The asymmetric propeller with squares, and some extensions

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The term ‘Asymmetric Propeller’ and studies on it appeared first in [1] by Bankoff, Erdos and Klamkin, it was in [2] by Alexanderson, and more recently in [3] by Gardner. The original propeller theorem refers to three congruent equilateral triangles that share the same vertex.

More expansions in turn are given, since equilateral triangles are not necessarily congruent [1], and they do not even need to have a common vertex, but only need to be erected at the vertices of an equilateral triangle [2]. Finally, Gardner gives a further generalisation by using three similar triangles erected at the vertices of a fourth triangle similar to the three given triangles [3] (see Figure 1).

Theorem 1 (Generalisation of the asymmetric propeller by Gardner [3]).

If ABC, AHJ, DBE and FGC are similar triangles, all labelled in the same sense and situated so that corresponding angles meet at the vertices of triangle ABC, then X, Y and Z are the midpoints of DF, GH and JE, are vertices of a triangle similar to the other four.

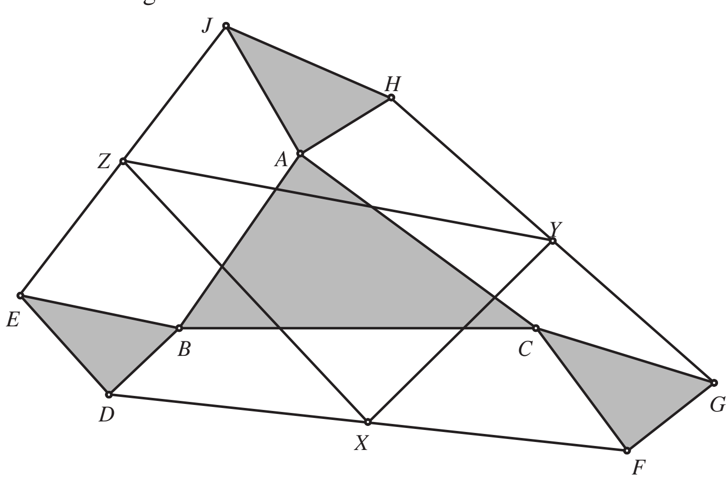

FIGURE 1: Further generalisation of the asymmetric propeller by Gardner

In [3] Gardner also mentions a propeller theorem which is not only with triangles, so for this paper, we propose a propeller theorem with four initial squares erected at the vertices of an initial square. We shall continue Gardner's idea, propose the asymmetric propeller for squares, and give additional generalisations and extensions. Interestingly, Theorem 2 would seem to form a bridge between the asymmetric propeller theorem (for squares), Van Aubel's theorem and Napoleon's theorem.

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Theorem 2

Let $ABCD$ be a square. Attach at vertices A, B, C and D four random squares $AA_1A_2A_3$, $BB_1B_2B_3$, $CC_1C_2C_3$ and $DD_1D_2D_3$ (labelled in the opposite sense to $ABCD$), with centres O_a, O_b, O_c and O_d, respectively. Let X, Y, Z and W be midpoints of segments A_1B_3, B_1C_3, C_1D_3 and D_1A_3, respectively. Then

(i) $XZ = YW$ and $XZ \perp YW$.

(ii) Midpoints of the segments XZ, YW, O_aO_c and O_bO_d are vertices of a square (see Figure 2).

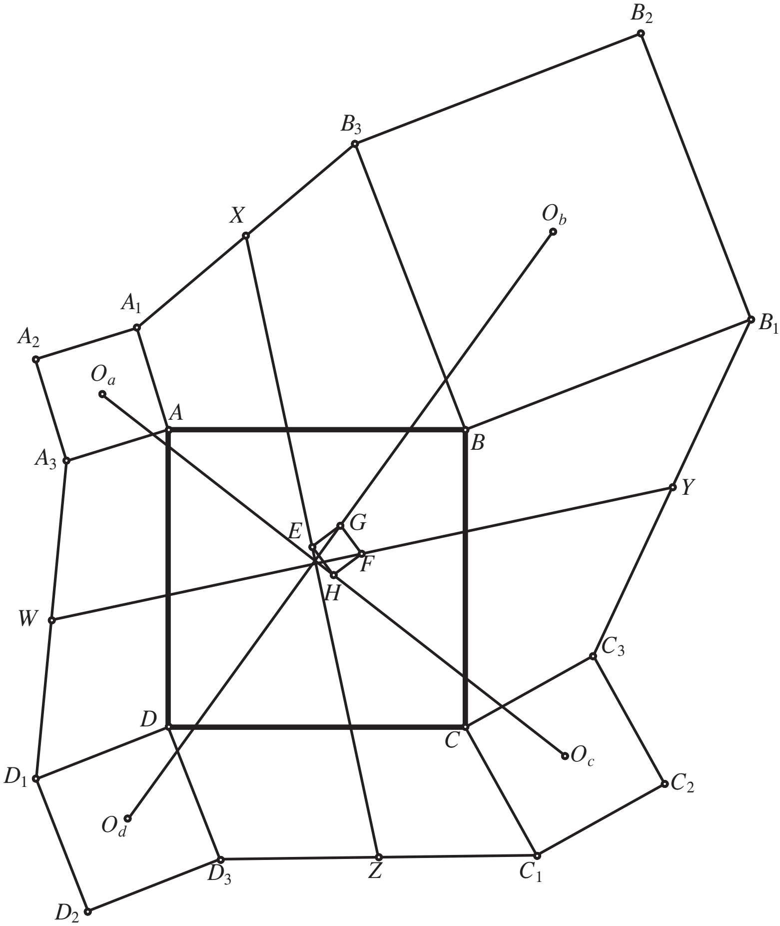

FIGURE 2: Asymmetric propeller with squares

Proof (see Figure 3):

(i) Throughout this Article, using the notation of [4], let $\overrightarrow{PQ}$ show the Euclidean vector connecting an initial point P with a terminal point Q.

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Let $\mathcal{R}$ be the 90° counter clockwise rotation of the plane (see [4]). Assume that the square $ABCD$ has a clockwise direction, we have

$$\overrightarrow{XZ} = \overrightarrow{XA_1} + \overrightarrow{A_1A} + \overrightarrow{AD} + \overrightarrow{DD_3} + \overrightarrow{D_3Z} \tag{1}$$

and

$$\overrightarrow{XZ} = \overrightarrow{XB_3} + \overrightarrow{B_3B} + \overrightarrow{BC} + \overrightarrow{CC_1} + \overrightarrow{C_1Z}. \tag{2}$$

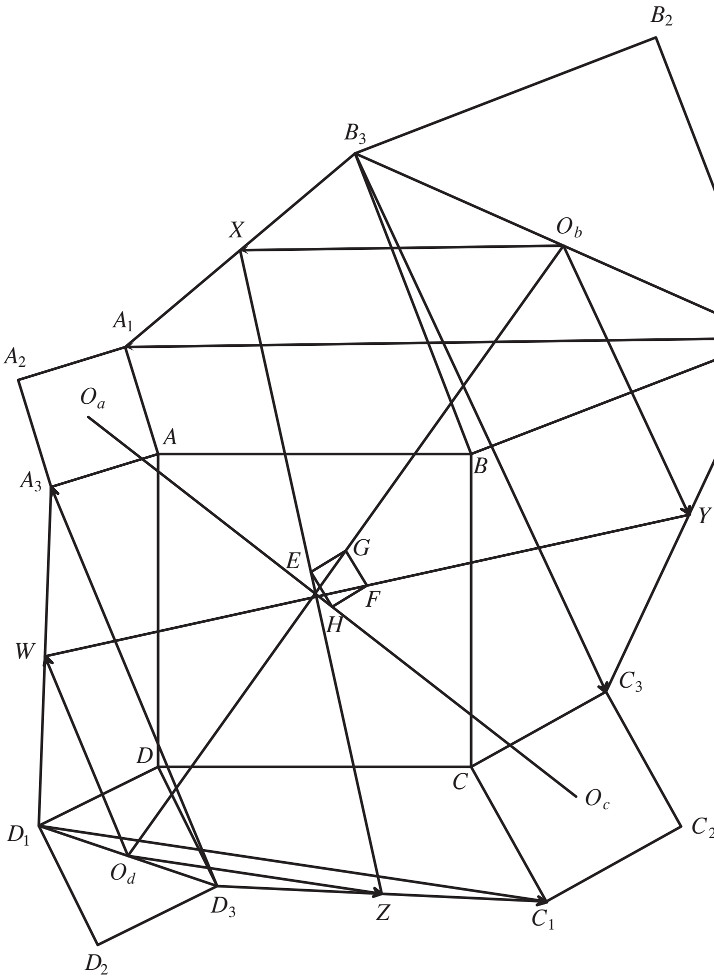

FIGURE 3: Proof of Theorem 2

Since X, Z are the midpoints of A_1B_3, C_1D_3, respectively, $\overrightarrow{XA_1} + \overrightarrow{XB_3} = \vec{0}$ and $\overrightarrow{ZC_1} + \overrightarrow{ZD_3} = \vec{0}$, combining with (1) and (2), we have

$$2\overrightarrow{XZ} = \overrightarrow{A_1A} + \overrightarrow{AD} + \overrightarrow{DD_3} + \overrightarrow{B_3B} + \overrightarrow{BC} + \overrightarrow{CC_1}. \tag{3}$$

Similarly we have

$$2\overrightarrow{WY} = \overrightarrow{A_3A} + \overrightarrow{AB} + \overrightarrow{BB_1} + \overrightarrow{D_1D} + \overrightarrow{DC} + \overrightarrow{CC_3}. \tag{4}$$

Since $\mathcal{R}$ is a linear mapping, from (3) we have

$$\begin{aligned} 2\mathcal{R}\left(\overrightarrow{XZ}\right) &= \mathcal{R}\left(2\overrightarrow{XZ}\right) = \mathcal{R}\left(\overrightarrow{A_1A} + \overrightarrow{AD} + \overrightarrow{DD_3} + \overrightarrow{B_3B} + \overrightarrow{BC} + \overrightarrow{CC_1}\right) \\ &= \mathcal{R}\left(\overrightarrow{A_1A}\right) + \mathcal{R}\left(\overrightarrow{AD}\right) + \mathcal{R}\left(\overrightarrow{DD_3}\right) + \mathcal{R}\left(\overrightarrow{B_3B}\right) + \mathcal{R}\left(\overrightarrow{BC}\right) + \mathcal{R}\left(\overrightarrow{CC_1}\right) \\ &= \overrightarrow{AA_3} + \overrightarrow{AB} + \overrightarrow{D_1D} + \overrightarrow{BB_1} + \overrightarrow{DC} + \overrightarrow{CC_3} = 2\overrightarrow{WY} \text{ (using (4)).} \end{aligned}$$

This implies that $\overrightarrow{WY} = \mathcal{R}\left(\overrightarrow{XZ}\right)$. Since $\mathcal{R}$ is a rotation of 90°, we obtain $XZ = WY$ and $XZ \perp WY$. This completes the proof of part (i).

(ii) Let E, F, G and H be the midpoints of segments XZ, YW, O_bO_d and O_aO_c, respectively. Since G, E are the midpoints of O_bO_d and XZ respectively, we have

$$\begin{aligned} 4\overrightarrow{GE} &= 2\overrightarrow{O_bX} + 2\overrightarrow{O_dZ} \\ &= \overrightarrow{B_1A_1} + \overrightarrow{D_1C_1} \\ &= \left(\overrightarrow{B_1B} + \overrightarrow{BA} + \overrightarrow{AA_1}\right) + \left(\overrightarrow{D_1D} + \overrightarrow{DC} + \overrightarrow{CC_1}\right). \end{aligned}$$

Using $\mathcal{R}$, we obtain

$$\begin{aligned} 4\mathcal{R}\left(\overrightarrow{GE}\right) &= \mathcal{R}\left(4\overrightarrow{GE}\right) = \mathcal{R}\left(\left(\overrightarrow{B_1B} + \overrightarrow{BA} + \overrightarrow{AA_1}\right) + \left(\overrightarrow{D_1D} + \overrightarrow{DC} + \overrightarrow{CC_1}\right)\right) \\ &= \left(\mathcal{R}\left(\overrightarrow{B_1B}\right) + \mathcal{R}\left(\overrightarrow{BA}\right) + \mathcal{R}\left(\overrightarrow{AA_1}\right)\right) + \left(\mathcal{R}\left(\overrightarrow{D_1D}\right) + \mathcal{R}\left(\overrightarrow{DC}\right) + \mathcal{R}\left(\overrightarrow{CC_1}\right)\right) \\ &= \left(\overrightarrow{B_3B} + \overrightarrow{BC} + \overrightarrow{AA_3}\right) + \left(\overrightarrow{D_3D} + \overrightarrow{DA} + \overrightarrow{CC_3}\right) \\ &= \left(\overrightarrow{B_3C} + \overrightarrow{AA_3}\right) + \left(\overrightarrow{D_3A} + \overrightarrow{CC_3}\right) \\ &= \overrightarrow{B_3C_3} + \overrightarrow{D_3A_3} \\ &= 2\overrightarrow{O_bY} + 2\overrightarrow{O_dW} \\ &= 4\overrightarrow{GF}. \end{aligned}$$

From this identity, triangle GEF is half of a square. Similarly, HEF is also half of a square, therefore $EFGH$ is a square. This completes the proof of part (ii).

We can further extend part (ii) of Theorem 2 as follows:

Theorem 3:

Let $ABCD$ be a quadrilateral. Attach at vertices A, B, C and D four parallelograms $AA_1A_2A_3$, $BB_1B_2B_3$, $CC_1C_2C_3$ and $DD_1D_2D_3$ with centres O_a, O_b, O_c and O_d, respectively. Let X, Y, Z and W be midpoints of segments A_1B_3, B_1C_3, C_1D_3 and D_1A_3, respectively. Then E, F, G and H, which are the

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midpoints of the segments XZ, YW, O_aO_c and O_bO_d respectively, are the vertices of a parallelogram (see Figure 4).

Proof:

We have

$$\begin{aligned} 4\overrightarrow{GF} &= 2\overrightarrow{O_dW} + 2\overrightarrow{O_bY} \\ &= \overrightarrow{D_3A_3} + \overrightarrow{B_3C_3} \\ &= \overrightarrow{D_3C_3} + \overrightarrow{B_3A_3} \\ &= 2\overrightarrow{ZO_c} + 2\overrightarrow{XO_a} \\ &= 4\overrightarrow{EH}. \end{aligned}$$

From this identity, $GFHE$ is a parallelogram. This completes the proof of Theorem 3.

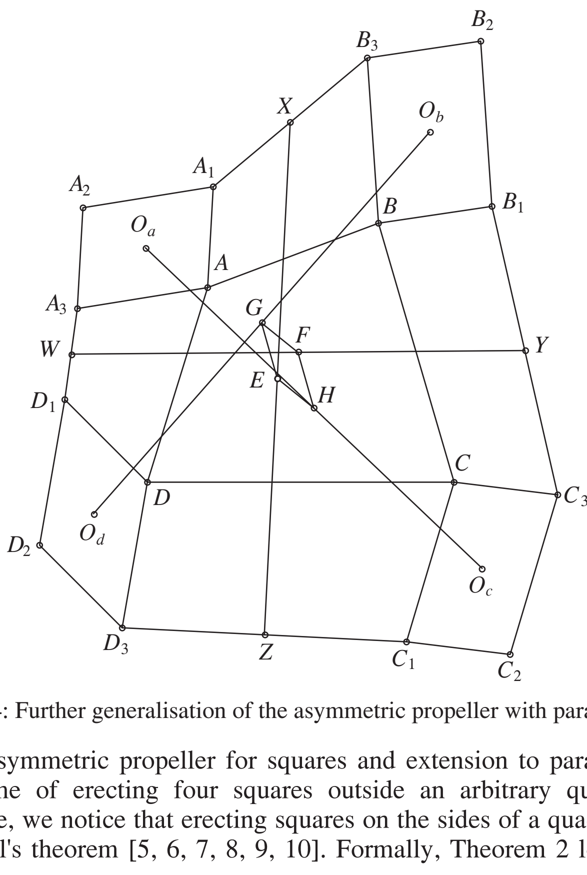

FIGURE 4: Further generalisation of the asymmetric propeller with parallelograms

The asymmetric propeller for squares and extension to parallelograms reminds me of erecting four squares outside an arbitrary quadrilateral. Meanwhile, we notice that erecting squares on the sides of a quadrilateral is Van Aubel's theorem [5, 6, 7, 8, 9, 10]. Formally, Theorem 2 looks pretty

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close to Van Aubel's theorem. This is also quite similar to Napoleon's theorem and its extensions (see [5, 8, 11]). The surprising thing is that it generates a square from any quadrilateral (see [9, 10]). Rotating vectors is still a useful tool to prove this.

It remains to consider the concept of the centroid of the n-point system in [12]: Given n points, identified by vectors, define their centroid to be the vector which is the average of the n points. So in the theorem below, the centroid of a quadrilateral can be understood as the centroid of a set of four points (vertices of this quadrilateral).

Theorem 4: Let $ABCD$ be an arbitrary quadrilateral. Erect four squares $ABEF$, $BCGH$, $CDKL$ and $DAPQ$ outside $ABCD$. Let X, Y, Z and W be the centroid of quadrilaterals $PFEH$, $EHGL$, $GLKQ$ and $KQPF$, respectively. Then

(i) quadrilateral $XYZW$ is a square,

(ii) the centre of the square $XYZW$ coincides with the centroid of quadrilateral $ABCD$ (see Figure 5).

Proof:

(i) Assume that the quadrilateral $ABCD$ has a clockwise direction. Since X, Y, Z and W are the centroids of quadrilaterals $PFHE$, $EHGL$, $GLKQ$ and $KQPF$, respectively, we have

$$4\overrightarrow{XY} = \overrightarrow{EE} + \overrightarrow{HH} + \overrightarrow{FG} + \overrightarrow{PL} = \overrightarrow{FG} + \overrightarrow{PL} \tag{5}$$

and

$$4\overrightarrow{XW} = \overrightarrow{FF} + \overrightarrow{PP} + \overrightarrow{EQ} + \overrightarrow{HK} = \overrightarrow{EQ} + \overrightarrow{HK}. \tag{6}$$

Because $ABEF$, $BCGH$, $CDKL$ and $DAPQ$ are square, so

$$\mathcal{R}\left(\overrightarrow{FB}\right) = \overrightarrow{AE},\ \mathcal{R}\left(\overrightarrow{BG}\right) = \overrightarrow{CH},\ \mathcal{R}\left(\overrightarrow{PD}\right) = \overrightarrow{QA},\ \mathcal{R}\left(\overrightarrow{DL}\right) = \overrightarrow{KC}. \tag{7}$$

Since $\mathcal{R}$ is a linear mapping, using (5), (6) and (7), we have

$$\begin{aligned}
4\mathcal{R}\left(\overrightarrow{XY}\right) &= \mathcal{R}\left(4\overrightarrow{XY}\right) \\
&= \mathcal{R}\left(\overrightarrow{FG} + \overrightarrow{PL}\right) \\
&= \mathcal{R}\left(\overrightarrow{FG}\right) + \mathcal{R}\left(\overrightarrow{PL}\right) \\
&= \mathcal{R}\left(\overrightarrow{FB} + \overrightarrow{BG}\right) + \mathcal{R}\left(\overrightarrow{PD} + \overrightarrow{DL}\right) \\
&= \left(\mathcal{R}\left(\overrightarrow{FB}\right) + \mathcal{R}\left(\overrightarrow{BG}\right)\right) + \left(\mathcal{R}\left(\overrightarrow{PD}\right) + \mathcal{R}\left(\overrightarrow{DL}\right)\right) \\
&= \left(\overrightarrow{AE} + \overrightarrow{CH}\right) + \left(\overrightarrow{QA} + \overrightarrow{KC}\right) \\
&= \left(\overrightarrow{AE} + \overrightarrow{QA}\right) + \left(\overrightarrow{CH} + \overrightarrow{KC}\right)
\end{aligned}$$

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$$= \overrightarrow{QE} + \overrightarrow{KH}$$

$$= -4\overrightarrow{XW}. \tag{8}$$

Because $\mathcal{R}$ is the rotation through angle 90° counter-clockwise and from (8) we find that W is the image of Y in a counter clockwise, rotation by 90° centre X. Thus XYW is half of a square. Similarly, we see that ZWY is the other half of a square so that $XYZW$ is a square.

(ii) Let I be the centre of the square $XYZW$. We will prove that I is also the centroid of $ABCD$. We have

$$\mathcal{R}\left(\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DA}\right) = \vec{0} \tag{9}$$

which is equivalent to

$$\mathcal{R}\left(\overrightarrow{AB}\right) + \mathcal{R}\left(\overrightarrow{BC}\right) + \mathcal{R}\left(\overrightarrow{CD}\right) + \mathcal{R}\left(\overrightarrow{DA}\right) = \vec{0} \tag{10}$$

or

$$\overrightarrow{AF} + \overrightarrow{BG} + \overrightarrow{CL} + \overrightarrow{DQ} = \vec{0}. \tag{11}$$

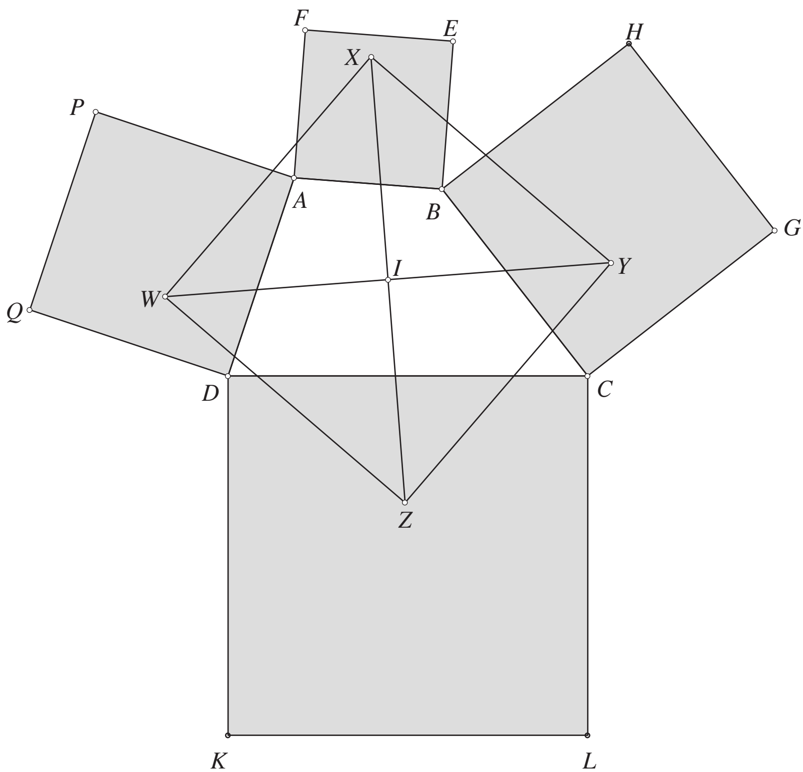

FIGURE 5: Proof of Theorem 4

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It is easily seen that

$$\overrightarrow{FE} + \overrightarrow{HG} + \overrightarrow{LK} + \overrightarrow{QP} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DA} = \vec{0}. \tag{12}$$

From (12), we have

$$\overrightarrow{IE} + \overrightarrow{IG} + \overrightarrow{IK} + \overrightarrow{IP} = \overrightarrow{IF} + \overrightarrow{IH} + \overrightarrow{IL} + \overrightarrow{IQ}. \tag{13}$$

From (11) and (13), we get that

$$\begin{aligned}\overrightarrow{IE} + \overrightarrow{IG} + \overrightarrow{IK} + \overrightarrow{IP} &= \overrightarrow{IF} + \overrightarrow{IH} + \overrightarrow{IL} + \overrightarrow{IQ} - \vec{0} \\ &= \overrightarrow{IF} + \overrightarrow{IH} + \overrightarrow{IL} + \overrightarrow{IQ} - \left(\overrightarrow{AF} + \overrightarrow{BH} + \overrightarrow{CL} + \overrightarrow{DQ}\right) \\ &= \overrightarrow{IA} + \overrightarrow{IB} + \overrightarrow{IC} + \overrightarrow{ID}. \end{aligned} \tag{14}$$

From (13) and (14), we see that

$$\overrightarrow{IA} + \overrightarrow{IB} + \overrightarrow{IC} + \overrightarrow{ID} = \tfrac{1}{2}\left(\overrightarrow{IE} + \overrightarrow{IG} + \overrightarrow{IK} + \overrightarrow{IP} + \overrightarrow{IF} + \overrightarrow{IH} + \overrightarrow{IL} + \overrightarrow{IQ}\right). \tag{15}$$

Since X, Z are the centroids of quadrilaterals $PFEH$, $GLKQ$, respectively, and I is the midpoint of XZ, we obtain

$$\begin{aligned}&\overrightarrow{IE} + \overrightarrow{IG} + \overrightarrow{IK} + \overrightarrow{IP} + \overrightarrow{IF} + \overrightarrow{IH} + \overrightarrow{IL} + \overrightarrow{IQ} \\ &= \left(\overrightarrow{IP} + \overrightarrow{IF} + \overrightarrow{IE} + \overrightarrow{IH}\right) + \left(\overrightarrow{IG} + \overrightarrow{IL} + \overrightarrow{IK} + \overrightarrow{IQ}\right) \\ &= 4\overrightarrow{IX} + 4\overrightarrow{IZ} \\ &= 4\left(\overrightarrow{IX} + \overrightarrow{IZ}\right) = \vec{0}. \end{aligned} \tag{16}$$

From (15) and (16), we have that

$$\overrightarrow{IA} + \overrightarrow{IB} + \overrightarrow{IC} + \overrightarrow{ID} = \vec{0} \tag{17}$$

i.e. I is also the centroid of quadrilateral $ABCD$. This completes the proof of Theorem 4.

Remark: The vector proofs are general, and also apply when the attached figures lie towards the inside as well as when the base quadrilateral is concave or crossed.

Acknowledgement: The author gratefully thanks the referee for the important comments and suggestions which helped the author improve this manuscript.

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10.1017/mag.2024.69
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