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THE π-FULL TIGHT RIESZ ORDERS ON $A(\Omega)$

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Let G be a lattice-ordered group (l-group), and let $t, u \in G^+$. We write $t \,\pi\, u$ if $t \wedge g = 1$ is equivalent to $u \wedge g = 1$, and say that a tight Riesz order T on G is π-full if $t \in T$ and $t \,\pi\, u$ imply $u \in T$. We study the set $\mathcal{T}_\pi$ of π-full tight Riesz orders on an l-permutation group (G, Ω), Ω a totally ordered set. In this description of results we assume that (G, Ω) is transitive. If Ω is discrete, we find that the l-group $A(\Omega)$ of all automorphisms of Ω has no π-full tight Riesz orders. Suppose now that Ω is dense, and let G be $A(\Omega)$ or any laterally complete l-subgroup of $A(\Omega)$. Then G has π-full tight Riesz orders, all of which are compatible, and all maximal tight Riesz orders on G are π-full. The π-full tight Riesz orders are precisely the sets consisting of G^+ with some non-empty normal set of minimal prime subgroups deleted, so that any maximal tight Riesz order must consist of G^+ with all conjugates of some minimal prime subgroup deleted. We obtain a one-to-one correspondence between $\mathcal{T}_\pi$ and the set $\mathcal{L}_\pi$ of all π-full l-ideals ($\neq G$) of G. Thus $A(\mathbb{R})$, $\mathbb{R}$ the real numbers, has precisely four π-full (compatible) tight Riesz orders, two of which are maximal. This investigation of tight Riesz orders sheds new light on minimal prime subgroups of laterally complete l-groups.

§1. Definitions. *We say that $g, k \in G^+ = \{g \in G \mid g \geq 1\}$ are* disjoint *(written $g \perp k$) if $g \wedge k = 1$. For any $S \subseteq G$, $S^\perp$ denotes the polar $\{g \in G : |g| \perp |s|$ for all $s \in S\}$, a convex l-subgroup of G. When $g, h \in G^+$ and $g^\perp = h^\perp$, we write $g \,\pi\, h$. (A π-class is sometimes known as a* filet *or* carrier *of G.) We shall say that a subset X of G^+ is* π-full *if $x \in X$ and $x \,\pi\, y$ imply $y \in X$; and that a convex l-subgroup C of G is* π-full *if C^+ is* π-full.

For any totally ordered set Ω, the group $A(\Omega)$ of order-automorphisms of Ω becomes an l-group under the pointwise order, whereby $f \leq g$ iff $\omega f \leq \omega g$ for all $\omega \in \Omega$. Of special interest are $\Omega = \mathbb{Z}$ (the integers) and $\Omega = \mathbb{R}$ (the real numbers). When G is an l-subgroup of $A(\Omega)$, i.e. a subgroup which is also a sublattice, (G, Ω) is called an *l-permutation group. Support*(g) means $\{\omega \in \Omega \mid \omega g \neq \omega\}$, and when a subset Σ of Ω contains support(g), we say that Σ *supports* g.

When (G, Ω) is an l-permutation group, $|g| \perp |k|$ iff support$(g) \cap$

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support$(k)=\varnothing$. If every non-singleton interval of Ω supports some $1\neq f\in G$, then $g\,\pi\,h$ if and only if the supports of g and h have the same topological closures (with respect to the order topology on Ω). The same description of π holds for $A(\Omega)$ whenever Ω is homogeneous, but as we shall not use this result, we omit the proof. However, the description is easily seen to fail for $A(\Omega)$ when Ω is $\mathbb{R}$ with each irrational replaced by a three point chain.

Let W denote $\{g\in G^+ \mid g^\perp=\{1\}\}$, the set of *weak units* of G. No π-full convex l-subgroup of G can meet W except for G itself.

If $W\nsubseteq\varnothing$, then W is a π-full normal filter on G^+, and is a tight Riesz order provided it is factorable. (For this and various subsequent statements, we assume that G contains more than one element.)

We recall now some well known facts. Let $g,h\in G^+$. Then $g^{\perp\perp}\cap h^{\perp\perp}=(g\wedge h)^{\perp\perp}$ [3, p. 2.5]. Hence $(g\wedge h)^\perp=g^\perp\cup h^\perp$, where $\cup$ denotes supremum in the lattice $\{g^\perp \mid g\in G^+\}$; for $g^\perp\cup h^\perp=(g^{\perp\perp}\cap h^{\perp\perp})^\perp=(g\wedge h)^{\perp\perp\perp}=(g\wedge h)^\perp$. Since also $(g\vee h)^\perp=g^\perp\cap h^\perp$, *$\pi$ is a lattice congruence on G^+*, i.e. $g_1\,\pi\,g_2$ and $h_1\,\pi\,h_2$ imply $(g_1\wedge h_1)\,\pi\,(g_2\wedge h_2)$ and dually; and also *they imply* $(g_1h_1)\,\pi\,(g_2h_2)$ because $(gh)\,\pi\,(g\vee h)$. Moreover, *$g^\perp\subseteq h^\perp$ implies $(g\wedge h)\,\pi\,h$* and $(g\vee h)\,\pi\,g$, and in particular if $w\in W$, $(w\wedge h)\,\pi\,h$ *and* $w\vee h\in W$; for $(g\wedge h)^\perp=(g^{\perp\perp}\cap h^{\perp\perp})^\perp=h^{\perp\perp\perp}=h^\perp$ and $(g\vee h)^\perp=g^\perp\cap h^\perp=g^\perp$.

A *tight Riesz order* on an l-group G is a *proper filter* T of G^+ (i.e. $g\geq t\in T$ implies $g\in T$ and $s,t\in T$ implies $s\wedge t\in T$, and $1\notin T$) which is *normal* (i.e. $T^g=T$ for all $g\in G$) and *factorable* (i.e. $T\subseteq T^2$, which forces $T=T^2$ since T is a filter). T is *compatible* if in addition inf $T=1$. The collection of tight Riesz orders on G will be ordered by inclusion, as will the collection of convex l-subgroups of G.

We mention here one example of a π-full tight Riesz order, due independently to Davis and Fox [5] and to Ball [2]: $W=\{g\in A^+(\mathbb{R}) \mid \text{support}(g)$ is dense in $\mathbb{R}\}$, the set of weak units of $A(\mathbb{R})$.

A *prime subgroup* of G is a convex l-subgroup P of G for which $g\wedge h=1$ implies $g\in P$ or $h\in P$. The *maximal tangents* of a tight Riesz order T are the convex l-subgroups of G maximal with respect to not meeting T, and they constitute a normal set of primes. We denote the set of maximal tangents of T by Max(T). Reilly [16] has shown that $T=G^+\backslash\cup\text{Max}(T)$. Thus each tight Riesz order T is obtained by deleting from G^+ a normal set of prime subgroups of G. Conversely, deletion from G^+ of a non-empty normal set of primes yields a normal proper filter on G^+ which is a tight Riesz order provided it is factorable. Factorability is generally the main difficulty in producing tight Riesz orders, and often this difficulty is overcome by assuming that G is divisible, which makes factorability obvious. However, $A(\Omega)$ is not always divisible.

Of great importance will be the *radical* Rad$(T)=\bigcap\text{Max}(T)$ of a tight Riesz order T. Rad(T) is an l-ideal of G. Ball [2] also treats Rad(T), which he writes as $M(T)$.

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PROPOSITION 1.1. *Let G be any l-group, and let $T \in \mathcal{T}_\pi$. Then each maximal tangent of T is π-full, and* Rad(T) *is π-full.*

Proof. Let M be a maximal tangent of T. Then $(M^+)\pi$, meaning $\{g \in G^+ \mid g\,\pi\,m$ for some $m \in M^+\}$, is a convex subsemigroup of G^+ containing 1. (If $h \leq g\,\pi\,m$, then $h\,\pi\,(g \vee m)$.) Hence the subgroup P of G generated by $(M^+)\pi$ is a convex l-subgroup of G which has $(M^+)\pi$ as its set of positive elements and thus is π-full. Since $M \cap T = \varnothing$ and T is π-full, $P \cap T = \varnothing$. Hence $M = P$, so that M is π-full. Since the maximal tangents of T are π-full, their intersection Rad(T) is also π-full.

Thus every π-full tight Riesz order is obtained by deleting from G^+ a non-empty normal set of π-full primes; and conversely given factorability. We shall find conditions under which every π-full subset of G^+ is factorable.

§2. Sectional quasi-pseudo-complementarity and minimal prime subgroups. We shall make crucial use of the following lattice-theoretic notion. An l-group G is *quasi-pseudo-complemented* (QPC) if $W \neq \varnothing$ and if for any $g \in G^+$, there exists $g^* \in G^+$ such that $g \wedge g^* = 1$ and $g \vee g^* \in W$. More generally, G is *sectionally quasi-pseudo-complemented* (SQPC) if for any $1 \leq g \leq h$, there exists $g^* \in G^+$ such that $g^* \leq h$, $g \wedge g^* = 1$, and $(g \vee g^*)\,\pi\,h$. This definition is unchanged by omission of the requirement that $g^* \leq h$; for g^* can be replaced by $g^* \wedge h$, and $g \vee (g^* \wedge h) = (g \vee g^*) \wedge (g \vee h) = ((g \vee g^*) \wedge h)\,\pi\,h$. A similar argument shows that if G is QPC, it is also SQPC; and the converse holds when $W \neq \varnothing$. Thus SQPC is just a generalization of QPC to the case in which G lacks weak units. Observe that if G is SQPC and F is any π-full filter, then for any $g \in G^+$, there exists $g^* \in G^+$ such that $g \wedge g^* = 1$ and $g \vee g^* \in F$.

The notion of SQPC has been treated by Spirason and Strzelecki [17] for vector lattices, by Keimel [13] for a number of algebraic systems, by Cornish [4] for lattices, and by Davis and Fox [5] for l-groups. A few of our results (2.3, 2.4, 2.5) can be found in these sources, though for convenience we prove them again here.

G is said to be *laterally complete* if each pairwise disjoint subset of G has a supremum; and to be *conditionally laterally complete* if each pairwise disjoint subset of G which is bounded above has a supremum. For any Ω, $A(\Omega)$ is laterally complete, and its l-ideal $B(\Omega) = \{g \in A(\Omega) \mid \text{support}(g) \text{ is bounded}\}$ is conditionally laterally complete.

Rick Ball and Colin Fox have independently observed that laterally complete l-groups are QPC. We prove here a slightly more general result.

PROPOSITION 2.1. *If G is laterally complete (resp., conditionally laterally complete), then G is QPC (resp., SQPC).*

Proof. We prove first that if G is conditionally laterally complete, then G is SQPC. Let $h \geq g \in G^+$. Let $\{g_i\}$ be maximal with respect to being a pairwise

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disjoint subset of G such that for each i, $g_i \perp g$ and $g_i \le h$. Since $\{g_i\}$ has h as an upper bound, $\sup\{g_i\}$ exists, and we denote it by g^*. Then $g \wedge g^* = 1$, for if not, then since $g_i \perp g$ for each i, $g^* g^{-1} \vee 1$ would be an upper bound of $\{g_i\}$ strictly smaller than g^*. Also $(g \vee g^*)\ \pi\ h$, for $g \vee g^* \le h$ and if there existed $k \in G$ such that $k \perp (g \vee g^*)$ but not $k \perp h$, then $k \wedge h$ would violate the maximality of $\{g_i\}$. This proves the proposition for conditionally laterally complete l-groups. For laterally complete l-groups, the proposition is established once it is shown that $W \neq \varnothing$, and that follows from an argument like the one that just produced g^*.

The minimal prime subgroups of G will play a crucial role in the study of $\mathcal{T}_\pi$. Since the minimal primes have $\{1\}$ as their intersection, G is never a minimal prime subgroup of itself (since we are assuming $G \neq \{1\}$.) We shall frequently use the well known fact that in any l-group G, *a prime P is minimal if and only if for each $p \in P^+$, there exists $g \in G^+ \backslash P$ such that $p \wedge g = 1$* [3, p. 2.13]. From this it is immediate that every minimal prime P is a union of polars, namely $P = \bigcup\{g^\perp \mid g \in G^+ \backslash P\}$. Thus in any l-group, *all minimal primes are π-full.* We prove now that the converse holds precisely when G is SQPC.

LEMMA 2.2 *Let C be a π-full convex l-subgroup of an l-group G, and let $1 < g \in G$. Then the π-full convex l-subgroup D of G generated by C and g has $D^+ = \{f \in G^+ \mid f \le f'\ \pi\ (c \vee g)$ for some $f' \in G^+,\ c \in C^+\}$.*

Proof. Clearly the set which we claim is D^+ is convex and contains 1. Moreover, it is a semigroup, for if $f_1'\ \pi\ (c_1 \vee g)$ and $f_2'\ \pi\ (c_2 \vee g)$, then $(f_1' f_2')\ \pi\ (f_1' \vee f_2')\ \pi\ ((c_1 \vee g) \vee (c_2 \vee g)) = (c_1 \vee c_2) \vee g$. Therefore the subgroup D generated by this set is a convex l-subgroup which has precisely this set as its positive set D^+. D^+ is π-full because if $h\ \pi\ f \le f'\ \pi\ (c \vee g)$, then $h \le (h \vee f')\ \pi\ (f \vee f') = f'\ \pi\ (c \vee g)$; i.e. D is π-full.

LEMMA 2.3. *Let G be any l-group, and let $1 < k \in G$. Let C be maximal among the π-full convex l-subgroups of G which do not contain k. Then C is prime.*

Proof. Suppose by way of contradiction that $g \wedge h = 1$ for some $g, h \in C$. Then the π-full convex l-subgroup generated by C and g contains k, so by Lemma 2.2, $k \le k_1'\ \pi\ (c_1 \vee g)$, and similarly $k \le k_2'\ \pi\ (c_2 \vee h)$, where $k_i' \in G^+$ and $c_i \in C$. Hence $k \le (k_1' \wedge k_2')\ \pi\ ((c_1 \vee g) \wedge (c_2 \vee h)) = (c_1 \wedge c_2) \vee (c_1 \wedge h) \vee (g \wedge c_2) \vee (g \wedge h) \in C$ since $g \wedge h = 1$, a contradiction.

COROLLARY 2.4. *Let C be a convex l-subgroup of an l-group G. Then C is π-full if and only if C is the intersection of a collection of π-full prime subgroups of G.*

THEOREM 2.5. *Let G be an l-group. Then the following are equivalent:*
(1) *G is* SQPC.

(2) *A prime subgroup $P \neq G$ is π-full if and only if it is a minimal prime subgroup.*

(3) *Every π-full convex l-subgroup $C \neq G$ is the intersection of the minimal primes containing it.*

Proof. Supose that G is SQPC. Let $P \neq G$ be a π-full prime, and suppose that M is another prime such that $M \subset P$. Pick $p \in P^+ \backslash M$ and pick $p^* \in G^+$ such that $p \wedge p^* = 1$ and $p \vee p^* \in G^+ \backslash P$ (possible since $G^+ \backslash P$ is a π-full filter.) Since $p \wedge p^* = 1$, $p^* \in M \subset P$—a contradiction.

Conversely, suppose that every π-full prime is a minimal prime. Let $1 \leq g \leq h$. The polar $g^\perp$ is a π-full convex l-subgroup of G. By Lemma 2.2, the π-full convex l-subgroup D of G generated by $g^\perp$ and g has as its positive elements precisely $\{f \in G^+ \mid f \leq f' \ \pi(c \vee g)$ for some $f' \in G^+$, $c \in (g^\perp)^\perp\}$. Then $h \in D$, for otherwise D could be enlarged to a π-full convex l-subgroup D' maximal with respect to not containing h; and then by Lemma 2.3, D' would be prime, but D' could not be a minimal prime because $g \in D'$ and $g^\perp \subseteq D'$. Since $h \in D$, $h \leq h' \ \pi \ (c \vee g)$ for some $h' \in G^+$, $c \in (g^\perp)^+$. Hence $h = (h \wedge h') \ \pi \ (h \wedge (c \vee g)) = (h \wedge c) \vee (h \wedge g) = (h \wedge c) \vee g$. Thus $((h \wedge c) \vee g) \ \pi \ h$ and $(h \wedge c) \wedge g \leq c \wedge g = 1$, i.e. $h \wedge c$ serves as g^*.

We have shown the equivalence of (1) and (2). By Corollary 2.4, (2) implies (3). Conversely, if $P \neq G$ is a π-full prime, then by (3), P is the intersection of the minimal primes containing it, each of which must be P since P is itself prime. Hence P is a minimal prime.

Let H be a convex l-subgroup of an l-group G. Let Min(H) be the set of minimal prime subgroups of G which contain H, and let $F(H) = G^+ \backslash \cup \operatorname{Min}(H)$. If H is an l-ideal of G and if $\operatorname{Min}(H) \neq \varnothing$, then $F(H)$ is a normal proper filter which is a tight Riesz order provided it is factorable. We denote by $\mathscr{L}_\pi$ the set of π-full l-ideals $H(\neq G)$ of G. If G is SQPC and $H \in \mathscr{L}_\pi$, then $\bigcap \operatorname{Min}(H) = H$ by Theorem 2.5, and $F(H)$ is a π-full tight Riesz order provided it is factorable.

Ball introduces a related notion in [2]. He deletes from G^+ the union of those primes which are minimal among the primes containing H, and calls the remainder $U(H)$. He shows that when $H \neq G$, $U(H) = \{x \in G^+ \mid g \in G^+$ and $x \wedge y \in H$ imply $y \in H\}$. We prove now that for π-full convex l-subgroups H in SQPC l-groups, $U(H) = F(H)$.

LEMMA 2.6. *Let H be a π-full convex l-subgroup of an l-group G. Then every P minimal among the prime subgroups of G which contain H is π-full. Hence $U(H)$ is π-full.*

Proof. Let P be minimal among the prime subgroups of G which contain H. Lemma 7 of [2] says that for every $p \in P^+$, there exists $x \in G^+ \backslash P$ such that $x \wedge p \in M$. Now let $g \ \pi \ p \in P^+$, and pick $x \in G^+ \backslash P$ such that $x \wedge p \in M$. Then

$(x \wedge g)\ \pi\ (x \wedge p)$, so since M is π-full, $x \wedge g \in M \subseteq P$. Since $x \in G^+ \backslash P$, $g \in P^+$. This shows that P^+ is π-full, i.e. P is π-full. Then clearly $U(H)$ is also π-full.

In view of Theorem 2.5, this lemma yields

PROPOSITION 2.7. *Let G be SQPC, and let $H \neq G$ be a π-full convex l-subgroup of G. Then every P minimal among the prime subgroups of G which contain H is in fact a minimal prime subgroup of G. Thus $F(H) = U(H) = \{x \in G^+ \mid y \in G^+$ and $x \wedge y \in H$ imply $y \in H\}$.*

PROPOSITION 2.8. *Let G be QPC. Let $g \in G^+$, and pick $g^* \in G^+$ such that $g \wedge g^* = 1$ and $g \vee g^* \in W$. Then each minimal prime subgroup of G contains precisely one of the two elements g and g^*. Furthermore, for any $T \in \mathcal{T}_\pi$, $g \in T$ if and only if $g^* \in \mathrm{Rad}(T)$.*

Proof. Let P be a minimal prime. Since $g \wedge g^* = 1$, either $g \in P$ or $g^* \in P$. If both lie in P, then $g \vee g^* \in P$, so that P meets W, which is impossible since P is minimal and thus π-full. Now let $T \in \mathcal{T}_\pi$. The maximal tangents of T are π-full by Proposition 1.1 and thus are minimal. If $g \in T$, then g lies outside all maximal tangents of T, so g^* lies in all of them and thus in their intersection $\mathrm{Rad}(T)$; and the converse is proved by reversing this argument.

Our nicest description of $\mathcal{T}_\pi$ (Theorem 4.1) requires that G be more than just SQPC, but many of its conclusions hold for all SQPC l-subgroups.

THEOREM 2.9. *Let G be SQPC (e.g. conditionally laterally complete). Then*
(1) *the following are equivalent for tight Riesz orders T:*
  (a) *$T \in \mathcal{T}_\pi$ (i.e. T is π-full.)*
  (b) *all maximal tangents of T are minimal prime subgroups of G.*
  (c) *$T \supseteq W$. (The equivalence of (c) holds when $W \neq \varnothing$; but $W \neq \varnothing$ does not guarantee that $W \in \mathcal{T}_\pi$.)*
(2) *each $T \in \mathcal{T}_\pi$ is compatible.*
(3) *for $T_1, T_2 \in \mathcal{T}_\pi$, $\mathrm{Max}(T_1) \subseteq \mathrm{Max}(T_2)$ iff $T_2 \subseteq T_1$.*
(4) *$T \to \mathrm{Rad}(T)$ is an o-embedding (preserving order both ways) of $\mathcal{T}_\pi$ into $\mathcal{L}_\pi$; and $\mathrm{Max}(T) = \mathrm{Min}(\mathrm{Rad}(T))$, so that $T = F(\mathrm{Rad}(T)) = \{x \in G^+ \mid y \in G^+$ and $x \wedge y \in \mathrm{Rad}(T)$ imply $y \in \mathrm{Rad}(T)\}$.*
(5) *for $T \in \mathcal{T}_\pi$, $\mathrm{Rad}(T) = \{1\}$ if and only if $T = W$.*

REMARK. The integers $\mathbb{Z}$ serve as a counterexample to various attempts to strengthen these conclusions. However, $\mathbb{Z}$ is in a sense the only kind of counterexample which does so; cf. Theorem 5.1.

Proof. Our proof that (a) implies (b) is essentially a part of the proof of Theorem 10 of Davis and Fox [5]. Let M be a maximal tangent of $T \in \mathcal{T}$, so that M is prime, and pick $t \in T$. For any $m \in M^+$, there exists $m^* \in G$ such that $m \vee m^* \in T$ and thus $m^* \notin M$; and such that $m \wedge m^* = 1$. Hence M is a minimal prime. The converse follows (in any l-group at all) from the fact that minimal primes are π-full.

It is easily seen that in any l-group, all π-full tight Riesz orders contain W. Conversely (assuming $W \neq \varnothing$), let $T \supseteq W$. Let M be a maximal tangent of T. For any $m \in M^+$, there exists $m^* \in G$ such that $m \vee m^* \in W$, so that $m^* \notin M$; and such that $m \wedge m^* = 1$. Hence M is a minimal prime.

For (2), let $T \in \mathcal{T}_\pi$. If $1 < g \in G$, then there exists $g^* \in G$ such that $g \vee g^* \in T$ and $g \wedge g^* = 1$, which guarantees $g^2 \not\leq g \vee g^*$. (If G is represented as an l-permutation group (G, Ω), then when $\omega < \omega g$, $\omega g^2 > \omega(g \vee g^*)$.) The rest of the argument we borrow from Ball [2]: Since T is factorable, $g \vee g^* = t_1 t_2$ for some $t_1, t_2 \in T$. If $g \leq$ each t_i, then $g^2 \leq t_1 t_2 = g \vee g^*$. Hence g is not a lower bound for T, showing that $\inf T = 1$.

Suppose $T_1 \subseteq T_2$, $T_i \in \mathcal{T}_\pi$. Let $M \in \operatorname{Max}(T_2)$. Then $M \cap T_1 = \varnothing$, so $M \subseteq M'$ for some $M' \in \operatorname{Max}(T_1)$. But M' is a minimal prime, so $M = M' \in \operatorname{Max}(T_1)$. This shows that $T_1 \subseteq T_2$ implies $\operatorname{Max}(T_2) \subseteq \operatorname{Max}(T_1)$, and the converse is clear.

Now we prove (4) by Proposition 1.1. $\operatorname{Rad}(T) \in \mathcal{L}_\pi$. Suppose $T_1 \subseteq T_2$, $T_i \in \mathcal{T}_\pi$. Then $\operatorname{Max}(T_2) \subseteq \operatorname{Max}(T_1)$, so that $\bigcap \operatorname{Max}(T_1) \subseteq \operatorname{Max}(T_2)$, i.e. $\operatorname{Rad}(T_1) \subseteq \operatorname{Rad}(T_2)$. On the other hand, suppose $t \in T_1 \backslash T_2$. Then $t \in M$ for some maximal tangent of T_2. Since M is a minimal prime, we may pick $t' \in G^+ \backslash M$ such that $t \wedge t' = 1$. Because $t \in T_1$, t lies in no maximal tangent of T_1, so t' lies in every maximal tangent of T_1 and thus in $\operatorname{Rad}(T_1)$. Thus $t' \in \operatorname{Rad}(T_1) \backslash \operatorname{Rad}(T_2)$. Hence $T \to \operatorname{Rad}(T)$ is an o-embedding.

Since all maximal tangents of T are minimal primes, $\operatorname{Max}(T) \subseteq \operatorname{Min}(\operatorname{Rad}(T))$. Now let $M \in \operatorname{Min}(\operatorname{Rad}(T))$. Then $M \cap T = \varnothing$. For if $t \in M \cap T$, pick $t^* \in G$ such that $t \vee t^* \in T \backslash M$ and $t \wedge t^* = 1$. Since $t \in T$, t lies in no maximal tangent of T, so t^* lies in each maximal tangent of T and thus in $\operatorname{Rad}(T)$, which is contained in M. But then $t \vee t^* \in M$, a contradiction. Hence $M \cap T = \varnothing$, so that M is contained in and thus equal to a maximal tangent P of T. Hence $\operatorname{Min}(\operatorname{Rad}(T)) \subseteq \operatorname{Max}(T)$.

Lastly, (5) is a translation of the fact that in any l-group, W is obtained by deleting from G^+ the union of any collection of minimal primes whose intersection is $\{1\}$.

§3. Factorability of π-full normal filters

LEMMA 3.1. *Let (G, Ω) be any l-permutation group, and let $\{g_i\}$ be any pairwise disjoint set for which $\sup\{g_i\}$ exists. Then this sup is pointwise on $\bigcup \operatorname{support}(g_i)$, i.e. if $\omega \in \operatorname{support}(g_i)$ for some i, then $\omega \sup\{g_i\} = \omega g_i$.*

Remark. Such a sup need not be pointwise throughout Ω. Let $G = \{g \in A(\mathbb{R}) \mid \exists n_g \in \mathbb{Z}^+$ such that $\forall \omega \in \mathbb{R}$, $(\omega + n_g)g = \omega g + n_g\}$. Let z be translation by $+1$ and pick $1 < f \in A(\mathbb{R})$ supported by the interval $(-\frac{1}{2}, +\frac{1}{2})$. For $i \geq 0$, let f_i be the (pointwise) sup of $\{f^{(z^n)} \mid n \in \mathbb{Z}^+$, and n can be divided by 2 precisely i

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times}. Thus, for example, f_0 "has a copy of f at each odd integer". Each $f_i \in G$, and $\{f_i\}$ has a sup in G, namely $f' \vee f$, where f' is the pointwise sup of $\{f_i\}$. ($f' \vee f$ "has a copy of f at each integer".) Hence $\sup\{f_i\}$ is not pointwise on support(f).

Proof of Lemma 3.1. Let $s = \sup\{g_i\}$. Suppose $\omega < \omega g_i < \omega s$. On the one hand, $sg_i^{-1} = \sup\{g_j \mid j \neq i\}$. But $\omega < \omega sg_i^{-1}$, so $(sg_i^{-1})g_i^{-1} \vee 1 < sg_i^{-1}$. Since $\{g_i\}$ is pairwise disjoint, $(sg_i^{-1})g_i^{-1} \vee 1$ is also an upper bound for $\{g_j \mid j \neq i\}$, a contradiction.

An *o-block* of an l-permutation group (G, Ω) is a convex subset Δ of Ω such that for any $g \in G$, either $\Delta g = \Delta$ or $\Delta g \cap \Delta = \emptyset$. If (G, Ω) is transitive and Ω is discrete, Ω is the disjoint union of o-blocks which are copies of the ordered set $\mathbb{Z}$ of integers. This is the kind of o-block which causes trouble, and we now generalize to the intransitive case. We shall say that an o-block Δ of (G, Ω) is a *$\mathbb{Z}$-block* if $\{g \in G \mid \Delta g = \Delta\}$, restricted to Δ, is o-isomorphic to the o-group $\mathbb{Z}$. (This does not force Δ to be o-isomorphic to the ordered set $\mathbb{Z}$). We shall refer to the element corresponding to $1 \in \mathbb{Z}$ as "translation by $+1$". Any transitive (G, Ω) on a dense Ω has no $\mathbb{Z}$-blocks.

Lemma 3.2. *Let (G, Ω) be a laterally complete l-permutation group, and suppose $t \in G^+$ has the property that there is no $\mathbb{Z}$-block Δ of (G, Ω) for which $\Delta t = \Delta$ and $t \mid \Delta$ is translation by $+1$. Then t can be factored as $t = t_1 t_2$ with $t_1, t_2 \in G^+$ and $t_1 \,\pi\, t \,\pi\, t_2$.*

Proof. Let $\{s_i\}$ be maximal with respect to

(a) for each i, $s_i \in G$ and $1 < s_i \leq t$,

(b) $s_i \perp s_j$ when $i \neq j$,

(c) for each i, $\{\omega \mid \omega < \omega s_i = \omega t\}$ does not support any $1 \neq g \in G$.

We use lateral completeness to form $t_2 = \sup\{s_i\} \leq t$ (except that if $\{s_i\} = \emptyset$, we let $t_2 = 1$.) We claim that $t_2 \,\pi\, t$. Suppose by way of contradiction that there exists $1 < u$ such that $u \perp t_2$ (and hence $u \perp s_i$ for all i), but not $u \perp t$; and with no loss of generality, suppose that $u \leq t$. Replacing u by $u^2 \wedge t$, we may assume also that for any ω such that $\omega < \omega u$, ωu is not the successor of ω in the orbit ωG except perhaps when $\omega u = \omega t$. Let $\Lambda = \{\omega \mid \omega < \omega u = \omega t\}$. If Λ does not support any $1 \neq g \in G$, we contradict the maximality of S. Thus we may pick $1 < g \in G$ with support$(g) \subseteq \Lambda$. We wish to pick $\omega \in$ support(g) and $h \in G^+$ such that $\omega < \omega h < \omega u$. We can do this unless for all $\omega \in$ support(g) ($\subseteq$ support(u)), ωu is the successor of ω in ωG and hence $\omega u = \omega t$. But in that case, when $\omega \in$ support(g), $\Delta = \{\rho \mid \omega t^{-n} \leq \rho \leq \omega t^n$ for some $n \in \mathbb{Z}^+\}$ would be an o-block of (G, Ω). By hypothesis, Δ could not be a $\mathbb{Z}$-block, so there would exist $k \in G$ which fixed ω and thus also each ωt^n, and moved strictly down some point v between some ωt^n and ωt^{n+1}. Then $vt^{-1} < \omega t^n < vk < v$, so that v would not be the successor of vt^{-1} in vG, a contradiction. Thus we may indeed pick $\omega \in$ support(g) and $h \in G^+$ such that $\omega < \omega h < \omega u$. Letting $w = u \wedge g \wedge h$, we have $1 \leq \omega \leq u$, $\omega < \omega w < \omega u$, and support$(w) \subseteq \Lambda$.

Among the pairwise disjoint subsets of G which contain w and whose elements have support contained in Λ, pick one which is maximal, and call it X. Use lateral completeness again to form $x = \sup X$, and let $y = (x \wedge u)^{-1} u \le u$. Then $1 \le y \le u \le t$, and for all i, $y \perp s_i$. Since $\omega \in \text{support}(w)$, Lemma 3.1 guarantees that $\omega x = \omega w < \omega u$, so that $1 < y$. Any $1 < f \in G$ supported by $\{\mu \mid \mu < \mu y = \mu t\} \subseteq \{\mu \mid \mu < \mu y = \mu u = \mu t\} \subseteq \{\mu \mid \mu x = \mu\} \cap \Lambda$ would violate the maximality of X, so y satisfies condition (c). This violates the maximality of $\{s_i\}$. Therefore $t_2 \pi t$. (Incidentally, this means that $\{s_i\}$ must have been non-empty.)

Let $t_1 = t t_2^{-1}$, so that $t_1 t_2 = t$ and $1 \le t_1 \le t$. We claim $t_1\ \pi\ t$. Suppose by way of contradiction that there exists $1 < u$ such that $u \perp t_1$ but not $u \perp t$, with $u \le t$. Then $\text{support}(u) \subseteq \{\omega \mid \omega < \omega t_2 = \omega t\}$. Recall that $t_2 = \sup\{s_i\}$, so that $t_2 u^{-1} \not\geq \text{some } s_i$, and thus $\text{support}(u) \cap \text{support}(s_i) \neq \varnothing$. Thus $u \wedge s_i > 1$. Also, $\text{support}(u \wedge s_i) = \text{support}(u) \cap \text{support}(s_i) \subseteq \{\omega \mid \omega < \omega t_2 = \omega t\} \cap \text{support}(s_i) \subseteq \{\omega \mid \omega < \omega s_i = \omega t\}$ since $s_i = t_2$ on $\text{support}(s_i)$ by Lemma 3.1. Letting $g = u \wedge s_i$, we contradict condition (c) for s_i, proving the lemma.

THEOREM 3.3. *Let (g, Ω) be a laterally complete l-permutation group having no $\mathbb{Z}$-blocks. Then every π-full subset of G^+ (and in particular every π-full normal filter) is factorable.*

Remark. The hypothesis of this theorem is satisfied by any *transitive* laterally complete (G, Ω) for which Ω is dense.

§4. The correspondence between $\mathcal{T}_\pi$ and $\mathcal{L}_\pi$

THEOREM 4.1. *Let (G, Ω) be a laterally complete l-permutation group having no $\mathbb{Z}$-blocks. Then all the conclusions of Theorem 2.9 obtain. Moreover:*

(1) W is the smallest member of $\mathcal{T}_\pi$, and the maximal tangents of W are precisely the minimal prime subgroups of G.

(2) $\mathcal{T}_\pi$ consists precisely of the sets $G^+ \backslash \cup\{M_i\}$, where $\{M_i\}$ is a non-empty normal set of minimal prime subgroups of G; and $\text{Rad}(G^+) \backslash \bigcup\{M_i\}) = \bigcap M_i$.

(3) Each maximal (compatible) tight Riesz order lies in $\mathcal{T}_\pi$, and has the form $G^+ \backslash \bigcup_{g \in G} M^g$ for some minimal prime subgroup M of G.

(4) $\mathcal{T}_\pi$ is closed under arbitrary intersection. For $T_1, T_2 \in \mathcal{T}_\pi$, $\text{Max}(T_1 \cap T_2) = \text{Max}(T_1) \cup \text{Max}(T_2)$; and if T_1 and T_2 are incomparable, $T_1 \cap T_2$ cannot have the form $G^+ \backslash \bigcup_{g \in G} M^g$ for any minimal prime subgroup M of G.

(5) $T \to \text{Rad}(T)$ is an o-isomorphism from $\mathcal{T}_\pi$ onto $\mathcal{L}_\pi$; its inverse is $H \to F(H) = \{x \in G^+ \mid y \in G^+ \text{ and } x \wedge y \in H \text{ imply } y \in H\}$.

Remark. The hypotheses of this theorem are satisfied by any *transitive* laterally complete (G, Ω) for which Ω is dense. But even then, as we shall see later, $G^+ \backslash \bigcup_{g \in G} M^g$ can fail to be maximal even when M is a minimal prime.

Proof. By Theorem 3.3, each π-full subset of G^+ is factorable. Hence W is a (π-full) tight Riesz order, and by Theorem 2.9 is the smallest such. By Theorem 2.9, the maximal tangents of W are minimal primes. Conversely, let

M be any minimal prime. Since M is π-full, $M \cap W = \emptyset$. Hence M is contained in and thus equal to a maximal tangent of W.

Each $T \in \mathcal{T}_\pi$ is simply $G^+ \setminus \bigcup \mathrm{Max}(T)$. The maximal tangents of T constitute a normal set of primes. By Proposition 1.1, they are π-full, and then by Theorem 2.5, they are minimal. Conversely, if $\{M_i\}$ is a non-empty normal set of minimal (hence π-full) primes, then $T = G^+ \setminus \bigcup \{M_i\}$ is a π-full normal proper filter, and being factorable by Theorem 3.3, it is a tight Riesz order. Each M_i is contained in and thus equal to a maximal tangent of T, so $\mathrm{Rad}(T) = \bigcap \mathrm{Max}(T) \subseteq \bigcap \{M_i\}$. Suppose by way of contradiction that there exists $1 < g \in \bigcap \{M_i\} \setminus \mathrm{Rad}(T)$, so that $g \notin P$ for some other maximal tangent P of T. Pick $g^* \in G$ such that $g \wedge g^* = 1$ and $g \vee g^* \in W$. Since $g \notin P$, we have $g^* \in P$ and thus $g^* \notin T$. On the other hand, g lies in each M_i, so by Proposition 2.8, g^* lies in no M_i, forcing $g^* \in T$, a contradiction. Therefore $\mathrm{Rad}(T) = \bigcap \{M_i\}$.

For (3), let T be a maximal tight Riesz order (and of course, a maximal compatible tight Riesz order is also maximal as a tight Riesz order.) Let P be a maximal tangent of T, and let M be a minimal prime contained in P. Then $G^+ \setminus \bigcup_{g \in G} M^g$ is a tight Riesz order containing T. By the maximality of T, $T = G^+ \setminus \bigcup_{g \in G} M^g$.

Since the intersection of any collection of π-full tight Riesz orders is non-empty (because it contains W) and π-full, it is factorable. Thus $\mathcal{T}_\pi$ is closed under intersection. We shall return to (4) after proving (5).

In view of Theorem 2.9, (5) will follow once we know that every $H \in \mathcal{L}_\pi$ is $\mathrm{Rad}(T)$ for some $T \in \mathcal{T}_\pi$. By part (2) of the present theorem, $F(H) = G^+ \setminus \bigcup \mathrm{Min}(H) \in \mathcal{T}_\pi$ and $\mathrm{Rad}(F(H)) = \bigcap \mathrm{Min}(H)$. By Theorem 2.5, $\bigcap \mathrm{Min}(H) = H$.

Finally, we finish (4). Let $T_1, T_2 \in \mathcal{T}_\pi$. By (3) of Theorem 2.9 $\mathrm{Max}(T_1 \cap T_2) \supseteq \mathrm{Max}(T_1) \cup \mathrm{Max}(T_2)$. Let $M \in \mathrm{Max}(T_1 \cap T_2)$. Then $M \supseteq \mathrm{Rad}(T_1 \cap T_2) = \mathrm{Rad}(T_1) \cap \mathrm{Rad}(T_2)$ by part (5). Since M is prime, M contains either $\mathrm{Rad}(T_1)$ or $\mathrm{Rad}(T_2)$. Since M is a minimal prime, part (4) of Theorem 2.9 guarantees that $M \in \mathrm{Max}(T_1) \cup \mathrm{Max}(T_2)$. Thus $\mathrm{Max}(T_1 \cap T_2) = \mathrm{Max}(T_1) \cup \mathrm{Max}(T_2)$. Suppose $T_1 \cap T_2 = G^+ \setminus \bigcup_{g \in G} M^g$ for some minimal prime M. Then $M \in \mathrm{Max}(T_1 \cap T_2) = \mathrm{Max}(T_1) \cup \mathrm{Max}(T_2)$, and we may suppose $M \in \mathrm{Max}(T_1)$. Then $T_1 \subseteq G^+ \setminus \bigcup_{g \in G} M^g = T_1 \cap T_2$, so $T_1 \subseteq T_2$.

Proposition 4.2 *Under the hypotheses of the previous theorem, let T be any tight Riesz order whatsoever. Then $T\pi$ is a (π-full) tight Riesz order, and $T\pi = T \vee W$ in the partially ordered set of all tight Riesz orders.*

Proof. It is easily checked that $T\pi$ is a π-full normal proper filter and thus factorable. Since a tight Riesz order F contains W if and only if F is π-full, $T\pi = T \vee W$.

§5. 0-2-homogeneous Ω. Ω is said to be *0-2-homogeneous* if for all $\beta <$

γ, $\sigma<\tau$, there exists $g\in A(\Omega)$ such that $\beta g=\sigma$ and $\gamma g=\tau$. Ω has *countable terminality* if Ω has a countable coinitial subset and a countable cofinal subset. For any Ω enjoying these two properties, $A(\Omega)$ has precisely three proper l-ideals [10]: $A_\lambda=\{g\in A(\Omega)\mid \text{support}(g) \text{ is bounded below}\}$, its dual A_ρ, and $B=A_\lambda\cap A_\rho$. Obviously these l-ideals are π-full. The set W of weak units of $A(\Omega)$ is $\{g\in A^+(\Omega)\mid \text{support}(g) \text{ is dense in } \Omega\}$. Let $T_\lambda=\{g\in A^+(\Omega)\mid \text{for some } \omega\in\Omega,\ \text{support}(g)\cap(-\infty,\omega) \text{ is dense in } (-\infty,\omega)\}$ and let T_ρ be the dual of T_λ. Part of the next theorem is due to Davis and Fox [5].

THEOREM 5.1. *Let Ω be o–2–homogeneous and have countable terminality (e.g. $\Omega=\mathbb{R}$). Then $A(\Omega)$ has precisely four π-full (compatible) tight Riesz orders: T_λ and T_ρ (which are the only two maximal tight Riesz orders on $A(\Omega)$), $T_\lambda\cap T_\rho$, and W. The maximal tangents of T_λ are precisely the minimal prime subgroups containing A_λ, so that* $\text{Rad}(T_\lambda)=A_\lambda$; *and dually for A_ρ.* $\text{Max}(T_\lambda\cap T_\rho)=\text{Max}(T_\lambda)\cup\text{Max}(T_\rho)$, *and* $\text{Rad}(T_\lambda\cap T_\rho)=B$. *The maximal tangents of W are precisely the minimal prime subgroups, and* $\text{Rad}(W)=\{1\}$.

Proof. $\mathcal{L}_\pi$ has precisely four members, and the characterization $F(H)=\{x\in G^+\mid y\in G^+ \text{ and } x\wedge y\in H \text{ imply } y\in H\}$ leads easily to the four π-full tight Riesz orders T_λ, T_ρ, $T_\lambda\cap T_\rho$, and W. The rest follows from Theorem 4.1.

For Ω as in the theorem, let M be a minimal prime of $A(\Omega)$ which does not contain B. (Such primes exist since the intersection of all minimal primes is $\{1\}$.) Then $A^+(\Omega)\setminus\bigcup_{g\in A(\Omega)} M^g$ has radical $\{1\}$, so $A^+(\Omega)\setminus\bigcup_{g\in A(\Omega)} M^g$ is the smallest π-full tight Riesz order W. Thus $A^+(\Omega)\setminus\bigcup_{g\in A(\Omega)} M^g$ can fail to be maximal even when M is a minimal prime.

COROLLARY 5.2. *Let Ω be as in the theorem. Let M and P be minimal prime subgroups of $A(\Omega)$, with $P\supseteq B$ but $M\not\supseteq B$. Then* $\bigcup_{g\in A(\Omega)} P^g\subset\bigcup_{g\in A(\Omega)} M^g$.

Proof. $P\supseteq B=\text{Rad}(T_\lambda\cap T_\rho)=\text{Rad}(T_\lambda)\cap\text{Rad}(T_\rho)$, so P contains either $\text{Rad}(T_\lambda)=A_\lambda$ or $\text{Rad}(T_\rho)=A_\rho$, say the former. Hence $(\bigcup_{g\in A(\Omega)} P^g)^+=A^+(\Omega)\setminus T_\lambda\subset A^+(\Omega)\setminus W=\bigcup_{g\in A(\Omega)} M^g$ by the above remarks.

It would be interesting to know whether all minimal primes P containing A_λ are conjugate. We have been unable to decide this.

EXAMPLE 5.3. Let $G=\{g\in A(\mathbb{R})\mid(\omega+1)g=\omega g+1,\ \forall\omega\in\mathbb{R}\}$. Then $W=\{g\in G\mid \text{support}(g) \text{ is dense in } \mathbb{R}\}$ is the only π-full (compatible) tight Riesz order on G.

REMARK. The same statement (and proof) apply to any "full and periodically o-primitive" l-permutation group; see [14] for definition and results. For the example as it stands, it was observed by Ball [2] that W is a tight Riesz order.

Proof. G is laterally complete [15, Theorem 19], and (G,Ω) has no proper

o-blocks and thus no $\mathbb{Z}$-blocks. Moreover, G is l-simple [11], so $\mathscr{L}_\pi$ contains only $\{1\}$.

EXAMPLE 5.3. Let $G=\{g \in A(\mathbb{R}) \mid \exists n_g \in Z^+$ such that $\forall \omega \in \mathbb{R}$, $(\omega+n_g)g = \omega g + n_g\}$. Then $W=\{g \in G \mid \text{support}(g)$ is dense in $\mathbb{R}\}$ is the only π-full (compatible) tight Riesz order on G.

Outline of proof. G is SPQC, on Theorem 2.9 applies. Since G is l-simple [12], G has at most one π-full tight Riesz order. Despite the fact that G is not laterally complete, it can be seen directly that W is factorable. Thus W is a π-full compatible tight Riesz order. A similar compatible tight Riesz order for G (not π-full) was found by Glass [8, Example 3].

Now we drop the hypothesis of countable terminality. The l-ideal structure of $A(\Omega)$ becomes vastly more complicated (Ball, [1]). First we prove a lemma due to Andrew Glass and Charles Holland [7, 314, proof omitted].

LEMMA 5.4. *Let Ω be o-2-homogeneous. Then all l-ideals of $A(\Omega)$ are π-full.*

Proof. Let H be an l-ideal of $A(\Omega)$, let $h \in H^+$, and $g \,\pi\, h$. Since every non-singleton interval of Ω supports some $1 \neq f \in A(\Omega)$, support(g) and support(h) have the same topological closures. Let $\omega \in \text{support}(g) \cap \text{support}(h)$, and let $\bar{\Omega}$ denote the Dedekind completion of Ω. Let $\bar{\omega}_0 = \omega$. For $n \geq 1$, let $\bar{\omega}_n$ be the smallest $\bar{\tau} \in \bar{\Omega}$ such that $\bar{\tau} \geq \bar{\omega}_{n-1}$ and $\bar{\tau}$ is fixed by g if n is odd (resp. fixed by h if n is even). (Thus $\bar{\omega}_1 = \sup\{\omega g^i \mid i \in \mathbb{Z}^+\}$.) Let $\bar{\beta}(\omega) = \sup\{\omega_n \mid n \in \mathbb{Z}^+\} \in \Omega$. Begin again at ω and define $\bar{\alpha}(\omega)$ dually. Then the Ω-interval $\Delta(\omega) = (\bar{\alpha}(\omega), \bar{\beta}(\omega))$ is o-2-homogeneous and has countable terminality, and $\bar{\alpha}(\omega)$ and $\bar{\beta}(\omega)$ are fixed by both g and h. Let $\tilde{g}$ denote the restriction of g to $\Delta(\omega)$, and $\tilde{h}$ the restriction of H. Since $g \,\pi\, h$, support$(\tilde{h})$ is bounded neither above nor below in $\Delta(\omega)$. By the proof of Theorem 6 in [10], $\tilde{g}$ is exceeded by the supremum of some two conjugates of $\tilde{h}$. Splicing together the various $\Delta(\omega)$'s, and bearing in mind that $g \,\pi\, h$, we see that g is exceeded by the supremum of some two conjugates of h. Hence $g \in H$, proving the lemma.

PROPOSITION 5.5. *Let Ω be o-2-homogeneous. Then $\mathscr{T}_\pi$ is o-isomorphic to the set of* all *l-ideals ($\neq A(\Omega)$) of $A(\Omega)$.*

For o-2-homogeneous Ω, Ball [1] has discovered a great deal about the l-ideal structure of $A(\Omega)$, and this information can be translated into information about $\mathscr{T}_\pi$. Every normal subgroup of $A(\Omega)$ is in fact an l-ideal, so $\mathscr{L}_\pi$ includes all normal subgroups $\neq A(\Omega)$. T_ρ is maximal if and only if Ω has a countable cofinal subset; and dually. (This was established by Davis and Fox in [5] and [6].) If Ω has coinitial and cofinal subsets of cardinality no greater than $\aleph_0$, each π-full tight Riesz order can be enlarged to a maximal (π-full) tight Riesz order. For other information see [1].

§6. $\mathcal{T}_\pi$ for arbitrary $A(\Omega)$

Even when $A(\Omega)$ has $\mathbb{Z}$-blocks, there is a one-to-one correspondence between $\mathcal{T}_\pi(A(\Omega))$ and $\mathcal{L}_\pi(A(\Omega) \mid \tilde{\Omega}, \tilde{\Omega})$, where $A(\Omega) \mid \tilde{\Omega}$ is the l-homomorphic image of $A(\Omega)$ obtained by restriction to a certain subset $\tilde{\Omega}$ of Ω. We need to delete enough of Ω so that the action of $A(\Omega)$ on the remainder $\tilde{\Omega}$ has no $\mathbb{Z}$-blocks. Let $\Omega_0 = \Omega$; when i is not a limit ordinal, let Ω_i be Ω_{i-1} with the $\mathbb{Z}$-blocks of $A(\Omega) \mid \Omega_{i-1}$ deleted; and when i is a limit ordinal, let Ω_i be $\bigcap_{j<i} \Omega_j$. Eventually $\Omega_i = \Omega_{i+1}$ for some i, this Ω_i we denote by $\tilde{\Omega}$. By induction, each Ω_i is a fixed block of $A(\Omega)$, i.e. $\Omega_i A(\Omega) = \Omega_i$. In particular, $\tilde{\Omega}$ is a fixed block of $A(\Omega)$. The action $A(\Omega) \mid \tilde{\Omega}$ is an l-homomorphic image of $A(\Omega)$. A word of warning: $A(\Omega) \mid \tilde{\Omega}$ need not be all of $A(\tilde{\Omega})$. Also, it may happen that $\tilde{\Omega} = \emptyset$, and in this case we establish the convention that $A(\Omega) \mid \tilde{\Omega} = \{1\}$, so that $\mathcal{L}_\pi(A(\Omega) \mid \tilde{\Omega}) = \emptyset$. When $S \subseteq A(\tilde{\Omega})$, we denote by $\mathrm{Ex}(S)$ the *expansion* $\{f \in A(\Omega) : f \mid \tilde{\Omega} \in S\}$; and of course Ex^+ denotes the set of positive elements of Ex. In the following theorem, the conclusions of Theorem 2.9 all obtain, but here we get also a different and more interesting description of $\mathcal{T}_\pi(A(\Omega))$.

THEOREM 6.1. *Let Ω be any totally ordered set. Then an o-isomorphism between $\mathcal{T}_\pi(A(\Omega))$ and $\mathcal{L}_\pi(A(\Omega) \mid \tilde{\Omega})$ is given by $T \to \mathrm{Rad}(T \mid \tilde{\Omega})$, the inverse map being $H \to Ex^+(F_{A(\Omega) \mid \tilde{\Omega}}(H))$. (In particular, $A(\Omega)$ has a π-full tight Riesz order if and only if $A(\Omega) \mid \tilde{\Omega} \neq \{1\}$.) When $T \leftrightarrow H$, $\mathrm{Max}(T) = \{\mathrm{Ex}(M) \mid M \in \mathrm{Max}(F_{A(\Omega) \mid \tilde{\Omega}}(H))\}$, and $\mathrm{Rad}(T) = \mathrm{Ex}(H)$.*

Remark. When Ω is homogeneous and discrete, this theorem guarantees that $A(\Omega)$ has no π-full tight Riesz orders (and its proof guarantees that no π-full proper filter on $A^+(\Omega)$ is factorable.) However, $A(\Omega)$ has other tight Riesz orders; indeed, unless Ω is o-isomorphic to $\mathbb{Z}$, $A(\Omega)$ has compatible tight Riesz orders (Glass, [9]). Whether or not Ω is homogeneous, some additional information about $\mathcal{T}_\pi$ can be obtained from Theorem 2.9, though we shall not bother to record it here.

Proof. Suppose first that $\tilde{\Omega} \neq \emptyset$. $A(\Omega) \mid \tilde{\Omega}$ is laterally complete because the pointwise supremum of any pairwise disjoint subset of $A(\Omega)|\tilde{\Omega}$ is again in $A(\Omega) \mid \tilde{\Omega}$. By construction, $A(\Omega) \mid \tilde{\Omega}$ has no $\mathbb{Z}$-blocks. Hence Theorem 4.1 applies to $(A(\Omega)|\tilde{\Omega}, \tilde{\Omega})$, in particular the correspondence between $\mathcal{T}_\pi(A(\Omega) \mid \tilde{\Omega})$ and $\mathcal{L}_\pi(A(\Omega) \mid \tilde{\Omega})$. For any $S \in \mathcal{T}_\pi$ $(A(\Omega) \mid \tilde{\Omega})$, we clearly have $\mathrm{Ex}(S) \in \mathcal{T}_\pi(A(\Omega))$. To establish the desired o-isomorphism, it remains to show that every $T \in \mathcal{T}_\pi(A(\Omega))$ arises in this way.

Let $T \in \mathcal{T}_\pi$, and let $t \in T$. Let t' be any element of $A^+(\Omega)$ which agrees with t on $\tilde{\Omega}$. We must show that $t' \in T$. Writing $t = t_1 t_2 (t_i \in T)$ and letting $s = t_1 \wedge t_2$, we have $s \in T$ such that $s^2 \leq t$. We let v be that element of $A(\Omega)$ which agrees with s except on subsets Σ of Ω which are maximal with respect to "$\Sigma \subseteq \Omega \backslash \tilde{\Omega}$ and Σ is convex in Ω" and satisfies $\Sigma s = \Sigma$; and on any such Σ, r is translation by $+1$

on any $\mathbb{Z}$-block $\Lambda \subseteq \Sigma$ for which $s \mid \Lambda \neq 1_\Lambda$, and fixes all other points of Σ. Then $r \pi s$; for if not, there would exist $1 \neq g \in A(\Omega)$ supported by a set consisting of some Σ of the above sort with its $\mathbb{Z}$-blocks deleted, and it is easily seen by induction (cf. the definition of $\tilde{\Omega}$) that this cannot happen. Since $r \pi s$, $r \in T$, so $r = r_1 r_2 (r_i \in T)$. On each $\mathbb{Z}$-block Λ for which r is translation by $+1$, either $r_1 = 1_\Lambda$ or $r_2 = 1_\Lambda$. Hence $r_1 \wedge r_2$ fixes all points in such Λ's, and thus fixes all points in Σ's of the above form; and $r_1 \wedge r_2 \leq s$ on the rest of Ω. But $r_1 \wedge r_2 \in T$, and $r_1 \wedge r_2 \leq t'$ (since for any Σ maximal with respect to the above conditions but for which $\Sigma s \neq \Sigma$, we have $\Sigma s > \Sigma$ since Σ is an o-block of $A(\Omega)$, and thus $\Sigma(r_1 \wedge r_2) \leq \Sigma s < \Sigma s^2 \leq \Sigma t = \Sigma t'$). Therefore $t' \in T$, as claimed.

This shows that if any $t \in T$ is changed arbitrarily off $\tilde{\Omega}$ (subject to remaining positive), then it remains in T. Hence $T = \mathrm{Ex}^+(T|\tilde{\Omega})$. Then since $1_{\tilde{\Omega}} \notin T$, $1_{\tilde{\Omega}} \notin T \mid \tilde{\Omega}$. It follows easily that $T \mid \tilde{\Omega} \in \mathcal{T}_\pi(A(\Omega) \mid \tilde{\Omega})$. Thus indeed $T = \mathrm{Ex}^+(S)$ for some $S \in \mathcal{T}_\pi(A(\Omega) \mid \tilde{\Omega})$. This establishes the o-isomorphism when $\tilde{\Omega} \neq \varnothing$. When $\tilde{\Omega} = \varnothing$ (so that $\mathcal{L}_\pi(A(\Omega) \mid \tilde{\Omega}) = \varnothing$ by convention), the argument in the preceding paragraph shows that any $T \in \mathcal{T}_\pi(A(\Omega))$ would contain 1_Ω, so that $\mathcal{T}_\pi(A(\Omega)) = \varnothing$.

Concerning maximal tangents, observe that if M is a maximal tangent of $T \in \mathcal{T}_\pi(A(\Omega))$, then since changing an element of T off $\tilde{\Omega}$ (to another positive element) leaves us in T, $\mathrm{Ex}(M \mid \tilde{\Omega}) < T = \varnothing$, so $M = \mathrm{Ex}(M \mid \tilde{\Omega})$. The rest we leave to the reader.

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There is some overlap with our work here and Bigard's work on π-full subgroups (which he calls z-subgroups) of an l-group. These results can be found in Lecture Notes in Mathematics, Springer Verlag, Volume 608, by A. Bigard, K. Keimel and S. Wolfenstein.

The referee points out that Lemma 3.1 is probably well-known. However, we know of no reference to it in any of the work on ordered permutation groups.

References

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