AUTOCLINISMS AND AUTOMORPHISMS OF FINITE GROUPS II

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In part I of this paper P. Hall's formula for finite stem groups was derived. Using results of C. R. Leedham-Green and S. McKay, a similar formula for isoclinic groups with arbitrary branch factor group is shown.

The main result of this paper is the following theorem, which appears without proof in [**1**, p. 203].

THEOREM. *Let* Γ *be an isoclinism family of finite groups,* Q *a finite abelian group and* $\mathrm{Acl}(\Gamma)$ *the autoclinism group of* Γ. *Then we have*

$$\frac{1}{|\mathrm{Acl}(\Gamma)\times \mathrm{Aut}(Q)|}=\sum \frac{1}{|\mathrm{Aut}(G)|},$$

where G *runs through a complete system of non-isomorphic groups in* Γ *with* Q *as branch factor group.*

Let G and H be groups (not necessarily finite). We call G and H *strongly isoclinic,* if there exists an isomorphism α from $G/(Z(G)\cap G')$ to $H/(Z(H)\cap H')$, which induces an isomorphism γ from G' to H'; α is called a *strong isoclinism,* if $G=H$ a *strong autoclinism.* It can be easily verified that a strong isoclinism induces isomorphisms α_1 from $G/Z(G)$ to $H/Z(H)$ and α_2 from G_{ab} to H_{ab}, where α_1 and α_2 "coincide" on $G/(G'Z(G))$, and which determine α. The pair (α_1, γ) is an "ordinary" isoclinism from G to H. The restriction of α_2 to $(G'Z(G))/G'$ is an isomorphism onto $(H'Z(H))/H'$. These quotients are called the *branch factor groups* of G and H, being invariant under strong isoclinism. In the terminology of P. Hall, strong isoclinism describes the "situation of the commutator quotients".

Let α be a strong autoclinism of G, $K=G/Z(G)$, Q the branch factor group of G, and τ the restriction of α_2 to Q. Then α determines an element $((\alpha_1, \gamma), \tau)$ of $\mathrm{Acl}(\Gamma)\times \mathrm{Aut}(Q)$. Let Φ denote the class of groups being strongly isoclinic to G, and $A(\Phi)$ the corresponding group of strong autoclinisms (which does not depend on the representatives of Φ). Then we have a homomorphism from $A(\Phi)$ to $\mathrm{Acl}(\Gamma)\times\mathrm{Aut}(Q)$, and it is easy to see that the kernel of this homomorphism is isomorphic to $\mathrm{Hom}(K_{ab}, Q)$. Let $L=G/(Z(G)\cap G')$ and $B=Z(G)\cap G'$, and we consider the central extension

$$C: 1\to B\to G\to L\to 1.$$

Then C determines an epimorphism from the Schur multiplier of L onto B, which corresponds to a coset Ω of $\mathrm{Ext}(L_{ab}, B)$, regarded as a subgroup of $H^2(L, B)$, (in part I the

Ext-group was denoted by $\overline{\text{sym}}$). It follows from Theorem I.2.1 in [**2**] and similar observations as in part I that the isomorphism classes of groups in Φ correspond to the orbits of $A(\Phi)$ on Ω, where $\alpha \in A(\Phi)$ acts on $H^2(L, B)$, resp. Ω via the action of α on L and γ on B. For finite groups we obtain in the same way as in part I the formula

$$\frac{1}{|A(\Phi)|} = \sum \frac{1}{|\mathrm{Aut}(H)|}, \tag{1}$$

where H runs through the isomorphism classes of groups in Φ.

Now we consider all groups in a family Γ with a fixed branch factor group Q, which are divided into certain classes Φ of strongly isoclinic groups. Let S be a (fixed) stem group in Γ with $K = S/Z(S)$. We consider all abelian extensions D of Q by K_{ab}:

$$D: 1 \to Q \to \bar{D} \to K_{ab} \to 1,$$

and denote for each D by $G(D)$ the direct product of S and $\bar{D}$ with amalgamated quotient K_{ab}. From Theorem II.3.2 in [**2**] we obtain that each group in Γ with Q as branch factor group is strongly isoclinic to some $G(D)$. In order to determine the isomorphism classes, we only have to decide, which groups $G(D)$ are in the same class Φ. The groups $G(D)$ are in one-to-one correspondence with the elements of $\mathrm{Ext}(K_{ab}, Q)$. As each autoclinism of S induces an automorphism of K_{ab}, we have an action of $\mathrm{Acl}(\Gamma) \times \mathrm{Aut}(Q)$ on $\mathrm{Ext}(K_{ab}, Q)$, and it is not very difficult to see that groups of the form $G(D)$ are strongly isoclinic, if and only if the corresponding elements of $\mathrm{Ext}(K_{ab}, Q)$ are conjugate under $\mathrm{Acl}(\Gamma) \times \mathrm{Aut}(Q)$. The corresponding stabilizers are the homomorphic images of the groups $A(\Phi)$. For finite groups we obtain

$$|\mathrm{Ext}(K_{ab}, Q)| = \sum \frac{|\mathrm{Acl}(\Gamma) \times \mathrm{Aut}(Q)|}{|A(\Phi)|/|\mathrm{Hom}(K_{ab}, Q)|},$$

where the sum is taken over all classes Φ of strongly isoclinic groups in Γ with Q as branch factor group, which yields

$$\frac{1}{|\mathrm{Acl}(\Gamma) \times \mathrm{Aut}(Q)|} = \sum \frac{1}{|A(\Phi)|}. \tag{2}$$

Now the theorem follows from (1) and (2).

REMARKS. The theorem above can also be obtained by a dual procedure, using Hall's "situation of the centrals". It is also possible to "extend" the formulae and the results on the isomorphism classes of groups in a family to isoclinism classes of arbitrary central extensions without any further complications.

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