibres of $\pi_{|T} : T \to E$. In other words, there are two fibres $\mathfrak{f}_{\alpha_j}$ and $\mathfrak{f}_{\alpha'_j}$ of $\pi : R \to E$, such that the intersection divisors $\mathfrak{f}_{\alpha_j} \cap T$ and $\mathfrak{f}_{\alpha'_j} \cap T$ belong to this g^1_2. Since $\mathfrak{f}_{\alpha_j} - \mathfrak{f}_{\alpha_i}$ restricts trivially to T, one has $\alpha'_j = \alpha_j \oplus \eta$ (see Remark 2.1). In particular, there are two uniquely defined points α_j and $\alpha_j \oplus \eta$ on E, such that the pairs

$$\begin{aligned}
&(\mathfrak{f}_{\alpha_j} + \mathfrak{e}_j, \ell) \text{ and } (\mathfrak{f}_{\alpha_j\oplus\eta} + \mathfrak{e}_j, \ell), \text{ if } \mathfrak{e}_j \subset \widetilde{R},\\
&(\mathfrak{f}_{\alpha_j}, \ell - \mathfrak{e}_j) \text{ and } (\mathfrak{f}_{\alpha_j\oplus\eta}, \ell - \mathfrak{e}_j), \text{ if } \mathfrak{e}_j \subset \widetilde{P},
\end{aligned}$$

define distinct numerically equivalent Cartier divisors on $X := \widetilde{R} \cup_T \widetilde{P}$. Hence, their difference is K_X (see, again, Remark 2.1).

Similarly, for four distinct (general) $y_i, y_j, y_k, y_l \in T$, the linear system of plane conics through y_i, y_j, y_k, y_l is again a pencil inducing a g^1_2 on T. As above, there are two fibres $\mathfrak{f}_{\alpha_{ijkl}}$ and $\mathfrak{f}_{\alpha_{ijkl}\oplus\eta}$ of $\pi : R \to E$, such that the divisors $\mathfrak{f}_{\alpha_{ijkl}} \cap T$ and $\mathfrak{f}_{\alpha_{ijkl}\oplus\eta} \cap T$ belong to this g^1_2. In particular, the pairs

$$\begin{aligned}
&(\mathfrak{f}_{\alpha_{ijkl}} + \mathfrak{e}_i + \mathfrak{e}_j + \mathfrak{e}_k + \mathfrak{e}_l, 2\ell) \text{ and } (\mathfrak{f}_{\alpha_{ijkl}\oplus\eta} + \mathfrak{e}_i + \mathfrak{e}_j + \mathfrak{e}_k + \mathfrak{e}_l, 2\ell), \text{ if } \mathfrak{e}_i, \mathfrak{e}_j, \mathfrak{e}_k, \mathfrak{e}_l \subset \widetilde{R},\\
&(\mathfrak{f}_{\alpha_{ijkl}}, 2\ell - \mathfrak{e}_i - \mathfrak{e}_j - \mathfrak{e}_k - \mathfrak{e}_l) \text{ and } (\mathfrak{f}_{\alpha_{ijkl}\oplus\eta}, 2\ell - \mathfrak{e}_i - \mathfrak{e}_j - \mathfrak{e}_k - \mathfrak{e}_l), \text{ if } \mathfrak{e}_i, \mathfrak{e}_j, \mathfrak{e}_k, \mathfrak{e}_l \subset \widetilde{P},
\end{aligned}$$

together with similar pairs when $\mathfrak{e}_i, \mathfrak{e}_j, \mathfrak{e}_k, \mathfrak{e}_l$ are distributed differently, define Cartier divisors on X. One may again check that their difference is K_X.

Considering instead, $y_i \in T \subset R = \mathrm{Sym}^2(E)$, we may write $y_i = p_i + (p_i \oplus \eta)$, for some $p_i \in E$. There are two sections in R passing through y_i, namely, $\mathfrak{s}_{p_i}$ and $\mathfrak{s}_{p_i\oplus\eta}$, cf. (1). Thus, the pairs

$$\begin{aligned}
&(\mathfrak{s}_{p_i} - \mathfrak{e}_i, 0) \text{ and } (\mathfrak{s}_{p_i\oplus\eta} - \mathfrak{e}_i, 0), \text{ if } \mathfrak{e}_i \subset \widetilde{R},\\
&(\mathfrak{s}_{p_i}, \mathfrak{e}_i) \text{ and } (\mathfrak{s}_{p_i\oplus\eta}, \mathfrak{e}_i), \text{ if } \mathfrak{e}_i \subset \widetilde{P}
\end{aligned}$$

define Cartier divisors on X. Again, one may check that their difference is K_X.

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Example 6.1. We consider $\widetilde{R} = \mathrm{Bl}_{y_1,y_2,y_3,y_4}(R)$ and $\widetilde{P} = \mathrm{Bl}_{y_5,\dots,y_9}(\mathbb{P}^2)$. Define

$$\begin{aligned} E_i^0 &:= (\mathfrak{s}_{p_i} - \mathfrak{e}_i, 0) \;\text{ for }\; i \in \{1,2,3,4\},\\ E_i^0 &:= (\mathfrak{f}_{\alpha_i}, \ell - \mathfrak{e}_i) \;\text{ for }\; i \in \{5,6,7,8\},\\ E_9^0 &:= (\mathfrak{s}_{p_9}, \mathfrak{e}_9),\\ E_{10}^0 &:= (\mathfrak{f}_{\alpha_{5678}}, 2\ell - \mathfrak{e}_5 - \mathfrak{e}_6 - \mathfrak{e}_7 - \mathfrak{e}_8). \end{aligned}$$

These are all Cartier divisors on $X = \widetilde{R} \cup_T \widetilde{P}$ by the above considerations. One may check that $(E_i^0)^2 = 0$ for all i and $E_i^0 \cdot E_j^0 = 1$ for all $i \neq j$. If X is a member of $\mathcal{D}^*$, then, arguing as in the proof of [23, Lemma 3.6], one may show that

$$E_1^0 + \cdots + E_{10}^0 - \xi \sim 3(E_9^0 + E_{10}^0 + E_{9,10}^0),$$

with ξ as in (7) and

$$E_{9,10}^0 = (\mathfrak{f}_{\alpha_9}, \ell - \mathfrak{e}_9).$$

Thus, we may similarly to (22) define

$$E_{i,j}^0 := \frac{1}{3}\left(E_1^0 + \cdots + E_{10}^0 - \xi\right) - E_i^0 - E_j^0 \;\text{ for each }\; i \neq j.$$

In particular, letting $y_{78} \in T \subset \mathbb{P}^2$ be the third intersection point of the line through y_7 and y_8 with T, and writing $y_{78} = p_{78} + (p_{78} \oplus \eta) \in T \subset \mathrm{Sym}^2(E)$ for some $p_{78} \in E$, we will use that

$$E_{5,6}^0 \sim (\mathfrak{s}_{p_{78}}, \ell - \mathfrak{e}_7 - \mathfrak{e}_8).$$

Note that $E_{9,10}^0$ and $E_{5,6}^0$ are Cartier divisors on any X in $\mathcal{D}$.

Remark 6.2. If $X = \widetilde{R} \cup_T \widetilde{P}$ belongs to $\mathcal{D}^*$, then, by Theorem 2.2, as it deforms to a general Enriques surface S, the sequence $(E_1^0, \dots, E_{10}^0)$ deforms to an isotropic 10-sequence $(E_1, \dots, E_{10})$ on S and each $E_{i,j}^0$ deforms to $E_{i,j}$, satisfying (22).

7. Proof of Theorem 1.1

We are now ready to finish the proof of our main result, Theorem 1.1. Keeping in mind that the various irreducible components of $\mathcal{E}_g$ are determined by the fundamental coefficients of the line bundles they parametrise (cf. Theorem 5.7), the proof will be divided in various cases depending on parity properties of the fundamental coefficients. Also recall that since we assume that we are not in $\mathcal{E}_g[2]$, at least one of the fundamental coefficients a_i is odd and $\varepsilon = 0$ (cf. Remark 5.6). In particular, the components we consider contain both (S, L) and $(S, L + K_S)$, so there is no need to distinguish between linear and numerical equivalence classes (cf. also [23, Theorem 1.1]).

The proof strategy will be as follows: given fundamental coefficients a_i, find a suitable line bundle with the same fundamental decomposition on some limit surface in terms of the isotropic divisors in Example 6.1, and apply Proposition 4.2 or 4.3 (and Remark 6.2). As mentioned in the beginning of Section 4, the existence of a rigid nodal elliptic curve will prove Theorem 1.1.

We will first treat three special cases in Sections 7.1–7.3 and then the three cases of Lemma 5.8 in Sections 7.4–7.6.

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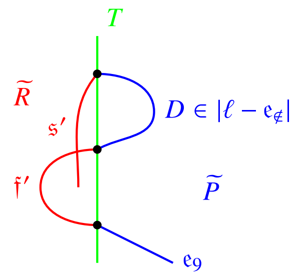
7.1. *The case* $a_1 = a_2 = 1$ *and* $a_i = 0$ *otherwise*

Consider the limit line bundle $L^0 = E_8^0 + E_9^0$, with E_8^0 and E_9^0 as in Example 6.1 (where we consider a general surface $\widetilde{R} \cup_T \widetilde{P}$ in $\mathcal{D}^*$). Note that the order in an isotropic 10-sequence does not matter, hence, we can choose E_8^0 and E_9^0 instead of E_1^0 and E_2^0. This fact will be used throughout the rest of the proof, without further mention. Then

$$L^0|_{\widetilde{R}} \equiv \mathfrak{f} + \mathfrak{s} \quad \text{and} \quad L^0|_{\widetilde{P}} \sim (\ell - \mathfrak{e}_8) + \mathfrak{e}_9.$$

In this case, there is no need to invoke Proposition 4.2 or 4.3: indeed, the linear system $|L|$ contains the following curve:



Here, $\mathfrak{f}'$ is the unique fibre passing through the point $y_9 = \mathfrak{e}_9 \cap T$ and D is the unique element of $|\ell - \mathfrak{e}_8|$ passing through the point y_9', such that $y_9 + y_9' = T \cap \mathfrak{f}'$; finally $\mathfrak{s}'$ is one of the two sections $\mathfrak{s}$ passing through the remaining intersection point of D with T.

Arguing as in the proof of Proposition 4.1, this curve can be deformed to a one-nodal rigid elliptic curve of arithmetic genus 2 in the linear system $|L|$ as $(\widetilde{R} \cup_T \widetilde{P}, L^0)$ deforms to (S, L).

7.2. *The case* $a_0 = a_9$ *and* $a_i = 0$ *otherwise*

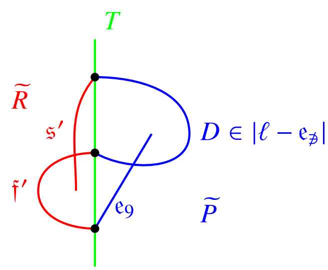
Both a_0 and a_9 are odd.

7.2.1. Subcase $a_0 = a_9 = 1$

Consider the limit line bundle $L^0 = E_{9,10}^0 + E_9^0$, with $E_{9,10}^0$ and E_9^0 as in Example 6.1 (as above). Then

$$L^0|_{\widetilde{R}} \equiv \mathfrak{f} + \mathfrak{s} \quad \text{and} \quad L^0|_{\widetilde{P}} \sim (\ell - \mathfrak{e}_9) + \mathfrak{e}_9.$$

There is, again, no need to invoke Proposition 4.2 or 4.3: indeed, the linear system $|L|$ contains the following curve, constructed as in the previous case:



Arguing, again, as in the proof of Proposition 4.1, this curve can be deformed to a rigid elliptic two-nodal curve of arithmetic genus 3 in the linear system $|L|$ as $(\widetilde{R} \cup_T \widetilde{P}, L^0)$ deforms to (S, L).

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7.2.2. Subcase $a_0 = a_9 \geq 3$
Consider the limit line bundle $L^0 = a_0(E^0_{5,6} + E^0_5)$, with $E^0_{5,6}$ and E^0_5 as in Example 6.1. Then

$$L' := L^0|_{\widetilde{R}} \equiv a_0\mathfrak{s} + a_0\mathfrak{f} \quad \text{and} \quad L'' := L^0|_{\widetilde{P}} \sim a_0(\ell - \mathfrak{e}_7 - \mathfrak{e}_8) + a_0(\ell - \mathfrak{e}_5) = a_0(2\ell - \mathfrak{e}_5 - \mathfrak{e}_7 - \mathfrak{e}_8).$$

We see that we might as well substitute $\widetilde{R}$ with R and $\widetilde{P}$ with $P' = \mathrm{Bl}_{y_5,y_7,y_8}(P)$ and consider $L^0 = (L', L'')$ as a line bundle on $R \cup_T P'$. We apply Proposition 4.2 with $k = 0$, $s = 0$ and $t = 3$.

Clearly L' satisfies condition ($\star$) and is odd and L'' is big and nef. Moreover, $\mathfrak{e}_i \cdot L'' > 0$ for $i \in \{5, 7, 8\}$. The conditions of Proposition 4.2 are satisfied, and we are done.

7.3. *The cases* $a_7 = a_9 = a_{10} = a_0 = 0$

At least one of the a_i is odd. Pick the minimal such and call it c_1. Reordering the remaining a_is, we have that a limit line bundle is of type

$$L^0 \equiv c_1E^0_1 + \sum_{i=5}^{8} c_iE^0_i + c_{10}E^0_{10},$$

with the E^0_i as in Example 6.1, where c_1 is the minimal odd coefficient, and we may also assume that $c_5 > 0$. Then

$$L' := L^0|_{\widetilde{R}} \equiv c_1(\mathfrak{s} - \mathfrak{e}_1) + \sum_{i=5}^{8} c_i\mathfrak{f} + c_{10}\mathfrak{f} = c_1\mathfrak{s} + (c_5 + c_6 + c_7 + c_8 + c_{10})\mathfrak{f} - c_1\mathfrak{e}_1,$$

$$L'' := L^0|_{\widetilde{P}} \sim \sum_{i=5}^{8} c_i(\ell - \mathfrak{e}_i) + c_{10}(2\ell - \mathfrak{e}_5 - \mathfrak{e}_6 - \mathfrak{e}_7 - \mathfrak{e}_8).$$

We see that we might as well substitute $\widetilde{R}$ with $R' = \mathrm{Bl}_{y_1}(R)$ and $\widetilde{P}$ with $P' = \mathrm{Bl}_{y_5,y_6,y_7,y_8}(P)$ and consider $L^0 = (L', L'')$ as a line bundle on $R' \cup_T P'$. We apply Proposition 4.2 with $s = 1$, $t = 4$ and $k = 0$. Conditions (i)–(iii) of ($\star$) are verified by L'; condition (iv) is equivalent to $c_5+c_6+c_7+c_8+c_{10} \geq 2$, which is verified unless $c_5 = 1$ and $c_6 = c_7 = c_8 = c_{10} = 0$. Since c_1 was assumed to be a minimal odd fundamental coefficient, we must have $c_1 = 1$ as well. This case is the one treated in Section 7.1. We may therefore assume that L' satisfies ($\star$). One readily checks that L' is odd (since c_1 is odd), and that L'' is big and nef (all components have square zero, intersect and can be represented by irreducible curves). This verifies conditions (i)–(iv) in Proposition 4.2. Since, for instance, $\mathfrak{e}_5 \cdot L'' = c_5 + c_{10} \geq c_5 > 0$, also condition (v) therein is satisfied. Hence, we are done by Proposition 4.2.

7.4. *Case where there are three distinct* $k, l, m \in \{1, \ldots, 7\}$*, such that* $a_i + a_k + a_l + a_m$ *is odd for* $i = 9$ *or* 10 *(case (i) in Lemma 5.8)*

Note that the cases among these with $a_0 = 0$ (whence also $a_9 = a_{10} = 0$) and $a_7 = 0$ fall into the cases treated in Section 7.3. We can therefore assume that

$$a_7 > 0 \quad (\text{whence } a_i > 0 \text{ for all } i \in \{1, \ldots, 7\}), \text{ if } a_0 = 0. \tag{29}$$

Similarly, the cases among these with $a_0 = a_9$ and all remaining $a_i = 0$ fall into the cases treated in Section 7.2. We can therefore assume that

$$a_0 \neq a_9, \quad \text{if} \quad a_i = 0 \quad \text{for all} \quad i \in \{1, \ldots, 7, 10\}. \tag{30}$$

We know that for $i = 9$ or 10, we can find indices k, l, m so that $a_i+a_k+a_l+a_m$ is odd. In this case, we take k, l, m so that $a_k + a_l + a_m$ is minimal with respect to this property. If we can do this for both $i = 9$

and 10 (with possibly different triples of indices k, l, m), we will pick $i \in \{9, 10\}$ so that $a_i + a_k + a_l + a_m$ is minimal. We rename these coefficients a_i, a_k, a_l, a_m as c_9, c_2, c_3, c_4, making sure that

$$c_2 \geq c_3 \geq c_4, \tag{31}$$

set $c_0 = a_0$, $c_{10} = \begin{cases} a_{10}, & \text{if } i = 9 \\ a_9, & \text{if } i = 10 \end{cases}$, and rename the remaining a_i as c_5, c_6, c_7, c_8 in such a way that

$$c_5 \geq c_6 \geq c_7 \geq c_8. \tag{32}$$

We thus have a limit line bundle

$$L^0 \equiv c_0 E^0_{9,10} + c_9 E^0_9 + c_{10} E^0_{10} + \sum_{i=2}^{8} c_i E^0_i,$$

with the E^0_i and $E^0_{9,10}$ as in Example 6.1, where, besides (31) and (32), one has

$$c_9 + c_{10} \geq c_0 \geq \max\{c_9, c_{10}\}, \tag{33}$$

$$c_9 + c_2 + c_3 + c_4 \;\text{ is odd}, \tag{34}$$

$$\text{there are no } i \in \{9, 10\}, k, l, m \in \{2, \dots, 8\}, \text{ such that} \tag{35}$$

$$c_i + c_k + c_l + c_m \text{ is odd and}$$

$$\begin{cases} c_k + c_l + c_m < c_2 + c_3 + c_4, \text{or} \\ c_k + c_l + c_m = c_2 + c_3 + c_4 \;\text{ and }\; c_i + c_k + c_l + c_m < c_9 + c_2 + c_3 + c_4. \end{cases}$$

Furthermore, (29) gives

$$c_i > 0 \;\text{ for all }\; i \in \{2, \dots, 8\}, \;\text{ if }\; c_0 = 0, \tag{36}$$

and (30) yields

$$c_0 \neq c_9 \;\text{ if }\; c_i = 0 \;\text{ for all }\; i \in \{2, \dots, 8, 10\}. \tag{37}$$

We define

$$\kappa := \sharp\{j \in \{2, 3, 4\} \mid c_j > 0\} \text{ and } \lambda := \sharp\{j \in \{5, 6, 7, 8\} \mid c_j > 0\}.$$

Claim 7.1. The following hold:

(i) If $c_0 = 0$, then $(\kappa, \lambda) = (3, 4)$.
(ii) If $\lambda \leq 2$, then $\kappa \leq 1$; moreover, $\kappa = 1$ implies $c_{10} \geq 2$.
(iii) If $(\kappa, \lambda, c_{10}) = (0, 0, 0)$, then $c_0 \neq c_9$.

Proof. Property (i) follows from condition (36).

Next assume $\lambda \leq 2$, that is, $c_7 = c_8 = 0$. Then properties (34) and (35) yield that $c_3 = c_4 = 0$, that is, $\kappa \leq 1$, as we now explain. Indeed, if c_4 is even and positive, we have that $c_9 + c_2 + c_3 + c_8$ is odd and $c_2 + c_3 + c_8 < c_2 + c_3 + c_4$, contradicting (35) (similarly for the cases where c_i is even and positive with $i = 2, 3$). From this, it follows that none among c_2, c_3, c_4 can be even and positive.

If c_3 and c_4 are odd, then $c_9 + c_2 + c_7 + c_8$ is odd and $c_2 + c_7 + c_8 < c_2 + c_3 + c_4$, contradicting (35) (similarly if c_2 and c_3 or c_2 and c_4 are odd). Hence, at most one among c_2, c_3, c_4 is odd.

In conclusion, at least two among c_2, c_3, c_4 must be zero, hence, $c_3 = c_4 = 0$ by (31), as we claimed.

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If $\kappa = 1$, we have $c_2 > 0$ and $c_9 + c_2$ is odd by (34). Condition (35) yields that c_9 is even (whence c_2 is odd), for otherwise, $c_9 = c_9 + c_3 + c_4 + c_8$ would be odd with $0 = c_3 + c_4 + c_8 < c_2 = c_2 + c_3 + c_4$. For the same reason, c_{10} is even, and condition (35) yields that $c_{10} \geq c_9$. By (33) and the fact that $c_0 > 0$ from (i), we must have $c_{10} > 0$. This proves (ii).

Finally, (iii) is a reformulation of property (37). □

Consider

$$\begin{aligned} L' := L^0|_{\widetilde{R}} &\equiv c_0\mathfrak{f} + c_9\mathfrak{s} + c_{10}\mathfrak{f} + \sum_{i=2}^{4} c_i(\mathfrak{s} - \mathfrak{e}_i) + \sum_{i=5}^{8} c_i\mathfrak{f} \\ &= (c_2 + c_3 + c_4 + c_9)\mathfrak{s} + (c_0 + c_5 + c_6 + c_7 + c_8 + c_{10})\mathfrak{f} - \sum_{i=2}^{4} c_i\mathfrak{e}_i \\ &= L'_0 + \sum_{i=2}^{\kappa+1} (\mathfrak{f} - \mathfrak{e}_i), \end{aligned}$$

where

$$L'_0 := (c_2 + c_3 + c_4 + c_9)\mathfrak{s} + (c_0 + c_5 + c_6 + c_7 + c_8 + c_{10} - \kappa)\mathfrak{f} - \sum_{i=2}^{\kappa+1} (c_i - 1)\mathfrak{e}_i$$

and $\sum_{i=2}^{\kappa+1} (\mathfrak{f} - \mathfrak{e}_i)$ is the sum of κ disjoint (-1)-curves. We note that we may consider L' as a line bundle on the blowup of R at κ points. Hence, we will eventually apply Proposition 4.2 with $k = s = \kappa$.

Claim 7.2. L'_0 verifies condition $(\star)$ and is odd.

Proof. Oddness is equivalent to condition (34). Conditions (i)–(iii) of $(\star)$ are easily checked. Condition (iv) is equivalent to

$$2c_0 + c_9 + 2c_{10} + 2\sum_{i=5}^{8} c_i - \kappa \geq 4. \tag{38}$$

If $c_0 = 0$, then $c_9 = c_{10} = 0$ by (33) and $(\kappa, \lambda) = (3, 4)$ by Claim 7.1(i), whence the left-hand side of (38) equals $2\sum_{i=5}^{8} c_i - \kappa \geq 8 - 3 = 5$, and we are done. Hence, we may assume that $c_0 > 0$ for the rest of the proof.

We note that, by (33),

$$2c_0 + c_9 + 2c_{10} + 2\sum_{i=5}^{8} c_i - \kappa \geq 3c_0 + c_{10} + 2\lambda - \kappa \geq 3 + c_{10} + 2\lambda - \kappa.$$

This, together with Claim 7.1(ii) tells us that (38) is always satisfied if $\lambda \geq 1$. Assume, therefore, that $\lambda = 0$. Then $\kappa \leq 1$, by Claim 7.1(ii). If $\kappa = 1$, then $c_{10} \geq 2$, by Claim 7.1(ii), and (38) is again satisfied. If $\kappa = 0$, we have that c_9 is odd by (34). If $c_{10} > 0$, (38) is satisfied. Otherwise, Claim 7.1(iii) yields $c_0 \geq 2$, whence (38) is again satisfied. □

Consider

$$\begin{aligned} L'' := L^0|_{\widetilde{P}} &\sim c_0(\ell - \mathfrak{e}_9) + c_9\mathfrak{e}_9 + c_{10}(2\ell - \mathfrak{e}_5 - \mathfrak{e}_6 - \mathfrak{e}_7 - \mathfrak{e}_8) + \sum_{i=5}^{8} c_i(\ell - \mathfrak{e}_i) \\ &= (c_0 - c_9)(\ell - \mathfrak{e}_9) + c_9\ell + c_{10}(2\ell - \mathfrak{e}_5 - \mathfrak{e}_6 - \mathfrak{e}_7 - \mathfrak{e}_8) + \sum_{i=5}^{8} c_i(\ell - \mathfrak{e}_i). \end{aligned}$$

The idea is now to apply Proposition 4.2 with $k = \kappa$.

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7.4.1. Subcase $\lambda = 3, 4$

We have $c_5 \geq c_6 \geq c_7 > 0$, by (32). Define

$$\begin{aligned} L_0''(3) &:= (c_0 - c_9)(\ell - \mathfrak{e}_9) + c_9\ell + c_{10}(2\ell - \mathfrak{e}_5 - \mathfrak{e}_6 - \mathfrak{e}_7 - \mathfrak{e}_8) + \\ &\quad + \sum_{i=5}^{7}(c_i - 1)(\ell - \mathfrak{e}_i) + c_8(\ell - \mathfrak{e}_8) + (\ell - \mathfrak{e}_5 - \mathfrak{e}_6) + (\ell - \mathfrak{e}_6 - \mathfrak{e}_7) + (\ell - \mathfrak{e}_7 - \mathfrak{e}_8), \\ L_0''(2) &= L_0''(3) + \mathfrak{e}_6, \\ L_0''(1) &= L_0''(3) + \mathfrak{e}_6 + \mathfrak{e}_7, \\ L_0''(0) &= L_0''(3) + \mathfrak{e}_6 + \mathfrak{e}_7 + \mathfrak{e}_8. \end{aligned} \tag{39}$$

Then one may check that

$$L'' = L_0''(\kappa) + \begin{cases} 0, & \text{if } \kappa = 0, \\ \mathfrak{e}_8, & \text{if } \kappa = 1, \\ \mathfrak{e}_7 + \mathfrak{e}_8, & \text{if } \kappa = 2, \\ \mathfrak{e}_6 + \mathfrak{e}_7 + \mathfrak{e}_8, & \text{if } \kappa = 3. \end{cases}$$

Claim 7.3. $L_0''(\kappa)$ is big and nef for all $\kappa \in \{0, 1, 2, 3\}$.

Proof. Since $\mathfrak{e}_i \cdot L_0''(3) > 0$, for $i \in \{6, 7, 8\}$, it suffices to verify that $L_0''(3)$ is big and nef. All divisors in the sum (39) are nef, except for the last three, which are irreducible. Nefness follows if the latter three intersect $L_0''(3)$ nonnegatively. We have

$$\begin{aligned} L_0''(3) \cdot (\ell - \mathfrak{e}_5 - \mathfrak{e}_6) &= c_0 + (c_7 - 1) + c_8 - 1 + 0 + 1 \geq 0, \\ L_0''(3) \cdot (\ell - \mathfrak{e}_7 - \mathfrak{e}_8) &= c_0 + (c_5 - 1) + (c_6 - 1) + 1 + 0 - 1 \geq 0. \end{aligned}$$

Finally

$$L_0''(3) \cdot (\ell - \mathfrak{e}_6 - \mathfrak{e}_7) = c_0 + (c_5 - 1) + c_8 + 0 - 1 + 0 \geq c_0 + c_8 - 1,$$

which is nonnegative, since, by Claim 7.1(i), either $c_0 > 0$, or $\lambda = 4$ (whence $c_8 > 0$). This proves nefness. Bigness is easily checked. □

We apply Proposition 4.2 with $k = s = \kappa$, $t = 5$ and $L_0'' = L_0(\kappa)$. What is left to be checked is condition (v). The set of additional $t - k = 5 - \kappa$ disjoint (-1)-curves on $\widetilde{P}$ is

$$\begin{cases} \mathfrak{e}_5, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8, \mathfrak{e}_9, & \text{if } \kappa = 0, \\ \mathfrak{e}_5, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_9, & \text{if } \kappa = 1, \\ \mathfrak{e}_5, \mathfrak{e}_6, \mathfrak{e}_9 & \text{if } \kappa = 2, \\ \mathfrak{e}_5, \mathfrak{e}_9 & \text{if } \kappa = 3, \end{cases}$$

and $\mathfrak{e}_5 \cdot L''(\kappa) = c_{10} + c_5 \geq c_5 > 0$, as $\lambda > 0$, verifying condition (v) in Proposition 4.2.

7.4.2. Subcase $\lambda \leq 2$

We have $c_0 > 0$, $c_7 = c_8 = 0$, and $\kappa \leq 1$, by Claim 7.1(i)–(ii).

If $\kappa = 0$, we apply Proposition 4.2 with $k = s = 0$ and $t = 5$. Condition (v) therein is satisfied, as for instance, $\mathfrak{e}_5 \cdot L'' = c_{10} + c_5$ and $\mathfrak{e}_9 \cdot L'' = c_0 - c_9$; indeed, if $\mathfrak{e}_5 \cdot L'' = 0$, then $c_{10} = c_5 = 0$, whence $\lambda = 0$, so Claim 7.1(iii) yields $\mathfrak{e}_9 \cdot L'' > 0$.

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If $\kappa = 1$, then $c_{10} \geq 2$, by Claim 7.1(ii). Write $L'' = L''_0 + \mathfrak{e}_9$, with

$$L''_0 := (c_0 - c_9)(\ell - \mathfrak{e}_9) + c_9\ell + (c_{10} - 1)(2\ell - \mathfrak{e}_5 - \mathfrak{e}_6 - \mathfrak{e}_7 - \mathfrak{e}_8) + \sum_{i=5}^{6} c_i(\ell - \mathfrak{e}_i) + (2\ell - \mathfrak{e}_5 - \mathfrak{e}_6 - \mathfrak{e}_7 - \mathfrak{e}_8 - \mathfrak{e}_9),$$

which is big and nef, since the only term with negative square is the last one, and one checks that $(2\ell - \mathfrak{e}_5 - \mathfrak{e}_6 - \mathfrak{e}_7 - \mathfrak{e}_8 - \mathfrak{e}_9) \cdot L''_0 = c_0 + c_9 + c_5 + c_6 - 1 \geq 0$. We apply Proposition 4.2 with $k = s = 1$ and $t = 5$. Condition (v), therein, is satisfied, as for instance, $\mathfrak{e}_5 \cdot L'' = c_{10} + c_5 \geq c_{10} \geq 2$.

7.5. *Case where $a_0 > 0$ is odd and all remaining a_i are even (case (ii) in Lemma 5.8)*

Since a_9, a_{10} are even and a_0 is odd, we have

$$a_9 + a_{10} > a_0 > a_9 \geq a_{10},$$

which implies $a_0 \geq 3$ and $a_9, a_{10} \geq 2$. Rearranging indices, we have a limit line bundle

$$L^0 \equiv c_0 E^0_{5,6} + \sum_{i=1}^{8} c_i E^0_i + c_{10} E^0_{10},$$

with the E^0_i and $E^0_{5,6}$, as in Example 6.1, where

$$c_5 + c_6 > c_0 > c_5 \geq c_6 \geq 2, \quad c_0 \text{ is odd}, c_5, c_6 \text{ are even}, \tag{40}$$

$$c_4 \leq c_3 \leq c_2 \leq c_1 \leq c_7 \leq c_8 \leq c_{10}, \quad \text{all even}. \tag{41}$$

We define

$$\kappa := \sharp\{j \in \{1,2,3,4\} \mid c_j > 0\}.$$

Consider

$$\begin{aligned} L' := L^0|_{\widetilde{R}} &\equiv c_0\mathfrak{s} + \sum_{i=1}^{4} c_i(\mathfrak{s} - \mathfrak{e}_i) + \sum_{i=5}^{8} c_i\mathfrak{f} + c_{10}\mathfrak{f} \\ &= (c_0 + c_1 + c_2 + c_3 + c_4)\mathfrak{s} + (c_5 + c_6 + c_7 + c_8 + c_{10})\mathfrak{f} - \sum_{i=1}^{\kappa} c_i\mathfrak{e}_i \\ &= L'_0 + \sum_{i=2}^{\kappa}(\mathfrak{f} - \mathfrak{e}_i), \end{aligned}$$

where

$$L'_0 := (c_0 + c_1 + c_2 + c_3 + c_4)\mathfrak{s} + (c_5 + c_6 + c_7 + c_8 + c_{10} - \kappa + 1)\mathfrak{f} - c_1\mathfrak{e}_1 - \sum_{i=2}^{\kappa}(c_i - 1)\mathfrak{e}_i$$

and $\sum_{i=2}^{\kappa}(\mathfrak{f} - \mathfrak{e}_i)$ is the sum of $\max\{0, \kappa - 1\}$ disjoint (-1)-curves. We note that we may consider L' as a line bundle on the blowup of R at κ points. Hence, we will eventually apply Proposition 4.2 with $s = \kappa$ and $k = \max\{0, \kappa - 1\}$.

Claim 7.4. L'_0 verifies condition $(\star)$ and is odd.

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Proof. Oddness follows since $c_0 + c_1 + c_2 + c_3 + c_4$ is odd by our assumptions (40) and (41). Conditions (i)–(iii) of ($\star$) are easily checked. Condition (iv) is equivalent to

$$c_0 + 2(c_5 + c_6 + c_7 + c_8 + c_{10}) \geq \kappa + 2.$$

This is verified since, by (40), the left-hand side is $\geq c_0 + 2c_5 + 2c_6 \geq 3 + 4 + 4 = 11$. □

We have

$$L'' := L^0|_{\widetilde{P}} \sim c_0(\ell - \mathfrak{e}_7 - \mathfrak{e}_8) + \sum_{i=5}^{8} c_i(\ell - \mathfrak{e}_i) + c_{10}(2\ell - \mathfrak{e}_5 - \mathfrak{e}_6 - \mathfrak{e}_7 - \mathfrak{e}_8).$$

We can view L'' as a line bundle on $\mathrm{Bl}_{y_5,y_6,y_7,y_8}(P)$. The idea is now to apply Proposition 4.2 with $k = \max\{0, \kappa - 1\}$, $s = \kappa$ and $t = 4$.

7.5.1. Subcase $c_7 = 0$

By (41), we have $c_1 = c_2 = c_3 = c_4 = 0$, whence $\kappa = 0$. We apply Proposition 4.2 with $s = k = 0$ and $t = 4$. Condition (v), therein, is satisfied, as for instance, $\mathfrak{e}_5 \cdot L'' = c_5 + c_{10} \geq c_5 > 0$ by (40).

7.5.2. Subcase $c_7 > 0$

By (41), we have $c_7, c_8, c_{10} \geq 2$.

Define

$$\begin{aligned} L_0''(3) &:= c_0(\ell - \mathfrak{e}_7 - \mathfrak{e}_8) + c_5(\ell - \mathfrak{e}_5) + \sum_{i=6}^{8}(c_i - 1)(\ell - \mathfrak{e}_i) + \\ &\quad + c_{10}(2\ell - \mathfrak{e}_5 - \mathfrak{e}_6 - \mathfrak{e}_7 - \mathfrak{e}_8) + (\ell - \mathfrak{e}_6 - \mathfrak{e}_7) + (\ell - \mathfrak{e}_7 - \mathfrak{e}_5) + (\ell - \mathfrak{e}_8 - \mathfrak{e}_6), \\ L_0''(2) &= L''(3) + \mathfrak{e}_5, \\ L_0''(1) &= L''(3) + \mathfrak{e}_5 + \mathfrak{e}_6, \\ L_0''(0) &= L''(3) + \mathfrak{e}_5 + \mathfrak{e}_6 + \mathfrak{e}_7. \end{aligned} \tag{42}$$

Then one may check that, for $j \in \{0, 1, 2, 3\}$:

$$L'' = L_0''(j) + \begin{cases} 0, & \text{if } j = 0, \\ \mathfrak{e}_7, & \text{if } j = 1, \\ \mathfrak{e}_6 + \mathfrak{e}_7, & \text{if } j = 2, \\ \mathfrak{e}_5 + \mathfrak{e}_6 + \mathfrak{e}_7, & \text{if } j = 3. \end{cases}$$

Claim 7.5. $L_0''(j)$ is big and nef for all $j \in \{0, 1, 2, 3\}$.

Proof. Since $\mathfrak{e}_i \cdot L_0''(3) > 0$, for $i \in \{5, 6, 7\}$, it suffices to verify that $L_0''(3)$ is big and nef. All divisors in the sum (42) are of nonnegative square, except for the first and the last three. We have, using (40) and the fact that $c_5, c_6, c_7, c_8, c_{10} \geq 2$:

$$\begin{aligned} L_0''(3) \cdot (\ell - \mathfrak{e}_7 - \mathfrak{e}_8) &= -c_0 + c_5 + (c_6 - 1) \geq 0, \\ L_0''(3) \cdot (\ell - \mathfrak{e}_6 - \mathfrak{e}_7) &= c_5 + (c_8 - 1) - 1 \geq 2 + 1 - 1 = 2, \\ L_0''(3) \cdot (\ell - \mathfrak{e}_7 - \mathfrak{e}_5) &= (c_6 - 1) + (c_8 - 1) - 1 + 1 \geq 1 + 1 + 0 \geq 2, \\ L_0''(3) \cdot (\ell - \mathfrak{e}_8 - \mathfrak{e}_6) &= c_5 + (c_7 - 1) + 1 - 1 \geq 2 + 1 + 0 = 3, \end{aligned}$$

which proves that $L_0''(3)$ is nef. It is easily verified that it is big. □

Now we apply Proposition 4.2 with $k = \max\{0, \kappa-1\} \le 3$, $s = \kappa$, $t = 4$ and $L_0'' = L_0(k)$. What is left to be checked is condition (v). This is satisfied because $\mathfrak{e}_8 \cdot L_0''(k) = c_0+c_8-1+c_{10}+1 = c_0+c_8+c_{10} > 0$.

7.6. *Case where* $a_0 > 0$ *and all remaining* a_i *are odd (case (iii) in Lemma 5.8)*

Rearranging indices, we have a limit line bundle

$$L^0 \equiv c_0 E^0_{9,10} + c_9 E^0_9 + c_{10} E^0_{10} + \sum_{i=1}^{7} c_i E^0_i,$$

with the E^0_i and $E^0_{9,10}$, as in Example 6.1, where

$$c_9 + c_{10} \ge c_0 \ge c_9 \ge c_{10} > 0, \quad c_9, c_{10} \ \text{ odd}, \tag{43}$$

$$0 < c_1 \le c_2 \le \cdots \le c_6 \le c_7, \quad \text{all odd}. \tag{44}$$

Consider

$$\begin{aligned} L' := L^0|_{\widetilde{R}} &\equiv c_0\mathfrak{f} + c_9\mathfrak{s} + c_{10}\mathfrak{f} + \sum_{i=1}^{4} c_i(\mathfrak{s} - \mathfrak{e}_i) + \sum_{i=5}^{7} c_i\mathfrak{f} \\ &= (c_1 + c_2 + c_3 + c_4 + c_9)\mathfrak{s} + (c_0 + c_5 + c_6 + c_7 + c_{10})\mathfrak{f} - \sum_{i=1}^{4} c_i\mathfrak{e}_i \\ &= L_0' + \sum_{i=2}^{4} (\mathfrak{f} - \mathfrak{e}_i), \end{aligned}$$

where

$$L_0' := (c_1 + c_2 + c_3 + c_4 + c_9)\mathfrak{s} + (c_0 + c_5 + c_6 + c_7 + c_{10} - 3)\mathfrak{f} - c_1\mathfrak{e}_1 - \sum_{i=2}^{4} (c_i - 1)\mathfrak{e}_i$$

and $\sum_{i=2}^{4} (\mathfrak{f} - \mathfrak{e}_i)$ is the sum of three disjoint (-1)-curves. We will eventually apply Proposition 4.3.

Claim 7.6. L_0' verifies condition $(\star)$ and is odd.

Proof. Oddness follows since $c_9 + c_1 + c_2 + c_3 + c_4$ is odd by our assumptions (43)–(44). Conditions (i)–(iv) of $(\star)$ easily follow from properties (43)–(44). □

We have

$$\begin{aligned} L'' := L^0|_{\widetilde{P}} &\sim c_0(\ell - \mathfrak{e}_9) + c_9\mathfrak{e}_9 + c_{10}(2\ell - \mathfrak{e}_5 - \mathfrak{e}_6 - \mathfrak{e}_7 - \mathfrak{e}_8) + \sum_{i=5}^{7} c_i(\ell - \mathfrak{e}_i) \\ &= L_0'' + \mathfrak{e}_6 + \mathfrak{e}_7 + \mathfrak{e}_8, \end{aligned}$$

with

$$\begin{aligned} L_0'' := &(c_0 - c_9)(\ell - \mathfrak{e}_9) + c_9\ell + c_{10}(2\ell - \mathfrak{e}_5 - \mathfrak{e}_6 - \mathfrak{e}_7 - \mathfrak{e}_8) + \sum_{i=5}^{7} (c_i - 1)(\ell - \mathfrak{e}_i) + \\ &+ (\ell - \mathfrak{e}_5 - \mathfrak{e}_6) + (\ell - \mathfrak{e}_6 - \mathfrak{e}_7) + (\ell - \mathfrak{e}_7 - \mathfrak{e}_8). \end{aligned}$$

Claim 7.7. L_0'' is big and nef.

Proof. All terms in the expression of L_0'' right above have nonnegative square, except for the last three. One computes

$$L_0'' \cdot (\ell - \mathfrak{e}_5 - \mathfrak{e}_6) = c_0 + (c_7 - 1) - 1 + 0 + 1 \geq c_0 > 0,$$
$$L_0'' \cdot (\ell - \mathfrak{e}_6 - \mathfrak{e}_7) = c_0 + (c_5 - 1) - 1 \geq 0,$$
$$L_0'' \cdot (\ell - \mathfrak{e}_7 - \mathfrak{e}_8) = c_0 + (c_5 - 1) + (c_6 - 1) + 1 - 1 \geq c_0 > 0,$$

which shows that L_0'' is nef. One easily computes that it is big. □

We apply Proposition 4.3. What is left to be checked is condition (iv): The additional disjoint (-1)-curves on $\widetilde{P}$ are $\mathfrak{e}_5$ and $\mathfrak{e}_9$, and we have

$$\mathfrak{e}_5 \cdot L_0'' = c_{10} + c_5 > c_{10} \geq c_0 - c_9 = \mathfrak{e}_9 \cdot L_0'',$$

by (43) and (44).

This concludes the proof of Theorem 1.1.

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