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ORIGINAL RESEARCH PAPER

On impact of largest claims reinsurance treaties on the ceding company's loss reserve

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Abstract

This article examines the impact of the largest claims reinsurance treaties on loss reserve of the ceding company. The largest claims reinsurance, known as LCR, and ECOMOR reinsurance treaties are considered to be the two most appropriate reinsurance treaties for large or catastrophe claims. Then, it studies the impact of such treaties on loss reserves. Through a simulation study, it shown that, under a more general situation, the LCR treaty can be a more efficient (in some sense, see below) treaty than the ECOMOR treaty for the ceding company.

Keywords: Loss reserve; Reinsurance; Mean square error of prediction; Simulation

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1. Introduction

Predicting loss reserves is a critical issue for an insurer, reinsurer, and regulator. Uncertainty of the loss reserve is one of the principal sources of risk in insurance/reinsurance companies. Therefore, an appropriate prediction of the loss reserves may help insurance/reinsurance companies in different directions, e.g., it may improve pricing methods, choose a reinsurance policy, etc.

Many insurers evaluate their pricing adequacy by the return on capital achieved by each line of business, say LoB. On the other hand, under the Solvency II framework, insurance companies must employ efficient methods (in some sense) for risk capital allocation. They employ different methods for each LoB, but there is a consensus that the allocated capital for a particular LoB should reflect the fact that the prediction of loss reserves is an uncertain task. Therefore, the allocated capital to such LoB should reflect magnitude of uncertainty. This observation shows how one may improve capital allocation and pricing adequacy by measuring loss reserve uncertainty, see Panning (2006) for more details.

Most of the regulatory frameworks (including Solvency II) and accounting standards (such as the International Financial Reporting Standard 17, known as IFRS 17) require a much higher level of information regarding the prediction of an outstanding claim for each LoB. The requirement can be understood as follows: if a reinsurance contract exists, the outstanding claims for both the cedent and reinsurer must be predicted separately, see England *et al.* (2019), Winkler & Kansal (2020), and Margraf *et al.* (2018), among others for more details.

A considerable amount of literature has been devoted to developing stochastic methods for the prediction of outstanding claims regardless of a reinsurance treaty. Some of these methods are the Mack method (Mack, 1993), the Bayesian method (Gisler & Wüthrich, 2008), the time series model (Buchwalder *et al.*, 2006), the Copulas method (Peters *et al.*, 2014), and Double Chain Ladder (Verrall *et al.*, 2010). As far as we know, a small amount of literature has studied the impact of reinsurance contracts on outstanding claims. Taylor (1982) was the first work that predicted the outstanding claims of an insurance portfolio, under an excess of loss reinsurance treaty. Hertig (1985) derived a prediction for ultimate claims and current IBNR reserves under some long-term reinsurance treaties and some mild assumptions on loss ratio distribution. Craighead (1994) is considered a reinsurer that has accepted several reinsurance treaties which have given rise to catastrophe losses. Under the assumption that such catastrophe losses follow a normal development pattern, he predicted the reinsurer's gross losses using two approaches (exposure totals and statistical modelling approaches). Margraf *et al.* (2018) considered an excess of loss treaty and studied its impact on the claim reserves. Riegel (2015) used the chain ladder method to predict uncertainty of a pricing approach under a long-tail quota shares reinsurance policy.

Murphy & McLennan (2006) developed a non-parametric framework to simulate the distribution of the ultimate position of large outstanding claims. Then, they employed an aggregate model to study the direct relationship between distribution of gross and net reinsurance loss reserves. Veprauskaite & Adams (2017) studied the relationship between loss reserving errors, leverage, and reinsurance in the UK's property-casualty insurance industry. They observed that financially weak insurance companies usually underestimate reserves to reduce leverage and so pre-empt costly regulatory scrutiny. Úbeda Inés (2020) considered the RBNS claims data under some reinsurance treaties. Then, using two well-known actuarial loss reserving methods (chain ladder and generalised linear mixed models), he predicted future claims payments and the corresponding mean square error, for each party.

As mentioned above, Craighead (1994) considered a normal development pattern for catastrophe losses, which may be valid for the whole reinsurer's portfolio, not just for a LoB. Unfortunately, this problem is not addressed by other authors.

Following the above discussion, this article focuses on the problem of predicting the cedent's loss reserves under the two largest claims reinsurance (LCR) and ECOMOR treaties, which are appropriate treaties for a LoB that may suffer from catastrophic losses. With three simulation examples, the Mean Square Error of Predictions, say MSEPs, of such two treaties are compared.

The rest of this article is structured as follows. Definitions and some basic notations that play vital roles in the rest of this article are presented in section 2. Theoretical findings of the article are represented in section 3. An application of our theoretical findings through a simulation study has been presented in section 4. Conclusions and suggestions have been given in section 5.

2. Preliminaries

Let $Y_1, Y_2, \cdots, Y_N$ be a sequence of i.i.d. random claim sizes which have a common cumulative distribution function $F(\cdot)$ and density function $f(\cdot)$. Moreover, suppose N is a random number of claims which takes values on $\{1, 2, \cdots\}$. Now suppose that $Y_{(1)}, Y_{(2)}, \cdots, Y_{(N)}$ stands for the order statistics, where, in particular, the smallest and the largest claim sizes are defined by $Y_{(1)} = min(Y_1, Y_2, \cdots, Y_N)$ and $Y_{(N)} = \max(Y_1, Y_2, \cdots, Y_N)$, respectively.

The kth moment of the mth largest claim sizes and the expectation of the cross product of the mth and zth largest claim sizes $(0 < m < z)$, respectively, are given by

$$E\Big(\big(Y_{(N-m+1)}\big)^k\Big) = \frac{1}{\Gamma(m)} \int_0^1 F^{-1}(v)^k \, [1-v]^{m-1} \, \psi_N^{(m)}(v) dv \tag{1}$$

$$\begin{aligned} E\big(Y_{(N-m+1)} Y_{(N-z+1)}\big) = \frac{1}{\Gamma(m)\Gamma(z-m)} \int_0^1 F^{-1}(v)(1-v)^{z-1} \psi_N^{(z)}(v) \\ \times \int_0^1 F^{-1}(1-u(1-v)) u^{m-1}(1-u)^{z-m-1} dv du \end{aligned} \tag{2}$$

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$$E\big(Y_1 Y_{(N-m+1)}\big) = \sum_{n\geq m} P(N=n) \sum_{h=1}^{3} T_h(n;m), \tag{3}$$

where

$$T_1(n;m) = \frac{(n-1)!}{(n-m-1)!(m-1)!}\int_0^1 F^{-1}(v)v^{n-m-1}(1-v)^{m-1}H(F^{-1}(v))dv,$$

$$T_2(n;m) = \frac{(n-1)!}{(n-m)!(m-2)!}\int_0^1 F^{-1}(v)v^{n-m}(1-v)^{m-2}(\mu - H(F^{-1}(v)))dv,$$

$$T_3(n;m) = \frac{(n-1)!}{(n-m)!(m-1)!}\int_0^1 F^{-1}(v)v^{n-m}(1-v)^{m-1}dv,$$

$v = F(Y_{(z)})$, $u = F(Y_{(m)})$, $\psi_N^{(m)}(v)$ stands for the mth derivative of $\psi_N(v) = \sum_{n=0}^{\infty} P(N=n)v^n$, with respect to v, and $H(z) = \int_0^z tf(t)dt, \forall z \geq 0$, which satisfies $\lim_{z\to\infty} H(z) = E(Y) = \mu$.

It should be noted that in the case of two-order statistics belonging to two different data sets, with unequal lengths, one cannot employ Equation (3). This situation may be studied for a specific case as follows.

Remark 1. Suppose that O and N are two independent random counts and write $M = O + N$. Then, the expectation of the cross product of two mth and zth order claim sizes can be calculated as follows:

$$\begin{aligned} E\big(Y_{(M-m+1)}Y_{(O-z+1)}\big) &= \sum_{n=0}^{\infty} E\big(Y_{(O+n-m+1)}Y_{(O-z+1)|N=n}\big)\, P(N=n) \\ &= \sum_{n=0}^{\infty} E\big(Y_{(O+n-m+1)}Y_{(O-z+1)}\big)\, P(N=n), \end{aligned}$$

where $E\big(Y_{(O+n-m+1)}Y_{(O-z+1)}\big)$ can be calculated using the Equation (2).

For more information on order statistics, we refer interested readers to Kremer (1982), Berglund (1998), and David & Nagaraja (2003).

In general insurance, insurance companies seek appropriate (in some sense) reinsurance protection to reduce and homogenise their risks. A reinsurance treaty is a form of an insurance contract, in which the reinsurer accepts to pay a portion of an insurer's risk by receiving a reinsurance premium (Payandeh Najafabadi & Panahi Bazaz, 2018). Besides regulatory obligations, there are several reasons which motivate an insurer to buy a reinsurance contract, see Albrecher *et al.* (2017), among others, for more details. In the reinsurance literature, the insurer is known as the first line insurer or ceding company. Suppose random claim Z can be decomposed as a sum of an insurance portion, Z^{In}, and a reinsurance portion, Z^{Re}, i.e., $Z = Z^{In} + Z^{Re}$, where $0 \leq Z^{In}$ and $Z^{Re} \leq Z$.

The largest claims reinsurance, say LCR, and the ECOMOR treaties are two reinsurance contracts that just cover some of the largest claims. Therefore, there are appropriate treaties for a LoB that may have the potential to receive some considerable large claims.

The following recall definition of an LCR treaty, and we refer interested readers to Ladoucette & Teugels (2006), Jiang & Tang (2008), and Fan *et al.* (2017), among others.

Definition 1. *Let $Y_1, Y_2, \cdots, Y_N$ be a sequence of independent and identical random claim sizes that have a common cumulative distribution function $F(\cdot)$ and a density function $f(\cdot)$. Moreover, suppose that $Y_{(1)}, Y_{(2)}, \cdots, Y_{(N)}$ stand for their corresponding order statistics. An LCR treaty that*

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Table 1. Standard Notations for an IBNR table, from Hindley (2017).

Notation	Description
i	Accident year, ranges from 1 to I
j	Development year, ranges from 0 to $I-1$
l	delay periods to pay a claim after being reported, ranges from 0 to d
d	The fixed maximal settlement delay, where
$N_{i,j}$	Total number of occurred claims in accident year i and reported in accounting year $i+j$
$N^{paid}_{i,j-l,l}$	Total number of paid claims that occurred at accident year i, reported at year $i+j-l$ and paid in accounting year $i+j$
$Y^{(k)}_{i,j}$	The k^{th} individual settled payment that occurred in accident year i and paid in j years later
$X_{i,j}$	All payments done for claims were incurred in year i and paid in accounting year $i+j$,
	well-known as the incremental claims for accident year in development year j
$C_{i,j}$	Cumulative claims for accident year at development year j
$\mathcal{D}_I$	The smallest σ−field generated on all available information $\mathcal{D}_I=\sigma\left(\{X_{i,j}\colon\ i+j\le I\}\right)$

covers the r largest claims is a reinsurance treaty without any priority that its insurer's portion and cedent's portion from random claims, respectively, are

$$X^{Re}(_{LCR(r)}) = \sum_{m=1}^{r} Y_{(N-m+1)} \text{ and } X^{In}(_{LCR(r)}) = \sum_{m=1}^{N-r} Y^*_{(m)}.$$

The ECOMOR treaty was introduced by Thepaut (1950), who extended the regular excess of loss treaty by letting: (1) its retention level be random and (2) just covering only that part of the r largest claims, see Kremer (1982) and Ladoucette & Teugels (2006), among others, for more details.

The following provides the general concept of the ECOMOR treaty.

Definition 2. *Let $Y_1, Y_2, \cdots, Y_N$ be a sequence of independent and identical random claim sizes that have a common cumulative distribution function $F(\cdot)$ and a density function $f(\cdot)$. Moreover, suppose that $Y_{(1)}, Y_{(2)}, \cdots, Y_{(N)}$ stand for their corresponding order statistics. The ECOMOR treaty covers only that part of the r largest claims that exceed the random retention $Y_{(N-r)}$. Therefore, its insurer's portion and cedent's portion from random claims, respectively, are*

$$X^{Re}(_{ECOMOR(r)}) = \sum_{m=1}^{r} Y_{(N-m+1)} - rY_{(N-r+1)} \text{ and } X^{In}(_{ECOMOR(r)}) = \sum_{m=1}^{N-r} Y_{(m)} + rY_{(N-r+1)}.$$

Consider an IBNR table that contains both observed developed claims (appeared in the upper triangle/trapezoid) and unobserved developed claims, say outstanding claims, (appeared in the lower triangle, say runoff triangle).

To make clear, Table 1, from Hindley (2017), represents all notations that will be used hereafter. Using notations represented in the Table 1, one may observe that

$$N^{paid}_{i,j} = \sum_{l=0}^{\min(j,d)} N^{paid}_{i,j-l,l}; \quad X_{i,j} = \sum_{k=1}^{N^{paid}_{i,j}} Y^{(k)}_{i,j} \text{ and } C_{i,j} = \sum_{k=1}^{j} X_{i,k}.$$

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3. Main Results

This section employs a model introduced by Verrall *et al.* (2010) and Martinez-Miranda *et al.* (2012, 2015) to predict the net loss reserve under the LCR and the ECOMOR treaties.

To begin, the LCR and the ECOMOR treaties are carefully recalled given by Definitions 1 and 2. Both the LCR and the ECOMOR treaties for a LoB that some of its claims do not settle immediately after its occurrence and develop j years later have to be formulated with concern. To derive the cedent's risk portion under these treaties for a loss developing environment, consider individual random claims $Y_{i,j}^{(k)}$, for $j = 0, 1, \cdots, I-1$, and $k = 1, 2, \cdots, N_{i,j}^{paid}$. Moreover, suppose that $Y_{i(1)}, Y_{i(2)}, \cdots, Y_{i\left(N_{i,j}^{paid}\right)}$ stands for the order statistics for $Y_{i,j}^{(k)}$, then

(1) the cedent's risk portion under an LCR treaty which covers the r largest claims is

$$
\begin{aligned}
X_{i,0}^{In}(\text{LCR(r)}) &= \sum_{k=1}^{N_{i,0}^{paid}} Y_{i,0}^{(k)} - \sum_{m=1}^{r} Y_{i(\xi_{i,0}-m+1)} \\
X_{i,1}^{In}(\text{LCR(r)}) &= \sum_{k=1}^{N_{i,1}^{paid}} Y_{i,1}^{(k)} - \left[\sum_{m=1}^{r} Y_{i(\xi_{i,1}-m+1)} - \sum_{m=1}^{r} Y_{i(\xi_{i,0}-m+1)}\right] \\
&\vdots \\
X_{i,J}^{In}(\text{LCR(r)}) &= \sum_{k=1}^{N_{i,J}^{paid}} Y_{i,J}^{(k)} - \left[\sum_{m=1}^{r} Y_{i(\xi_{i,J}-m+1)} - \sum_{m=1}^{r} Y_{i(\xi_{i,J-1}-m+1)}\right]
\end{aligned}
\tag{4}
$$

(2) the cedent's risk portion under an ECOMOR treaty which covers the r largest claims is

$$
\begin{aligned}
X_{i,0}^{In}(\text{ECOMOR(r)}) &= \sum_{k=1}^{N_{i,0}^{paid}} Y_{i,j}^{(k)} - \left(\sum_{m=1}^{r} Y_{i(\xi_{i,0}-m+1)} - rY_{i(\xi_{i,0}-r+1)}\right) \\
X_{i,1}^{In}(\text{ECOMOR(r)}) &= \sum_{k=1}^{N_{i,1}^{paid}} Y_{i,j}^{(k)} - \left(\sum_{m=1}^{r} \left(Y_{i(\xi_{i,1}-m+1)} - Y_{i(\xi_{i,0}-m+1)}\right)\right. \\
&\qquad \left. - r\left(Y_{i(\xi_{i,1}-r+1)} - Y_{i(\xi_{i,0}-r+1)}\right)\right) \\
&\vdots \\
X_{i,J}^{In}(\text{ECOMOR(r)}) &= \sum_{k=1}^{N_{i,J}^{paid}} Y_{i,j}^{(k)} - \left(\sum_{m=1}^{r} \left(Y_{i(\xi_{i,J}-m+1)} - Y_{i(\xi_{i,J-1}-m+1)}\right)\right. \\
&\qquad \left. - r\left(Y_{i(\xi_{i,J}-r+1)} - Y_{i(\xi_{i,J-1}-r+1)}\right)\right)
\end{aligned}
\tag{5}
$$

where $\xi_{i,j} = \sum_{h=0}^{j} N_{i,h}^{paid}$ and $\xi_{i,0} = N_{i,0}$.

Hereafter, the discussion is based on the following model assumption.

Model Assumption 1. *Suppose the individual random claims are stated by* $Y_{i,j}^{(k)}$, *for* $j = 0, 1, \cdots, I-1$, *and* $k = 1, 2, \cdots, N_{i,j}^{paid}$. *Moreover, suppose that:*

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(A_1) *The number of claims incurred in accident year i and reported in year $i+j$, say $N_{i,j}^{report}$, follows a Poisson distribution with intensity $\alpha_i\beta_j$, where $\sum_{j=0}^{I-1}\beta_j = 1$.*
(A_2) *For given $N_{i,j}^{report,}$ the random vector $\left(N_{i,j,0}^{paid}, \cdots, N_{i,j,I-1}^{paid}\right)$ has the Multinomial distribution with parameters $\left(N_{i,j}^{report}; p_0^*, \cdots, p_{I-1}^*\right)$, where delay probabilities $p_0^*, \cdots, p_{I-1}^*$ satisfy $\sum_{l=0}^{I-1} p_l^* = 1$.*
(A_3) *For all $i = 1, 2, \cdots, I$, $j = 0, 2, \cdots, I-1$ and $k = 1, 2, \cdots, N_{i,j}^{paid}$, the individual discounted payments, $Y_{i,j}^{(k)}/\gamma_i$, are mutually independent with the common distribution $F(\cdot)$, where γ_i stands for an inflation index in accident year i. Moreover, $E\left(Y_{i,j}^{(k)}\right) = \gamma_i\mu$ and $Var\left(Y_{i,j}^{(k)}\right) = \gamma_i^2\sigma^2$.*
(A_4) *$Y_{i,j}^{(k)}$ and $N_{i,j}^{report}$ are two independent random variables.*
(A_5) *$\mathcal{D}_I$ stands update filtration based on the past information at observation time I*
(A_6) *Claims are settled just with a single payment.*

Under distributional assumptions, given by Model Assumption 1, one may observe that $E\left(N_{i,j}^{paid}|\mathcal{D}_I\right) = \sum_{l=0}^{\min(j,d)} N_{i,j-l}^{report}\, p_l^*$ and $Var\left(N_{i,j}^{paid}|\mathcal{D}_I\right) = \sum_{l=0}^{\min(j,d)} N_{i,j-l}^{report}\, p_l^*\left(1-p_l^*\right)$. Therefore:

(1) The conditional expectation $E\left(X_{i,j}|\mathcal{D}_I\right)$ is

$$
\begin{aligned}
E\left(X_{i,j}|\mathcal{D}_I\right) &= E\left(\sum_{k=1}^{N_{i,j}^{paid}} Y_{i,j}^{(k)}\middle|\mathcal{D}_I\right) = E\left(E\left(\sum_{k=1}^{N_{i,j}^{paid}} Y_{i,j}^{(k)}|N_{i,j}^{paid}\right)\middle|\mathcal{D}_I\right) = E\left(N_{i,j}^{paid}|\mathcal{D}_I\right)E\left(Y_{i,j}^{(k)}\right)\\
&= \gamma_i\mu\sum_{l=0}^{\min(j,d)} N_{i,j-l}^{report}\, p_l^*.
\end{aligned}
\tag{6}
$$

(2) The conditional variance $Var\left(X_{i,j}|\mathcal{D}_I\right)$ is

$$
\begin{aligned}
Var\left(X_{i,j}|\mathcal{D}_I\right) &= E\left(Var\left(X_{i,j}|N_{i,j}^{paid}\right)\middle|\mathcal{D}_I\right) + Var\left(E\left(X_{i,j}|N_{i,j}^{paid}\right)\middle|\mathcal{D}_I\right)\\
&= E\left(Var\left(\sum_{k=1}^{N_{i,j}^{paid}} Y_{i,j}^{(k)}|N_{i,j}^{paid}\right)\middle|\mathcal{D}_I\right) + Var\left(E\left(\sum_{k=1}^{N_{i,j}^{paid}} Y_{i,j}^{(k)}|N_{i,j}^{paid}\right)\middle|\mathcal{D}_I\right)\\
&= E\left(N_{i,j}^{paid}|\mathcal{D}_I\right)Var\left(Y_{i,j}^{(1)}\right) + Var\left(N_{i,j}^{paid}|\mathcal{D}_I\right)\left[E\left(Y_{i,j}^{(1)}\right)\right]^2\\
&= \sum_{l=0}^{\min(j,d)} N_{i,j-l}^{report}\, p_l^*\gamma_i^2\sigma^2 + \sum_{l=0}^{\min(j,d)} N_{i,j-l}^{report}\, p_l^*\left(1-p_l^*\right)\gamma_i^2\mu^2\\
&= \sum_{l=0}^{\min(j,d)} N_{i,j-l}^{report}\, p_l^*\gamma_i^2\mu^2\left[\eta^2 + \left(1-p_l^*\right)\right],
\end{aligned}
\tag{7}
$$

where η stands for coefficient of variation for random variable $Y_{i,j}^{(k)}/\gamma_i$.

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Under an LCR reinsurance treaty and Model Assumption 1, the following theorem develops the best prediction for the cedent's (and reinsurer's) portion for random claim $X_{i,j}$.

Theorem 1. *Suppose* $X_{i,j}^{In}(\text{LCR}(r))$ *(resp.* $X_{i,j}^{Re}(\text{LCR}(r))$*) stands for the cedent's (resp. reinsurer's) share portion for random claim* $X_{i,j}$*, under an LCR reinsurance treaty which recovers just the r largest claims, say LCR(r). Then, under Model Assumption 1 and the LCR(r) treaty, given the information in* $\mathcal{D}_I$*:*

(1) *The best prediction for* $X_{i,j}^{In}(\text{LCR}(r))$*, say* $\hat{X}_{i,j}^{In}(\text{LCR}(r))$*, and its corresponding conditional mean square error, respectively, are*

$$E\Big(X_{i,j}^{In}(\text{LCR}(r))|\mathcal{D}_I\Big) = \gamma_i\mu \sum_{l=0}^{\min(j,d)} N_{i,j-l}^{report} p_l^* - \sum_{m=1}^{r} \frac{1}{\Gamma(m)} \int_0^1 F^{-1}(v)\,[1-v]^{m-1} g_\xi(v,m)dv$$

$$\begin{aligned} MSEP_{\mathcal{D}_I}\Big(X_{i,j}^{In}(\text{LCR}(r)), \hat{X}_{i,j}^{In}(\text{LCR}(r))\Big) = & \sum_{l=0}^{\min(j,d)} N_{i,j-l}^{report} p_l^* \gamma_i^2\mu^2\left(\eta^2 + \left(1-p_l^*\right)\right) + \sigma_{LCR(r)}^2 \\ & -2\sum_{l=0}^{\min(j,d)} N_{i,j-l}^{report} p_l^* \sum_{m=1}^{r}\left[\sum_{k\geq m} P\big(\xi_{i,j}=k\big)\sum_{h=1}^{3} T_h(k;m)\right] \\ & +2\sum_{l=0}^{\min(j,d)} N_{i,j-l}^{report} p_l^* \sum_{m=1}^{r}\left[\sum_{k\geq m} P\big(\xi_{i,j-1}=k\big)\sum_{h=1}^{3} T_h(k;m)\right] \\ & +2\gamma_i\mu\sum_{l=0}^{\min(j,d)} N_{i,j-l}^{report} p_l^* \Bigg[\sum_{m=1}^{r}\frac{1}{\Gamma(m)}\int_0^1 F^{-1}(v) \\ & \times [1-v]^{m-1} g_\xi(v,m)dv\Bigg]. \end{aligned}$$

(2) *The best prediction for* $X_{i,j}^{Re}(\text{LCR}(r))$*, say* $\hat{X}_{i,j}^{Re}(\text{LCR}(r))$*, and its corresponding conditional mean square error, respectively, are*

$$E\Big(X_{i,j}^{Re}(\text{LCR}(r))|\mathcal{D}_I\Big) = \sum_{m=1}^{r} \frac{1}{\Gamma(m)} \int_0^1 F^{-1}(v)\,[1-v]^{m-1} g_\xi(v,m)dv$$

$$MSEP_{\mathcal{D}_I}\Big(X_{i,j}^{Re}(\text{LCR}(r)), \hat{X}_{i,j}^{Re}(\text{LCR}(r))\Big) = \sigma_{LCR(r)}^2$$

where $T_h(\cdot\,;\,\cdot)$, $h=1,2,3$, *are given in* Equation (3), $\xi_{i,j}=\sum_{k=0}^{j} N_{i,k}^{paid}$, $g_\xi(v,m)=\psi_{\xi_{i,j}}^{(m)}(v)-\psi_{\xi_{i,j-1}}^{(m)}(v)$ *and* $\sigma_{\text{LCR}(r)}^2$ *stands for* $Var\Big(X_{i,j}^{Re}(\text{LCR}(r))|\mathcal{D}_I\Big)$ *which is given by Lemma 1 in the Appendix.*

Proof. For the first section of Part (1) observe that $E\Big(X_{i,j}^{In}(\text{LCR}(r))|\mathcal{D}_I\Big)\Big) = E\big(X_{i,j}|\mathcal{D}_I\big) - E\Big(X_{i,j}^{Re}(\text{LCR}(r))|\mathcal{D}_I\Big)$. The first expectation has been given by Equation (6). The second expectation may be restated as

$$E\left(X_{i,j}^{Re}(\text{LCR(r)})|\mathcal{D}_I\right) = E\left(\sum_{m=1}^{r} Y_{i(\xi_{i,j}-m+1)}|\mathcal{D}_I\right) - E\left(\sum_{m=1}^{r} Y_{i(\xi_{i,j-1}-m+1)}\right)|\mathcal{D}_I\Big)$$
$$= \sum_{m=1}^{r} \frac{1}{\Gamma(m)} \int_0^1 F^{-1}(v)\,[1-v]^{m-1}\,\psi_{\xi_{i,j}}^{(m)}(v)dv$$
$$- \sum_{m=1}^{r} \frac{1}{\Gamma(m)} \int_0^1 F^{-1}(v)\,[1-v]^{m-1}\,\psi_{\xi_{i,j-1}}^{(m)}(v)dv,$$

where the second equation arrives from the application of Equation (1).

For the second section of Part (1), one may decompose the conditional MSEP as

$$MSEP_{\mathcal{D}_I}\left(X_{i,j}^{In}(\text{LCR(r)}), \hat{X}_{i,j}^{In}(\text{LCR(r)})\right) = Var\left(X_{i,j}^{In}(\text{LCR(r)})|\mathcal{D}_I\right)$$
$$+E\left[\left(\hat{X}_{i,j}^{In}(\text{LCR(r)}) - E\left(X_{i,j}^{In}(\text{LCR(r)})|\mathcal{D}_I\right)\right)^2 |\mathcal{D}_I\right]$$
$$= Var\left(X_{i,j}^{In}(\text{LCR(r)})|\mathcal{D}_I\right) + \left(\hat{X}_{i,j}^{In}(\text{LCR(r)}) - E\left(X_{i,j}^{In}(\text{LCR(r)})|\mathcal{D}_I\right)\right)^2$$
$$= Var\left(X_{i,j}^{In}(\text{LCR(r)})|\mathcal{D}_I\right)$$

where the second expression obtained from the fact that both $\hat{X}_{i,j}^{In}(\text{LCR(r)})$ and $E\left(X_{i,j}^{In}(\text{LCR(r)})|\mathcal{D}_I\right)$ are $\mathcal{D}_I$-measurable and the third expression obtained from $\hat{X}_{i,j}^{In}(\text{LCR(r)}) = E\left(X_{i,j}^{In}(\text{LCR(r)})|\mathcal{D}_I\right)$.

Now, observe that

$$Var\left(X_{i,j}^{In}(\text{LCR(r)})|\mathcal{D}_I\right) = Var(X_{i,j}|\mathcal{D}_I) + Var\left(X_{i,j}^{Re}(\text{LCR(r)})|\mathcal{D}_I\right) - 2Cov\left(X_{i,j}, X_{i,j}^{Re}(\text{LCR(r)})|\mathcal{D}_I\right).$$

The conditional variance $Var(X_{i,j}|\mathcal{D}_I)$ has been given by Equation (7), and other terms are given by Lemma 1, in the Appendix.

Proof of Part (2) is similar. □

The following theorem develops the best prediction for the cedent's (and reinsurer's) portion for random claim $X_{i,j}$, under an ECOMOR reinsurance treaty and Model Assumption 1.

Theorem 2. *Suppose $X_{i,j}^{In}(\text{ECOMOR(r)})$ (resp. $X_{i,j}^{Re}(\text{ECOMOR(r)})$) stands for the cedent's (resp. reinsurer's) share portion for random claim $X_{i,j}$, under an ECOMOR reinsurance treaty which recovers just the r largest claims, say ECOMOR(r). Then, under Model Assumption 1 and the ECOMOR(r) treaty, given the information in $\mathcal{D}_I$:*

(1) The best prediction for $X_{i,j}^{In}(\text{ECOMOR(r)})$, say $\hat{X}_{i,j}^{In}(\text{ECOMOR(r)})$, and its corresponding conditional mean square error, respectively, are

$$E\left(X_{i,j}^{In}(\text{ECOMOR(r)})|\mathcal{D}_I\right) = \gamma_i\mu \sum_{l=0}^{\min(j,d)} N_{i,j-l}^{report}\, p_l^*$$
$$- \sum_{m=1}^{r} \frac{1}{\Gamma(m)} \int_0^1 F^{-1}(v)\,[1-v]^{m-1}\,g_\xi(v,m)dv$$
$$+ \frac{r}{\Gamma(r)} \int_0^1 F^{-1}(v)\,[1-v]^{r-1}\,g_\xi(v,r)dv$$

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$$MSEP_{\mathcal{D}_I}\left(X^{In}_{i,j}(\text{ECOMOR}(r)), \hat{X}^{In}_{i,j}(\text{ECOMOR}(r))\right)$$
$$= \sum_{l=0}^{\min(j,d)} N^{report}_{i,j-l} p^*_l \gamma_i^2 \mu^2 \left(\eta^2 + (1-p^*_l)\right) + \sigma^2_{\text{ECOMOR}(r)}$$
$$-2 \sum_{l=0}^{\min(j,d)} N^{report}_{i,j-l} p^*_l \sum_{m=1}^{r} \left[P(\xi_{i,j}=k) - P(\xi_{i,j-1}=k)\right] \sum_{h=1}^{3} W_h(k;m)$$
$$+2\gamma_i\mu \sum_{l=0}^{\min(j,d)} N^{report}_{i,j-l} p^*_l \sum_{m=1}^{r} \frac{1}{\Gamma(m)} \int_0^1 F^{-1}(v)\,[1-v]^{m-1} g_\xi(v,m)dv$$
$$+2\gamma_i\mu \sum_{l=0}^{\min(j,d)} N^{report}_{i,j-l} p^*_l \frac{r}{\Gamma(r)} \int_0^1 F^{-1}(v)\,[1-v]^{r-1} g_\xi(r,m)(v)dv,$$

(2) *The best prediction for* $X^{Re}_{i,j}(\text{LCR}(r))$*, say* $\hat{X}^{Re}_{i,j}(\text{LCR}(r))$*, and its corresponding conditional mean square error, respectively, are*

$$E\left(X^{Re}_{i,j}(\text{ECOMOR}(r))|\mathcal{D}_I\right) = \sum_{m=1}^{r} \frac{1}{\Gamma(m)} \int_0^1 F^{-1}(v)\,[1-v]^{m-1} g_\xi(v,m)dv$$
$$-\frac{r}{\Gamma(r)} \int_0^1 F^{-1}(v)\,[1-v]^{r-1} g_\xi(v,r)dv$$
$$MSEP_{\mathcal{D}_I}\left(X^{Re}_{i,j}(\text{ECOMOR}(r)), \hat{X}^{Re}_{i,j}(\text{ECOMOR}(r))\right) = \sigma^2_{\text{ECOMOR}(r)},$$

where $T_h(\cdot;\cdot)$, $h=1,2,3$, *are given in* Equation (3), $\xi_{i,j} = \sum_{k=0}^{j} N^{paid}_{i,k}$, $g_\xi(v,m) = \psi^{(m)}_{\xi_{i,j}}(v) - \psi^{(m)}_{\xi_{i,j-1}}(v)$, $\sigma^2_{\text{ECOMOR(r)}}$ *stands for* $Var\left(X^{Re}_{i,j}(\text{ECOMOR(r)})|\mathcal{D}_I\right)$ *which is given by Lemma 2 in the Appendix and* $W_h(k;m) = T_h(k;m) - T_h(k;r)$.

Proof. For the first section of Part (1) observe that $E\left(X^{In}_{i,j}(\text{ECOMOR(r)})|\mathcal{D}_I)\right) = E(X_{i,j}|\mathcal{D}_I) - E\left(X^{Re}_{i,j}(\text{ECOMOR(r)})|\mathcal{D}_I\right)$. The first expectation has been given by Equation (6). A similar argument as provided in proof of Part (1) (Theorem, 1) leads to

$$E\left(X^{Re}_{i,j}(\text{ECOMOR(r)})|\mathcal{D}_I\right) = E\left(\sum_{m=1}^{r} Y_{i(\xi_{i,j}-m+1)} - rY_{i(\xi_{i,j}-r+1)}\right.$$
$$\left. - \sum_{m=1}^{r} Y_{i(\xi_{i,j-1}-m+1)} + rY_{i(\xi_{i,j-1}-r+1)}|\mathcal{D}_I\right)$$
$$= \sum_{m=1}^{r} E\left(Y_{i(\xi_{i,j}-m+1)} - rY_{i(\xi_{i,j}-r+1)}\right)$$
$$- \sum_{m=1}^{r} E\left(Y_{i(\xi_{i,j-1}-m+1)} - rY_{i(\xi_{i,j-1}-r+1)}\right)$$

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$$
\begin{aligned}
&= \sum_{m=1}^{r} \frac{1}{\Gamma(m)} \int_0^1 F^{-1}(v)\,[1-v]^{m-1}\,\psi^{(m)}_{\xi_{i,j}}(v)dv \\
&\quad - \frac{r}{\Gamma(r)} \int_0^1 F^{-1}(v)\,[1-v]^{r-1}\,\psi^{(r)}_{\xi_{i,j}}(v)dv \\
&\quad - \sum_{m=1}^{r} \frac{1}{\Gamma(m)} \int_0^1 F^{-1}(v)\,[1-v]^{m-1}\,\psi^{(m)}_{\xi_{i,j-1}}(v)dv \\
&\quad + \frac{r}{\Gamma(r)} \int_0^1 F^{-1}(v)\,[1-v]^{r-1}\,\psi^{(r)}_{\xi_{i,j-1}}(v)dv
\end{aligned}
$$

For Part (2), similar proof of Part (2), in Theorem 2, the conditional MSEP can be restated as

$$
\begin{aligned}
MSEP_{\mathcal{D}_I}\left(X^{In}_{i,j}(\text{ECOMOR(r)}), \hat{X}^{In}_{i,j}(\text{ECOMOR(r)})\right) &= Var\Big(X^{In}_{i,j}(\text{ECOMOR(r)})|\mathcal{D}_I\Big) \\
&\quad + \Big(\hat{X}^{In}_{i,j}(\text{ECOMOR(r)}) \\
&\quad - E\Big(X^{In}_{i,j}(\text{ECOMOR(r)})|\mathcal{D}_I\Big)\Big)^2 \\
&= Var\Big(X^{In}_{i,j}(\text{ECOMOR(r)})|\mathcal{D}_I\Big)
\end{aligned}
$$

where the second expression obtained from the fact that both $\hat{X}^{In}_{i,j}(\text{ECOMOR(r)})$ and $E\Big(X^{In}_{i,j}(\text{ECOMOR(r)})|\mathcal{D}_I\Big)$ are $\mathcal{D}_I$-measurable and the third expression obtained from $\hat{X}^{In}_{i,j}(\text{ECOMOR(r)}) = E\Big(X^{In}_{i,j}(\text{ECOMOR(r)})|\mathcal{D}_I\Big)$. Now, observe that

$$
\begin{aligned}
Var\Big(X^{In}_{i,j}(\text{ECOMOR(r)})|\mathcal{D}_I\Big) &= Var\big(X_{i,j}|\mathcal{D}_I\big) + Var\Big(X^{Re}_{i,j}(\text{ECOMOR(r)})|\mathcal{D}_I\Big) \\
&\quad - 2Cov\Big(X_{i,j}, X^{Re}_{i,j}(\text{ECOMOR(r)})|\mathcal{D}_I\Big).
\end{aligned}
$$

The conditional variance $Var\big(X_{i,j}|\mathcal{D}_I\big)$ has been given by Equation (7), and other terms are given by Lemma 2, in the Appendix.

4. Simulation Study

This section develops a simulation study to (**1**) show practical application of the findings and (**2**) make a comparison between two reinsurance treaties.

Several authors developed some simulation algorithms to simulate a full IBNR table, see Stanard (1985), Bühlmann *et al.* (1980), Vaughan (1998), Narayan & Warthen (2000), Schiegl (2002), Stelljes (2006), among others, for more details. For the simulation study, Schiegl's (2002) simulation algorithm was adjusted, and the following numerical procedure was employed.

Numerical Procedure 1. *Using the following three steps, implement a numerical study:*

- ***Step 1)*** *Using Algorithm 1 simulates $M = 10,000$ full IBNR tables;*
- ***Step 2)*** *For each simulated full IBNR table, remove the lower triangle, say observe runoff triangle, and using Algorithm 2 (resp. Algorithm 3) to predict $X^{In}_{i,j}(LCR(r))$ (resp. $X^{In}_{i,j}(ECOMOR(r))$) for the runoff triangle.*
- ***Step 3)*** *Evaluate the Mean Square Error of Prediction, say MSEP, using the observation and the predicted runoff triangles.*

Algorithm 1 shows how $N^{report}_{i,j}$, $Y^{(k)}_{i,j}$ and consequently $X_{i,j}$ are simulated regardless of the cedent and reinsurer portions, in an IBNR table.

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Algorithm 1: Generate a Full IBNR table which contains some additional information.

Input: Number of IBNR's row/column I, parameters $(\alpha_i, \beta_j, p_l^*, \gamma_i)$ as well as distributional parameters for the single payment $Y_{i,j}^{(k)}$.

Output: A full IBNR table which contains information about $N_{i,j}^{report}$, $Y_{i,j}^{(k)}$ and $X_{i,j}$

Set $i \leftarrow 1$;
while $i \leq I$ **do**
  Use the Poisson distribution (with intensity α_i) to generate the number of claims for the accident year i, and call it N_i;
  for $j \leftarrow 0$ ***to*** $I - 1$ **do**
    Using the Multinomial distribution with parameters $(N_i, \beta_0, \cdots, \beta_{I-1})$, to generate vector $(N_{i,0}^{report}, \cdots, N_{i,j}^{report})'$;
    for $l \leftarrow 0$ ***to*** d **do**
      Use the Multinomial distribution with parameters $(N_{i,j}^{report}, p_0^*, \cdots, p_d^*)$, to generate vector $(N_{i,j-0,0}^{paid}, \cdots, N_{i,j-d,d}^{paid})'$;
      Set $N_{i,j}^{paid} = \sum_{l=0}^{min(j,d)} N_{i,j-l,l}^{paid}$;
    for $k \leftarrow 1$ ***to*** $N_{i,j}^{paid}$ **do**
      Use distribution of the single payment to generate single payments $Y_{i,j}^{(k)}$;
      Set $X_{i,j} = \sum_{k=1}^{N_{i,j}^{paid}} Y_{i,j}^{(k)}$;
  Set $i \leftarrow i + 1$.

The Cedent's portion for outstanding claims under the LCR(r) treaty has been given by Algorithm 2. The Cedent's portion for outstanding claims under the ECOMOR(r) treaty has been given by Algorithm 2.

Before providing some examples, the following definition is recalled.

Definition 3. *The incomplete gamma function is defined by* $\Gamma(a, t) = \int_0^t e^{-z} z^{a-1} dz$.

It is known that the regular gamma function can be concluded by $\Gamma(a) = \Gamma(a, \infty)$, the gamma function. Moreover,

$$\sum_{m=1}^{r} \frac{\Gamma(m+b, z)}{\Gamma(m)} = \frac{r\Gamma(b+r+1, z)}{\Gamma(r+1)}. \tag{8}$$

The (Type I) Pareto distribution has a considerable application in a wide range of sciences, including social, actuarial, and financial sciences. The Pareto distribution is characterised by its scale parameter τ and tail index θ.

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Algorithm 2: $X_{i,j}^{In}(LCR(r))$: Cedent's portion for outstanding claims under the LCR(r) treaty.

Input: Single payments $Y_{i,j}^{(k)}$; number of payments $N_{i,j}^{paid}$ and number of largest claims cover by an LCR treaty , r.

Output: $X_{i,j}^{In}(LCR(r))$

Set $i \leftarrow 1$, $j \leftarrow 0$ and

while $i \leq I$ **do**

  Set $\xi_{i,0} = N_{i,0}^{paid}$;

  $X_{i,0}^{In}(LCR(r)) \leftarrow \sum_{k=1}^{\xi_{i,0}} Y_{i,0}^{(k)} - \sum_{m=1}^{r} Y_{i(\xi_{i,0}-m+1)}$;

  while $j \leq I-1$ **do**

    Set $j \leftarrow j+1$ and $\xi_{i,j} \leftarrow \xi_{i,j-1} + N_{i,j}^{paid}$;

    If $\xi_{i,j} \leq r$, then $X_{i,j}^{In}(LCR(r)) \leftarrow 0$;

    Otherwise, $X_{i,j}^{In}(LCR(r)) \leftarrow \sum_{k=1}^{N_{i,j}^{paid}} Y_{i,j}^{(k)} - \left[\sum_{m=1}^{r} Y_{i(\xi_{i,j}-m+1)} - \sum_{m=1}^{r} Y_{i(\xi_{i,j-1}-m+1)}\right]$;

  Set $i \leftarrow i+1$.

In the following, the Numerical Procedure 1 is employed against the Pareto distribution.

Example 1. Suppose for all $i = 1, 2, \cdots, I$, $j = 0, 1, \cdots, I-1$ and $k = 1, 2, \cdots, N_{i,j}^{paid}$, the individual discounted payments, $Y_{i,j}^{(k)}/\gamma_i$, are mutually independent with common Pareto distribution (with parameters θ and τ), where γ_i stands for an inflation index in accident year i. In other words, $P\big(T_{i,j} \leq t_{ij}\big) = 1 - (t_{ij}/\tau)^{-\theta}$, for $t_{ij} > \tau$, $E\Big(Y_{i,j}^{(k)}\Big) = \gamma_i \theta \tau/(\theta - 1)$ (for $\theta > 1$) and $Var\Big(Y_{i,j}^{(k)}\Big) = \theta \tau^2 \gamma_i^2 / \big((\theta-1)^2(\theta-2)\big)$ (for $\theta > 2$), where $T_{i,j} = Y_{i,j}^{(k)}/\gamma_i$.

Under this distributional assumption, given by this example, the results of the Theorems 1 and (2) may be simplified as follows.

$$E\Big(X_{i,j}^{In}(\text{LCR(r)})|\mathcal{D}_I\Big) = \frac{\theta\tau\gamma_i}{\theta-1} \sum_{l=0}^{\min(j,d)} N_{i,j-l}^{report} p_l^* - \left[\sum_{m=1}^{r} M_1\left(\phi_{i,j}, m\right) - \sum_{m=1}^{r} M_1\left(\phi_{i,j-1}, m\right)\right]$$

$$E\Big(X_{i,j}^{Re}(\text{LCR(r)})|\mathcal{D}_I\Big) = \sum_{m=1}^{r} M_1\left(\phi_{i,j}, m\right) - \sum_{m=1}^{r} M_1\left(\phi_{i,j-1}, m\right)$$

$$\begin{aligned} MSEP_{\mathcal{D}_I}\Big(X_{i,j}^{In}(\text{LCR(r)}), \hat{X}_{i,j}^{In}(\text{LCR(r)})\Big) = & \sum_{l=0}^{\min(j,d)} N_{i,j-l}^{report} p_l^* \left(\frac{\theta\tau\gamma_i}{\theta-1}\right)^2 \left[\frac{1}{\theta(\theta-2)} + 1 - p_l^*\right] + \sigma^2_{\text{LCR(r)}} \\ & - 2 \sum_{l=0}^{\min(j,d)} N_{i,j-l}^{report} p_l^* \sum_{m=1}^{r} \sum_{k=m+1}^{N_i} \left[P\big(\xi_{i,j} = k\big) - P\big(\xi_{i,j-1} = k\big)\right] \sum_{h=1}^{3} T_h(m;k) \\ & + 2\frac{\theta\tau\gamma_i}{\theta-1} \sum_{l=0}^{\min(j,d)} N_{i,j-l}^{report} p_l^* \left[\sum_{m=1}^{r} M_1\left(\phi_{i,j}, m\right) - \sum_{m=1}^{r} M_1\left(\phi_{i,j-1}, m\right)\right] \end{aligned}$$

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Algorithm 3: $X^{In}_{i,j}(ECOMOR(r))$: Cedent's portion for outstanding claims under the ECOMOR(r) treaty.

Input: Single payments $Y^{(k)}_{i,j}$; number of payments $N^{paid}_{i,j}$ and number of largest claims cover by an ECOMOR treaty , r.

Output: $X^{In}_{i,j}(ECOMOR(r))$

Set $i \leftarrow 1$, $j \leftarrow 0$ and

while $i \leq I$ **do**

  Set $\xi_{i,0} = N^{paid}_{i,0}$;

  $X^{In}_{i,0}(ECOMOR(r)) \leftarrow \sum_{k=1}^{\xi_{i,0}} Y^{(k)}_{i,j} - (\sum_{m=1}^{r} Y_{i(\xi_{i,0}-m+1)} - rY_{i(\xi_{i,0}-r+1)})$;

  while $j \leq I-1$ **do**

    Set $j \leftarrow j+1$ and $\xi_{i,j} \leftarrow \xi_{i,j-1} + N^{paid}_{i,j}$;

    If $\xi_{i,j} \leq r$, then $X^{In}_{i,j}(ECOMOR(r)) \leftarrow 0$;

    Otherwise,

    $X^{In}_{i,j}(ECOMOR(r)) \leftarrow \sum_{k=1}^{N^{paid}_{i,j}} Y^{(k)}_{i,j} - \left[\sum_{m=1}^{r} \left(Y_{i(\xi_{i,j}-m+1)} - Y_{i(\xi_{i,j-1}-m+1)}\right) + r\left(Y_{i(\xi_{i,j}-r+1)} - Y_{i(\xi_{i,j-1}-r+1)}\right)\right]$;

  Set $i \leftarrow i+1$.

$$
\begin{aligned}
MSEP_{\mathcal{D}_I}\left(X^{Re}_{i,j}(\mathrm{LCR(r)}), \hat{X}^{Re}_{i,j}(\mathrm{LCR(r)})\right) &= \sum_{m=1}^{r} M_2\left(\phi_{i,j}, m\right) - \left(\sum_{m=1}^{r} M_1\left(\phi_{i,j}, m\right)\right)^2 + \sum_{m=1}^{r} M_2\left(\phi_{i,j-1}, m\right) \\
&\quad - \left(\sum_{m=1}^{r} M_1\left(\phi_{i,j-1}, m\right)\right)^2 - 2r\sum_{m=1}^{r}\sum_{k=m+1}^{N_i} P\left(\xi_{i,j}=k\right)\sum_{h=1}^{3} T_h(m;k) \\
&\quad + 2\sum_{m=1}^{r} M_1\left(\phi_{i,j}, m\right)\sum_{m=1}^{r} M_1\left(\phi_{i,j-1}, m\right) =: \sigma^2_{\mathrm{LCR(r)}}
\end{aligned}
$$

and

$$
\begin{aligned}
E\left(X^{In}_{i,j}(\mathrm{ECOMOR(r)})|\mathcal{D}_I\right) &= \frac{\theta\tau\gamma_i}{\theta-1}\sum_{l=0}^{\min(j,d)} N^{report}_{i,j-l}\, p^*_l - \left[\sum_{m=1}^{r} M_1\left(\phi_{i,j}, m\right) - \sum_{m=1}^{r} U_1\left(\phi_{i,j}, m\right)\right] \\
&\quad + \left[\sum_{m=1}^{r} M_1\left(\phi_{i,j-1}, m\right) - \sum_{m=1}^{r} U_1\left(\phi_{i,j-1}, m\right)\right]
\end{aligned}
$$

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$$E\Big(X_{i,j}^{Re}(\text{ECOMOR(r)})|\mathcal{D}_I\Big) = \left[\sum_{m=1}^{r} M_1\left(\phi_{i,j}, m\right) - \sum_{m=1}^{r} U_1\left(\phi_{i,j}, m\right)\right] - \left[\sum_{m=1}^{r} M_1\left(\phi_{i,j-1}, m\right) - \sum_{m=1}^{r} U_1\left(\phi_{i,j-1}, m\right)\right]$$

$$\begin{aligned}
MSEP_{\mathcal{D}_I}\left(X_{i,j}^{In}(\text{ECOMOR(r)}), \hat{X}_{i,j}^{In}(\text{ECOMOR(r)})\right) &= \sum_{l=0}^{\min(j,d)} N_{i,j-l}^{report} p_l^* \left(\frac{\theta\tau\gamma_i}{\theta-1}\right)^2 \left[\frac{1}{\theta(\theta-2)} + 1 - p_l^*\right] \\
&+ \sigma^2_{\text{ECOMOR(r)}} \\
&- 2\sum_{l=0}^{\min(j,d)} N_{i,j-l}^{report} p_l^* \sum_{m=1}^{r} \sum_{k_1=m+1}^{N_i} P(\xi_{i,j}=k_1) \sum_{h=1}^{3} T_h(k_1;m) \\
&+ 2\sum_{l=0}^{\min(j,d)} N_{i,j-l}^{report} p_l^* \sum_{k_2=r+1}^{N_i} P(\xi_{i,j-1}=k_2) \sum_{h=1}^{3} T_h(k_2;r) \\
&+ 2\frac{\theta\tau\gamma_i}{\theta-1}\left[\sum_{m=1}^{r} M_1\left(\phi_{i,j}, m\right) - \sum_{m=1}^{r} U_1\left(\phi_{i,j}, m\right)\right] \\
&+ 2\frac{\theta\tau\gamma_i}{\theta-1}\left[\sum_{m=1}^{r} M_1\left(\phi_{i,j-1}, m\right) - \sum_{m=1}^{r} U_1\left(\phi_{i,j-1}, m\right)\right],
\end{aligned}$$

$$\begin{aligned}
MSEP_{\mathcal{D}_I}\left(X_{i,j}^{Re}(\text{ECOMOR(r)}), \hat{X}_{i,j}^{Re}(\text{ECOMOR(r)})\right) &= \sigma^2_{\text{LCR(r)}} + r^2 L_2\left(\phi_{i,j}, r\right) - r^2\left(L_1\left(\phi_{i,j}, r\right)\right)^2 \\
&+ r^2 L_2\left(\phi_{i,j-1}, r\right) - r^2\left(L_1\left(\phi_{i,j-1}, r\right)\right)^2 \\
&+ 2r^2 L_1\left(\phi_{i,j}, r\right) L_1\left(\phi_{i,j-1}, r\right) - 2rL_2\left(\phi_{i,j}, r\right) \\
&\quad - 2rL_2\left(\phi_{i,j-1}, r\right) \\
&+ 2r\sum_{m=1}^{r} \sum_{k=m+1}^{N_i} P\big(\xi_{i,j}=k\big) \sum_{h=1}^{3} T_h(k;m) \\
&+ 2rE\Big(X_{i,j}^{Re}(\text{LCR(r)})|\mathcal{D}_I\Big) L_1\left(\phi_{i,j}, r\right) \\
&\quad - 2rE\Big(X_{i,j}^{Re}(\text{LCR(r)})|\mathcal{D}_I\Big) L_1\left(\phi_{i,j-1}, r\right) \\
&=: \sigma^2_{\text{ECOMOR(r)}}
\end{aligned}$$

where $\phi_{i,j} = E(\xi_{i,j})$ and

$$\sum_{m=1}^{r} M_\nu(a,m) = (\tau\gamma_i)^\nu (a)^{\frac{\nu}{\theta}} \frac{r\Gamma(r-\frac{\nu}{\theta}+1, a)}{\Gamma(r+1)}$$

$$\sum_{m=1}^{r} U_\nu(a,m) = (\tau\gamma_i)^\nu (a)^{\frac{\nu}{\theta}} \frac{r\Gamma(r-\frac{\nu}{\theta}, a)}{\Gamma(r)}$$

$$L_\nu(a,b) = \frac{a^b}{\Gamma(b)} \int_y y^\nu f(y)(1-F(y))^{b-1} e^{-a(1-F(y))} dy.$$

Now, the Numerical Procedure 1 is employed with parameters given by Table 2, to (**1**) predict outstanding claims $X_{i,j}^{In}(\text{LCR(r)})$ (resp. $X_{i,j}^{In}(ECOMOR(r))$) and (**2**) compare these two reinsurance treaties. Tables 3 and 4, respectively, present the prediction of outstanding claims

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Table 2. Simulation parameters.

α_i	120	118	96	85	79	75	64	60	58	54
β_j	0.63	0.18	0.07	0.03	0.03	0.02	0.01	0.01	0.02	0.01
γ_i	1	0.77	0.73	0.89	0.78	0.78	0.66	0.74	0.70	0.82
p_l	0.7	0.1	0.05	0.05	0.03	0.03	0.02	0.02	–	–

Table 3. Mean of loss reserve net of LCR(r) treaty for Pareto distribution with parameters θ and τ.

	$\tau=1$				$\tau=2$			
	$\theta=3$		$\theta=4$		$\theta=3$		$\theta=4$	
LCR(r)	Reserve	MSEP	Reserve	MSEP	Reserve	MSEP	Reserve	MSEP
LCR(6)	118	9,311	107	4,918	238	38,058	217	18,811
LCR(8)	117	12,951	106	7,265	236	53,122	216	27,507
LCR(10)	116	17,128	106	10,019	234	70,428	215	37,665
LCR(15)	114	29,790	104	18,598	229	122,926	211	69,169

Table 4. Mean of loss reserve net of ECOMOR(r) treaty for Pareto distribution with parameters θ and τ.

	$\tau=1$				$\tau=2$			
	$\theta=3$		$\theta=4$		$\theta=3$		$\theta=4$	
ECOMOR (r)	Reserve	MSEP	Reserve	MSEP	Reserve	MSEP	Reserve	MSEP
ECOMOR(6)	122	24,551	109	13,774	246	110,869	222	62,282
ECOMOR(8)	122	36,195	109	21,505	245	165,116	222	97,604
ECOMOR(10)	121	49,122	109	30378	244	225,882	221	138,360
ECOMOR(15)	120	86,288	109	56,931	243	402,664	220	261,238

$X^{In}_{i,j}(\text{LCR(r)})$ (resp. $X^{In}_{i,j}(ECOMOR(r))$) and the MSEP for the LCR(r) and the ECOMOR(r) treaties for different r.

As Tables 3 and 4 show (**1**) for a fixed shape parameter τ, the reserve and the MSEP increase as the scale parameter increases, (**2**) for a fixed scale parameter θ, the reserve and the MSEP increase as the shape parameter increases, (**3**) the cedent's MSEP under the LCR(r) treaty is smaller than such amount under the ECOMOR(r), and (**4**) for both treaties, the amount of cedent's MSEP increases as the number of claims covered by the reinsurer increases.

The Fréchet distribution is an appropriate distribution to model "heavy tail" (or "fat tail") phenomena. The probability density function, for Fréchet distribution with the location parameter, μ^*, the scale parameter, σ^* and the shape parameter, α^* is

$$f(s)=\frac{\alpha^*}{\sigma^*}\left(\frac{s-\mu^*}{\sigma^*}\right)^{-1-\alpha^*}e^{\left(\frac{s-\mu^*}{\sigma^*}\right)^{-\alpha^*}}$$

Mean and variance are

$$\mu_{Frechet}=\mu^*+\sigma^*\Gamma\left(1-\frac{1}{\alpha^*}\right) \qquad (9)$$

$$\sigma^2_{Frechet}=\sigma^{*2}\left[\Gamma\left(1-\frac{2}{\alpha^*}\right)-\left(\Gamma\left(1-\frac{1}{\alpha^*}\right)\right)^2\right]$$

Application of the Numerical Procedure 1 whenever individual discounted payments are sampled from Fréchet distribution.

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Example 2. Suppose for all $i=1,2,\cdots,I$, $j=0,1,\cdots,I-1$ and $k=1,2,\cdots,N_{i,j}^{paid}$, the individual discounted payments, $\frac{Y_{i,j}^{(k)}}{\gamma_i}$, are mutually independent with common Fréchet distribution (with parameter μ^*, σ^* and α^*), where γ_i stands for an inflation index in accident year i. In other words, $E\left(Y_{i,j}^{(k)}\right)=\gamma_i\mu_{Frechet}$ and $Var\left(Y_{i,j}^{(k)}\right)=\gamma_i^2\sigma_{Frechet}^2$ where $\mu_{Frechet}$ and $\sigma_{Frechet}^2$ are given by Equation (9).

Under the above distributional assumption, the results of Theorems 1 and 2 can be simplified as

$$E\left(X_{i,j}^{In}(\text{LCR(r)})|\mathcal{D}_I\right)=\gamma_i\mu_{Frechet}\sum_{l=0}^{\min(j,d)}N_{i,j-l}^{report}p_l^*-\sum_{m=1}^{r}H_{Q_1}\left(\phi_{i,j},m\right)+\sum_{m=1}^{r}H_{Q_1}\left(\phi_{i,j-1},m\right)$$

$$E\left(X_{i,j}^{Re}(\text{LCR(r)})|\mathcal{D}_I\right)=\sum_{m=1}^{r}H_{Q_1}\left(\phi_{i,j},m\right)-\sum_{m=1}^{r}H_{Q_1}\left(\phi_{i,j-1},m\right)$$

$$\begin{aligned}MSEP_{\mathcal{D}_I}\left(X_{i,j}^{In}(\text{LCR(r)}),\hat{X}_{i,j}^{In}(\text{LCR(r)})\right)=&\sum_{l=}^{\min(j,d)}N_{i,j-l}^{report}p_l^*\gamma_i^2\mu_{Frechet}^2\left(\eta^2+\left(1-p_l^*\right)\right)+\sigma_{\text{LCR(r)}}^2\\&-2\sum_{l=}^{\min(j,d)}N_{i,j-l}^{report}p_l^*\sum_{m=1}^{r}\sum_{k=m+1}^{N_i}\left[P\left(\xi_{i,j}=k\right)\right.\\&\left.-P\left(\xi_{i,j-1}=k\right)\right]\sum_{h=1}^{3}T_h(k;m)\\&+2\left[\gamma_i\mu_{Frechet}\sum_{l=0}^{\min(j,d)}N_{i,j-l}^{report}p_l^*\right]\left[\sum_{m=1}^{r}H_{Q_1}\left(\phi_{i,j},m\right)\right.\\&\left.-\sum_{m=1}^{r}H_{Q_1}\left(\phi_{i,j-1},m\right)\right]\end{aligned}$$

$$\begin{aligned}MSEP_{\mathcal{D}_I}\left(X_{i,j}^{Re}(\text{LCR(r)}),\hat{X}_{i,j}^{Re}(\text{LCR(r)})\right)=&\sum_{m=1}^{r}H_{Q_2}\left(\phi_{i,j},m\right)-\left(\sum_{m=1}^{r}H_{Q_1}\left(\phi_{i,j},m\right)\right)^2\\&+\sum_{m=1}^{r}H_{Q_2}\left(\phi_{i,j-1},m\right)-\left(\sum_{m=1}^{r}H_{Q_1}\left(\phi_{i,j-1},m\right)\right)^2\\&-2r\sum_{m=1}^{r}\sum_{k=m+1}^{N_i}P\left(\xi_{i,j}=k\right)\sum_{h=1}^{3}T_h(k;m)\\&+2\sum_{m=1}^{r}H_{Q_1}\left(\phi_{i,j},m\right)\sum_{m=1}^{r}H_{Q_1}\left(\phi_{i,j-1},m\right)=:\sigma_{\text{LCR(r)}}^2\end{aligned}$$

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and

$$E\Big(X^{In}_{i,j}(\text{ECOMOR(r)})|\mathcal{D}_I\Big) = \gamma_i\mu_{Frechet}\sum_{l=0}^{\min(j,d)} N^{report}_{i,j-l}\,p^*_l - \sum_{m=1}^{r} H_{Q_1}\left(\phi_{i,j},m\right) + \sum_{m=1}^{r} H_{Q_1}\left(\phi_{i,j-1},m\right) + rH_{Q_1}\left(\phi_{i,j},r\right) - rH_{Q_1}\left(\phi_{i,j-1},r\right)$$

$$E\Big(X^{Re}_{i,j}(\text{ECOMOR(r)})|\mathcal{D}_I\Big) = \sum_{m=1}^{r} H_{Q_1}\left(\phi_{i,j},m\right) - \sum_{m=1}^{r} H_{Q_1}\left(\phi_{i,j-1},m\right) - rH_{Q_1}\left(\phi_{i,j},r\right) + rH_{Q_1}\left(\phi_{i,j-1},r\right)$$

$$\begin{aligned}MSEP_{\mathcal{D}_I}\left(X^{In}_{i,j}(\text{ECOMOR(r)}),\hat{X}^{In}_{i,j}(\text{ECOMOR(r)})\right) = &\sum_{l=}^{\min(j,d)} N^{report}_{i,j-l}\,p^*_l\gamma_i^2\mu^2_{Frechet}\left(\eta^2+\left(1-p^*_l\right)\right) + \sigma^2_{\text{ECOMOR(r)}}\\ &-2\sum_{l=0}^{\min(j,d)} N^{report}_{i,j-l}\,p^*_l\sum_{m=1}^{r}\sum_{k_1=m+1}^{N_i} P(\xi_{i,j}=k_1)\times\sum_{h=1}^{3} T_h(k_1;m)\\ &+2\sum_{l=0}^{\min(j,d)} N^{report}_{i,j-l}\,p^*_l\sum_{k_2=r+1}^{N_i} P(\xi_{i,j-1}=k_2)\times\sum_{h=1}^{3} T_h(k_2;r)\\ &+2\gamma_i\mu_{Frechet}\sum_{l=0}^{\min(j,d)} N^{report}_{i,j-l}\,p^*_l\left[\sum_{m=1}^{r} H_{Q_1}\left(\phi_{i,j},m\right) - \sum_{m=1}^{r} H_{Q_1}\left(\phi_{i,j-1},m\right)\right]\\ &-2r\gamma_i\mu_{Frechet}\sum_{l=0}^{\min(j,d)} N^{report}_{i,j-l}\,p^*_l\left[H_{Q_1}\left(\phi_{i,j},r\right) - H_{Q_1}\left(\phi_{i,j-1},r\right)\right]\end{aligned}$$

$$MSEP_{\mathcal{D}_I}\left(X^{Re}_{i,j}(\text{ECOMOR(r)}),\hat{X}^{Re}_{i,j}(\text{ECOMOR(r)})\right) = \sigma^2_{\text{LCR(r)}} + r^2H_{Q_2}\left(\phi_{i,j},r\right) - r^2H^2_{Q_1}\left(\phi_{i,j},r\right) + r^2H_{Q_2}\left(\phi_{i,j-1},r\right) - r^2H^2_{Q_1}\left(\phi_{i,j-1},r\right)$$

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Table 5. Mean of loss reserve net of LCR(r) treaty for Frechet distribution with parameters $\mu*, \sigma^*$ and α^*.

	$\mu^*=0, \sigma^*=1$				$\mu*=0, \sigma^*=2$			
	$\alpha^*=3$		$\alpha^*=4$		$\alpha^*=3$		$\alpha^*=4$	
ECOMOR (r)	Reserve	MSEP	Reserve	MSEP	Reserve	MSEP	Reserve	MSEP
LCR(6)	79	9,137	74	5,224	162	35,579	150	19,991
LCR(8)	78	12,420	73	7,605	159	48,148	149	28,838
LCR(10)	77	16,125	73	10,373	157	62,277	148	39,088
LCR(15)	75	27,143	71	18,849	152	104,141	144	71,420

Table 6. Mean of loss reserve net of ECOMOR(r) treaty for Frechet distribution with parameters $\mu*, \sigma^*$ and α^*.

	$\mu^*=0, \sigma^*=1$				$\mu*=0, \sigma^*=2$			
	$\alpha^*=3$		$\alpha^*=4$		$\alpha^*=3$		$\alpha^*=4$	
ECOMOR (r)	Reserve	MSEP	Reserve	MSEP	Reserve	MSEP	Reserve	MSEP
ECOMOR(6)	83	19,636	77	12,312	169	76,770	155	47,630
ECOMOR(8)	83	29,399	76	19,454	169	114,393	153	74,628
ECOMOR(10)	78	40,152	76	28,032	168	157,186	154	106,822
ECOMOR(15)	82	71,912	76	53,654	167	279,327	158	210,814

$$+2r^2 H_{Q_1}\left(\phi_{i,j}, r\right) H_{Q_1}\left(\phi_{i,j-1}, r\right) - 2rH_{Q_2}\left(\phi_{i,j}, r\right) - 2rH_{Q_2}\left(\phi_{i,j-1}, r\right)$$
$$+2r \sum_{m=1}^{r} \sum_{k=m+1}^{N_i} P\big(\xi_{i,j}=k\big) \sum_{h=1}^{3} T_h(k; m)$$
$$+2rE\Big(X_{i,j}^{In}(\text{LCR(r)})|\mathcal{D}_I\Big)\left(H_{Q_1}\left(\phi_{i,j}, r\right) - H_{Q_1}\left(\phi_{i,j-1}, r\right)\right) =: \sigma^2_{\text{ECOMOR(r)}}$$

where $H_{Q_\nu}(a,b) = \frac{a^b}{(b-1)!} Q_\nu(a,b)$ and $Q_\nu(a,b) = \int_y y^\nu f(y)(1-F(y))^{m-1} e^{-a(1-F(y))} dy$.

Now, the Numerical Procedure 1 is employed to (**1**) predict outstanding claims $X_{i,j}^{In}(\text{LCR(r)})$ (resp. $X_{i,j}^{In}(ECOMOR(r))$) and (**2**) compare these two reinsurance treaties. Tables 5 and 6, respectively, present prediction of outstanding claims $X_{i,j}^{In}(\text{LCR(r)})$ (resp. $X_{i,j}^{In}(ECOMOR(r))$) and the MSEP for the LCR(r) and the ECOMOR(r) treaties for different r. As Tables 4 and 5 show (**1**) for a fixed shape parameter, the reserve and the MSEP increase as the scale parameter increases, (**2**) for a fixed scale parameter, the reserve and the MSEP increase as the shape parameter increases, (**3**) the cedent's MSEP under the LCR(r) treaty is smaller than such amount under the ECOMOR(r), and (**4**) for both treaties, the amount of cedent's MSEP increases as the number of claims covered by the reinsurer increases.

The Weibull distribution is a continuous probability distribution that is an excellent candidate whenever large claims in the portfolio are addressed. The probability density function for a Weibull distribution with the scale parameter, λ^*, and the shape parameter, θ^*, is

$$f(s) = \frac{\theta^*}{\lambda^*}\left(\frac{s}{\lambda^*}\right)^{\theta^*-1} e^{-\left(\frac{s}{\lambda^*}\right)^{\theta^*}}, \quad \forall\, s \geq 0.$$

Moreover, mean and variance for such a Weibull distribution, respectively, are

$$\mu_W = \lambda^*\Gamma\left(1+\frac{1}{\theta^*}\right) \tag{10}$$
$$\sigma^2_W = \lambda^{*2}\left[\Gamma\left(1+\frac{2}{\theta^*}\right) - \Gamma^2\left(1+\frac{1}{\theta^*}\right)\right].$$

In the following, the Numerical Procedure 1 is employed against the Weibull distribution.

Example 3. Suppose for all $i = 1, 2, \cdots, I, j = 0, 1, \cdots, I-1$ and $k = 1, 2, \cdots, N^{paid}_{i,j}$, the individual discounted payments, $Y^{(k)}_{i,j}/\gamma_i$, are mutually independent with common Weibull distribution (with parameter λ^* and θ^*), where γ_i stands for an inflation index in accident year i.

Under the above distributional assumption,the results of Theorems 1 and 2 can be simplified as

$$E\left(X^{In}_{i,j}(\text{LCR(r)})|\mathcal{D}_I\right) = \gamma_i\mu_W \sum_{l=0}^{\min(j,d)} N^{report}_{i,j-l}\, p^*_l - \sum_{m=1}^{r}\left(G_1\left(\phi_{i,j}, m\right) - G_1\left(\phi_{i,j-1}, m\right)\right)$$

$$E\left(X^{Re}_{i,j}(\text{LCR(r)})|\mathcal{D}_I\right) = \sum_{m=1}^{r}\left(G_1\left(\phi_{i,j}, m\right) - G_1\left(\phi_{i,j-1}, m\right)\right)$$

$$\begin{aligned}MSEP_{\mathcal{D}_I}\left(X^{In}_{i,j}(\text{LCR(r)}), \hat{X}^{In}_{i,j}(\text{LCR(r)})\right) = & \sum_{l=0}^{\min(j,d)} N^{report}_{i,j-l}\, p^*_l \gamma_i^2 \mu_W^2\left(\eta^2 + \left(1-p^*_l\right)\right) + \sigma^2_{\text{LCR(r)}} \\ & -2\sum_{l=0}^{\min(j,d)} N^{report}_{i,j-l}\, p^*_l \sum_{m=1}^{r}\sum_{k=m+1}^{N_i}\left[\left(P\left(\xi_{i,j}=k\right)\right.\right. \\ & \left.-P\left(\xi_{i,j-1}=k\right)\right]\sum_{h=1}^{3} T_h(k;m) \\ & +2\left[\gamma_i\mu_W\sum_{l=0}^{\min(j,d)} N^{report}_{i,j-l}\, p^*_l\right]\left[\sum_{m=1}^{r}\left(G_1\left(\phi_{i,j}, m\right)\right.\right. \\ & \left.\left.-G_1\left(\phi_{i,j-1}, m\right)\right)\right]\end{aligned}$$

$$\begin{aligned}MSEP_{\mathcal{D}_I}\left(X^{Re}_{i,j}(\text{LCR(r)}), \hat{X}^{Re}_{i,j}(\text{LCR(r)})\right) = & \sum_{m=1}^{r} G_2\left(\phi_{i,j}, m\right) - \left(\sum_{m=1}^{r} G_1\left(\phi_{i,j}, m\right)\right)^2 \\ & + \sum_{m=1}^{r} G_2\left(\phi_{i,j-1}, m\right) - \left(\sum_{m=1}^{r} G_1\left(\phi_{i,j-1}, m\right)\right)^2 \\ & -2r\sum_{m_1=1}^{r}\sum_{k=m+1}^{N_i} P\left(\xi_{i,j}=k\right)\sum_{h=1}^{3} T_h(k;m) \\ & +2\left[\sum_{m=1}^{r}\left(G_1\left(\phi_{i,j}, m\right)\right.\right]\left[\sum_{m=1}^{r}\left(G_1\left(\phi_{i,j-1}, m\right)\right.\right] =: \sigma^2_{\text{LCR(r)}},\end{aligned}$$

and

$$E\Big(X_{i,j}^{In}(\text{ECOMOR(r)})|\mathcal{D}_I\Big) = \gamma_i\mu_W \sum_{l=0}^{\min(j,d)} N_{i,j-l}^{report} p_l^* - \sum_{m=1}^{r}\left(G_1\left(\phi_{i,j}, m\right)\right.$$

$$\left.-G_1\left(\phi_{i,j-1}, m\right)\right)$$

$$+r\left(\left(G_1\left(\phi_{i,j}, r\right) - G_1\left(\phi_{i,j-1}, r\right)\right)\right.$$

$$E\Big(X_{i,j}^{Re}(\text{ECOMOR(r)})|\mathcal{D}_I\Big) = \sum_{m=1}^{r}\left(G_1\left(\phi_{i,j}, m\right) - G_1\left(\phi_{i,j-1}, m\right)\right)$$

$$-r\Big(\left(G_1\left(\phi_{i,j}, r\right) - G_1\left(\phi_{i,j-1}, r\right)\right)$$

$$MSEP_{\mathcal{D}_I}\left(X_{i,j}^{In}(\text{ECOMOR(r)}), \hat{X}_{i,j}^{In}(\text{ECOMOR(r)})\right) = \sum_{l=0}^{\min(j,d)} N_{i,j-l}^{report} p_l^* \gamma_i^2 \mu_W^2\left(\eta^2 + \left(1 - p_l^*\right)\right)$$

$$+\sigma^2_{\text{ECOMOR(r)}}$$

$$-2\sum_{l=0}^{\min(j,d)} N_{i,j-l}^{report} p_l^* \left(\sum_{m=1}^{r}\sum_{k_1=m+1}^{\infty} P(\xi_{i,j} = k_1)\right.$$

$$\times\sum_{h=1}^{3}(T_h(k_1, m)$$

$$-\sum_{k_2=r+1}^{N_i} P(\xi_{i,j-1} = k_2)\sum_{h=1}^{3}(T_h(k_2, r)))$$

$$+2\gamma_i\mu_W \sum_{l=0}^{\min(j,d)} N_{i,j-l}^{report} p_l^* \sum_{m=1}^{r}\left(G_1\left(\phi_{i,j}, m\right)\right.$$

$$\left.-G_1\left(\phi_{i,j-1}, m\right)\right)$$

$$-2r\gamma_i\mu_W \sum_{l=0}^{\min(j,d)} N_{i,j-l}^{report} p_l\left(G_1\left(\phi_{i,j}, r\right)\right.$$

$$\left.-G_1\left(\phi_{i,j-1}, r\right)\right)$$

$$MSEP_{\mathcal{D}_I}\left(X_{i,j}^{Re}(\text{ECOMOR(r)}), \hat{X}_{i,j}^{Re}(\text{ECOMOR(r)})\right) = \sigma^2_{\text{LCR(r)}} + r^2 G_2\left(\phi_{i,j}, r\right) - r^2 G_1^2\left(\phi_{i,j}, r\right)$$

$$+r^2 G_2\left(\phi_{i,j-1}, r\right)$$

$$-r^2 G_1^2\left(\phi_{i,j-1}, r\right) + 2r^2 G_1\left(\phi_{i,j}, r\right) G_1\left(\phi_{i,j-1}, r\right)$$

$$+2r\sum_{m_1=1}^{r}\sum_{k_1=m_1+1}^{N_i} P(\xi_{i,j} = k_1)\sum_{h=1}^{3} T_h(k_1; m)$$

$$-2rG_2\left(\phi_{i,j}, r\right) - 2rG_2\left(\phi_{i,j-1}, r\right)$$

$$+2rE\Big(X_{i,j}^{Re}(\text{LCR(r)})|\mathcal{D}_I\Big)\left(G_1\left(\phi_{i,j}, r\right)\right.$$

$$\left.-G_1\left(\phi_{i,j-1}, r\right)\right) =: \sigma^2_{\text{ECOMOR(r)}},$$

Table 7. Mean of loss reserve net of LCR(r) treaty for Weibull distribution with parameters λ^* and θ^*.

	$\lambda^*=1$				$\lambda^*=2$			
	$\theta^*=0.5$		$\theta^*=1$		$\theta^*=0.5$		$\theta^*=1$	
LCR(r)	Reserve	MSEP	Reserve	MSEP	Reserve	MSEP	Reserve	MSEP
LCR(6)	135	91,651	79	5,893	263	393,807	158	25,960
LCR(8)	128	101,673	77	7,691	249	450,080	155	34,842
LCR(10)	122	111,060	75	9,584	236	505,284	151	44,415
LCR(15)	109	133,578	70	14,729	211	643,655	143	71,030

Table 8. Mean of loss reserve net of ECOMOR(r) treaty for Weibull distribution with parameters λ^* and θ^*.

	$\lambda^*=1$				$\lambda^*=2$			
	$\theta^*=0.5$		$\theta^*=1$		$\theta^*=0.5$		$\theta^*=1$	
LCR(r)	Reserve	MSEP	Reserve	MSEP	Reserve	MSEP	Reserve	MSEP
ECOMOR(6)	159	154,415	84	15,022	311	671,287	169	64,853
ECOMOR(8)	155	178,561	83	21,754	304	800,727	169	95,426
ECOMOR(10)	152	197,029	84	28,568	296	907,282	169	126,678
ECOMOR(15)	143	231,131	84	44,516	279	1,139,440	169	201,980

where $G_\nu(a,b)=\frac{a^b}{(b-1)!}\frac{\theta^*}{(\lambda^*)^{\theta^*}}\int_0^\infty y^{\nu+\theta^*-1}e^{-b(\frac{y}{\lambda^*})^{\theta^*}-ae^{-\left(\frac{y}{\lambda^*}\right)^{\theta^*}}}dy$ and μ_W and σ_W^2 are given by Equation (10).

Now, the Numerical Procedure 1 is employed to (**1**) predict outstanding claims $X_{i,j}^{In}(\text{LCR(r)})$ (resp. $X_{i,j}^{In}(ECOMOR(r))$) and (**2**) compare these two reinsurance treaties. Tables 7 and 8, respectively, present prediction of outstanding claims $X_{i,j}^{In}(\text{LCR(r)})$ (resp. $X_{i,j}^{In}(ECOMOR(r))$) and the MSEP for the LCR(r) and the ECOMOR(r) treaties for different r. As Tables 7 and 8 show (**1**) for a fixed shape parameter, the reserve and the MSEP increase as the scale parameter increases, (**2**) for a fixed scale parameter, the reserve and the MSEP increase as the shape parameter increases, (**3**) the cedent's MSEP under the LCR(r) treaty is smaller than such amount under the ECOMOR(r), and (**4**) for both treaties, the amount of cedent's MSEP increases as the number of claims covered by the reinsurer increases.

5. Conclusions and Suggestions

Reinsurance has an important role in insurance companies' solvency. It can reduce the probability of a cedent's ruin. Insurance companies should use reinsurance to reduce their risk. The type of reinsurance treaty has an important role in risk management and investment decision-making. In this article, new mathematical results have been derived that are associated with the cedent's net loss reserves considering LCR(r) and ECOMOR(r) treaties. LCR(r) and ECOMOR(r) covers are not popular in the reinsurance world, but the results of Theorem 1 and Theorem 2 provide a useful tool for assessing the impact of very large claims on the cedent's portfolio.

The findings of this article indicate that the LCR(r) treaty is always more efficient than ECOMOR(r) treaty for the cedent. The loss reserve net of the LCR(r) treaty produces a smaller MSEP in a single triangle simulation and the mean of it in 10,000 iterations for the cedent.

Conclusions of this article are based on synthetic data. Of course, the next interesting step is to see whether or not the conclusions also hold for real data and other stochastic loss reserve methods. The results of this article can be extended to other types of reinsurance treaties, different scenarios for inflation, different stochastic loss reserve models, etc.

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In case that there is we have real data rather than simulated ones, based on Verrall (1991) recommendation, the following steps are suggested to estimate unknown parameters.

Step 1: Employ the standard chain ladder model against $N_{i,j}$, to estimate the development factor $\hat{\lambda}_j$, for $j = 1, 2, \cdots, I-1$;

Step 2: Estimates β_j and α_i, by

$$\hat{\beta}_0 = \frac{1}{\prod_{l=1}^{I-1} \hat{\lambda}_l}$$

$$\hat{\beta}_j = \frac{\hat{\lambda}_j - 1}{\prod_{l=j}^{I-1} \hat{\lambda}_l}, \ for\ j = 1, 2, \cdots, I-1;$$

$$\hat{\alpha}_i = \sum_{j=0}^{I-i} N_{i,j}^{report} \prod_{j=I-i+1}^{I-1} \hat{\lambda}_j.$$

Step 3: Employ the standard chain ladder model against $X_{i,j}$, reported by the paid triangle to estimate $\tilde{\beta}_j$ and $\tilde{\alpha}_i$ for $j = 0, 1, \cdots, I-1$ and $i + j \leq I$.

Step 4: Set the following system of equations.

$$\tilde{\beta}_j = \sum_{l=0}^{j} \beta_{j-l} p_l^* \ for\ j = 0, 1, \cdots, I-1.$$

Now employ the estimated $\hat{\beta}_j$ and $\hat{\tilde{\beta}}_j$ to estimate $\hat{p}_0^*, \cdots, \hat{p}_{I-1}^*$. All estimated $\hat{p}_l$ has to be non-negative and satisfy $\sum_l^{I-1} \hat{p}_l^* = 1$. Therefore, negative values should be removed, and the last non-negative value should be adjusted to get condition $\sum_l \hat{p}_l^* = 1$.

Step 5: Use the maximum likelihood method against likelihood of $\frac{Y_{i,j}^{(k)}}{\gamma_i}$ to estimate distributional parameters of $Y_{i,j}^{(k)}$.

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Appendix

Lemma 1. *Under Model Assumption 1 and the LCR(r) treaty, given the information $\mathcal{D}_I$, we have:*

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(1) The conditional variance $Var\left(X_{i,j}^{Re}(_{LCR(r)})|\mathcal{D}_I\right)$ *is*

$$
\begin{aligned}
&= \sum_{m=}^{r} \frac{1}{\Gamma(m)} \int_0^1 F^{-1}(v)^2\,[1-v]^{m-1}\,\psi_{\xi_{i,j}}^{(m)}(v)dv \\
&- \left(\sum_{m=1}^{r} \frac{1}{\Gamma(m)} \int_0^1 F^{-1}(v)\,[1-v]^{m-1}\,\psi_{\xi_{i,j}}^{(m)}(v)dv\right)^2 \\
&+ \sum_{m=}^{r} \frac{1}{\Gamma(m)} \int_0^1 F^{-1}(v)^2\,[1-v]^{m-1}\,\psi_{\xi_{i,j-1}}^{(m)}(v)dv \\
&- \left(\sum_{m=1}^{r} \frac{1}{\Gamma(m)} \int_0^1 F^{-1}(v)\,[1-v]^{m-1}\,\psi_{\xi_{i,j-1}}^{(m)}(v)dv\right)^2 \\
&-2r \sum_{m=1}^{r} \sum_{k=m+1}^{N_i} P\left(\xi_{i,j}=k\right) \sum_{h=1}^{3} T_h(k;m) \\
&+2\left[\sum_{m=1}^{r} \frac{1}{\Gamma(m)} \int_0^1 F^{-1}(v)\,[1-v]^{m-1}\,\psi_{\xi_{i,j}}^{(m)}(v)dv\right] \\
&\quad\times\left[\sum_{m=1}^{r} \frac{1}{\Gamma(m)} \int_0^1 F^{-1}(v)\,[1-v]^{m-1}\,\psi_{\xi_{i,j-1}}^{(m)}(v)dv\right]
\end{aligned}
$$

(2) The conditional covariance $Cov(X_{i,j}, X_{i,j}^{Re}(_{LCR(r)})|\mathcal{D}_I)$ *is*

$$
\begin{aligned}
&\sum_{m=1}^{r} \sum_{n_{ij}=0}^{\infty} n_{ij} P\left(N_{i,j}^{paid}=n_{ij}\right) \sum_{k\geq m}^{\infty} P\left(\xi_{i,j-1}=k\right) \sum_{h=1}^{3} T_h(k;m) \\
&- \sum_{m=1}^{r} \sum_{n_{ij}=0}^{\infty} n_{ij} P\left(N_{i,j}^{paid}=n_{ij}\right) \sum_{k=m}^{\infty} P\left(\xi_{i,j-1}=k\right) \sum_{h=1}^{3} T_h(k;m) \\
&- \left[\sum_{m=1}^{r} \frac{1}{\Gamma(m)} \int_0^1 F^{-1}(v)\,[1-v]^{m-1}\left[\psi_{\xi_{i,j}}^{(m)}(v)-\psi_{\xi_{i,j-1}}^{(m)}(v)\right]dv\right]\left[\gamma_i\mu \sum_{l=0}^{\min(j,d)} N_{i,j-l}^{report}\,p_l^*\right],
\end{aligned}
$$

where $\xi_{i,j} = \sum_{h=0}^{j} N_{i,h}^{paid}$.

Proof. For Part (1) observe that

$$
\begin{aligned}
Var\left(X_{i,j}^{Re}(_{LCR(r)})|\mathcal{D}_I\right) &= Var\left(\sum_{m=1}^{r} Y_{i(\xi_{i,j}-m+1)} - \sum_{m=1}^{r} Y_{i(\xi_{i,j-1}-m+1)}|\mathcal{D}_I\right) \\
&= \overbrace{Var\left(\sum_{m=1}^{r} Y_{i(\xi_{i,j}-m+1)}|\mathcal{D}_I\right)}^{I} + \overbrace{Var\left(\sum_{m=1}^{r} Y_{i(\xi_{i,j-1}-m+1)}|\mathcal{D}_I\right)}^{II} \\
&- \overbrace{2Cov\left(\sum_{m=1}^{r} Y_{i(\xi_{i,j}-m+1)}, \sum_{m=1}^{r} Y_{i(\xi_{i,j-1}-m+1)}|\mathcal{D}_I\right)}^{III}.
\end{aligned}
$$

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The first expression, indicated by I, can be simplified as

$$
\begin{aligned}
Var\left(\sum_{m=1}^{r} Y_{i(\xi_{i,j}-m+1)}|\mathcal{D}_I\right) &= E\left(\sum_{m=1}^{r} (Y_{i(\xi_{i,j}-m+1)})^2\right) + 2E\left(\sum_{m=2}^{r}\sum_{k=1}^{m-1} Y_{i(\xi_{i,j}-k+1)}Y_{i(\xi_{i,j}-m+1)}|\mathcal{D}_I\right) \\
&\quad - \left(E\left(\sum_{m=1}^{r} Y_{i(\xi_{i,j}-m+1)}|\mathcal{D}_I\right)\right)^2 \\
&= E\left(\sum_{m=1}^{r} (Y_{i(\xi_{i,j}-m+1)})^2|\mathcal{D}_I\right) - \left(E\left(\sum_{m=1}^{r} Y_{i(\xi_{i,j}-m+1)}|\mathcal{D}_I\right)\right)^2 \\
&= \sum_{m=1}^{r} E\Big((Y_{i(\xi_{i,j}-m+1)})^2|\mathcal{D}_I\Big) - \left(\sum_{m=1}^{r} E\Big(Y_{i(\xi_{i,j}-m+1)}|\mathcal{D}_I\Big)\right)^2 \\
&= \sum_{m=1}^{r} \frac{1}{\Gamma(m)} \int_0^1 F^{-1}(v)^2\,[1-v]^{m-1}\,\psi^{(m)}_{\xi_{i,j}}(v)dv \\
&\quad - \left(\sum_{m=1}^{r} \frac{1}{\Gamma(m)} \int_0^1 F^{-1}(v)\,[1-v]^{m-1}\,\psi^{(m)}_{\xi_{i,j}}(v)dv\right)^2,
\end{aligned}
$$

where the second equation is obtained from $E(\sum_{m=2}^{r}\sum_{k=1}^{m-1} Y_{i(\xi_{i,j}-k+1)}Y_{i(\xi_{i,j}-m+1)}) = 0$, reported by Seal (1969, chapter 5), and the fourth equation is obtained from Equation (1).

Similarly, the second expression, indicated by II, will be

$$
\begin{aligned}
Var\left(\sum_{m=1}^{r} Y_{i(\xi_{i,j-1}-m+1)}|\mathcal{D}_I\right) &= \sum_{m=1}^{r} \frac{1}{\Gamma(m)} \int_0^1 F^{-1}(v)^2\,[1-v]^{m-1}\,\psi^{(m)}_{\xi_{i,j-1}}(v)dv \\
&\quad - \left(\sum_{m=1}^{r} \frac{1}{\Gamma(m)} \int_0^1 F^{-1}(v)\,[1-v]^{m-1}\,\psi^{(m)}_{\xi_{i,j-1}}(v)dv\right)^2,
\end{aligned}
$$

To evaluate the third expression, indicated by III, observe that

$$
\begin{aligned}
\frac{Part(III)}{2} &= E\left(\sum_{m=1}^{r} Y_{i(\xi_{i,j}-m+1)} \sum_{m=1}^{r} Y_{i(\xi_{i,j-1}-m+1)}|\mathcal{D}_I\right) \\
&\quad - E\left(\sum_{m=1}^{r} Y_{i(\xi_{i,j}-m+1)}|\mathcal{D}_I\right) E\left(\sum_{m=1}^{r} Y_{i(\xi_{i,j-1}-m+1)}|\mathcal{D}_I\right) \\
&= E\left(\sum_{m_1=1}^{r}\sum_{m_2=1}^{r} Y_{i(\xi_{i,j}-m_1+1)}Y_{i(\xi_{i,j-1}-m_2+1)}|\mathcal{D}_I\right) \qquad (11) \\
&\quad - \left[\sum_{m=1}^{r} E\Big(Y_{i(\xi_{i,j}-m+1)}|\mathcal{D}_I\Big)\right]\left[\sum_{m=1}^{r} E\Big(Y_{i(\xi_{i,j-1}-m+1)}|\mathcal{D}_I\Big)\right]
\end{aligned}
$$

$$= r\sum_{m=1}^{r}\sum_{k=m+1}^{N_i} P\left(\xi_{i,j}=k\right)\sum_{h=1}^{3} T_h(k;m)$$
$$-\left[\sum_{m=1}^{r}\frac{1}{\Gamma(m)}\int_0^1 F^{-1}(v)\,[1-v]^{m-1}\,\psi^{(m)}_{\xi_{i,j}}(v)dv\right]$$
$$\times\left[\sum_{m=1}^{r}\frac{1}{\Gamma(m)}\int_0^1 F^{-1}(v)\,[1-v]^{m-1}\,\psi^{(m)}_{\xi_{i,j-1}}(v)dv\right],$$

where the last equation is obtained from Equations (1) and (3).
For Part (2) observe that

$$E\left(X_{i,j}X^{Re}_{i,j}(\text{LCR(r)})|\mathcal{D}_I\right) = E\left(\sum_{k=1}^{N^{paid}_{i,j}} Y^{(k)}_{i,j}\left(\sum_{m=1}^{r} Y_{i(\xi_{i,j}-m+1)}-\sum_{m=1}^{r} Y_{i(\xi_{i,j-1}-m+1)}\right)|\mathcal{D}_I\right)$$
$$=\sum_{m=1}^{r} E\left(E\left(Y_{i(\xi_{i,j}-m+1)}\sum_{k=1}^{N^{paid}_{i,j}} Y^{(k)}_{i,j}|N^{paid}_{i,j}\right)|\mathcal{D}_I\right)$$
$$-\sum_{m=1}^{r} E\left(E\left(Y_{i(\xi_{i,j-1}-m+1)}\sum_{k=1}^{N^{paid}_{i,j}} Y^{(k)}_{i,j}|N^{paid}_{i,j}\right)|\mathcal{D}_I\right)$$
$$=\sum_{m=1}^{r} E\left(N^{paid}_{i,j}E\left(Y_{i(\xi_{i,j}-m+1)}Y^{(1)}_{i,j}|N^{paid}_{i,j}\right)|\mathcal{D}_I\right)$$
$$-\sum_{m=1}^{r} E\left(N^{paid}_{i,j}E\left(Y_{i(\xi_{i,j-1}-m+1)}Y^{(1)}_{i,j}|N^{paid}_{i,j}\right)|\mathcal{D}_I\right)$$
$$=\sum_{m=1}^{r}\sum_{n_{i,j}=1}^{\xi_{i,j}-\xi_{i,j-1}} n_{i,j}P\left(N^{paid}_{i,j}=n_{i,j}\right)\sum_{k=m+1}^{N_i} P\left(\xi_{i,j}=k\right)\sum_{h=1}^{3} T_h(k;m)$$
$$-\sum_{m=1}^{r}\sum_{n_{i,j}=1}^{\xi_{i,j}-\xi_{i,j-1}} n_{i,j}P\left(N^{paid}_{i,j}=n_{i,j}\right)\sum_{k=m+1}^{N_i} P\left(\xi_{i,j-1}=k\right)\sum_{h=1}^{3} T_h(k;m)$$

Therefore,

$$Cov\left(X_{i,j}X^{Re}_{i,j}(\text{LCR(r)})|\mathcal{D}_I\right)=\sum_{m=1}^{r}\sum_{n_{i,j}=1}^{\xi_{i,j}-\xi_{i,j-1}} n_{i,j}P\left(N^{paid}_{i,j}=n_{i,j}\right)\sum_{k=m+1}^{N_i} P\left(\xi_{i,j}=k\right)\sum_{h=1}^{3} T_h(k;m)$$
$$-\sum_{m=1}^{r}\sum_{n_{i,j}=1}^{\xi_{i,j}-\xi_{i,j-1}} n_{i,j}P\left(N^{paid}_{i,j}=n_{i,j}\right)\sum_{k=m+1}^{N_i} P\left(\xi_{i,j-1}=k\right)\sum_{h=1}^{3} T_h(k;m)$$
$$-\left[\sum_{m=1}^{r}\frac{1}{\Gamma(m)}\int_0^1 F^{-1}(v)\,[1-v]^{m-1}\left[\psi^{(m)}_{\xi_{i,j}}(v)-\psi^{(m)}_{\xi_{i,j-1}}(v)\right]dv\right]$$
$$\times\left[\gamma_i\mu\sum_{l=0}^{\min(j,d)} N^{report}_{i,j-l}\,p^*_l\right]$$

$$
\begin{aligned}
E\Big(X_{i,j}X_{i,j}^{Re}(\text{LCR(r)})|\mathcal{D}_I\Big) &= E\left(\sum_{k=1}^{N_{i,j}^{paid}} Y_{ij}^{(k)}\left(\sum_{m=1}^{r} Y_{i(\xi_{i,j}-m+1)} - \sum_{m=1}^{r} Y_{i(\xi_{i,j-1}-m+1)}\right)|\mathcal{D}_I\right)\\
&= \sum_{m=1}^{r} E\left(E\left(Y_{i(\xi_{i,j}-m+1)}\sum_{k=1}^{N_{i,j}^{paid}} Y_{i,j}^{(k)}|N_{i,j}^{paid}\right)|\mathcal{D}_I\right)\\
&\quad - \sum_{m=1}^{r} E\left(E\left(Y_{i(\xi_{i,j-1}-m+1)}\sum_{k=1}^{N_{i,j}^{paid}} Y_{i,j}^{(k)}|N_{i,j}^{paid}\right)|\mathcal{D}_I\right)\\
&= \sum_{m=1}^{r} E\Big(N_{i,j}^{paid}E\Big(Y_{i(\xi_{i,j-1}+N_{i,j}^{paid}-m+1)}Y_{i,j}^{(1)}|N_{i,j}^{paid}\Big)|\mathcal{D}_I\Big)\\
&\quad - \sum_{m=1}^{r} E\Big(N_{i,j}^{paid}E\Big(Y_{i(\xi_{i,j-1}-m+1)}Y_{i,j}^{(1)}|N_{i,j}^{paid}\Big)|\mathcal{D}_I\Big)\\
&= \sum_{m=1}^{r}\sum_{n_{ij}=0}^{\infty} n_{ij}P\Big(N_{i,j}^{paid}=n_{ij}\Big)\sum_{k=m}^{\infty} P\big(\xi_{i,j}=k\big)\sum_{h=1}^{3} T_h(k;m)\\
&\quad - \sum_{m=1}^{r}\sum_{n_{ij}=0}^{\infty} n_{ij}P\Big(N_{i,j}^{paid}=n_{ij}\Big)\sum_{k=m}^{\infty} P\big(\xi_{i,j-1}=k\big)\sum_{h=1}^{3} T_h(k;m).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
Cov(X_{i,j}, X_{i,j}^{Re}(\text{LCR(r)})|\mathcal{D}_I) &= \sum_{m=1}^{r}\sum_{n_{ij}=0}^{\infty} n_{ij}P\Big(N_{i,j}^{paid}=n_{ij}\Big)\sum_{k=m}^{\infty} P\big(\xi_{i,j}=k\big)\sum_{h=1}^{3} T_h(k;m)\\
&\quad - \sum_{m=1}^{r}\sum_{n_{ij}=0}^{\infty} n_{ij}P\Big(N_{i,j}^{paid}=n_{ij}\Big)\sum_{k=m}^{\infty} P\big(\xi_{i,j-1}=k\big)\sum_{h=1}^{3} T_h(k;m)\\
&\quad - \left[\sum_{m=1}^{r}\frac{1}{\Gamma(m)}\int_0^1 F^{-1}(v)\,[1-v]^{m-1}\left[\psi_{\xi_{i,j}}^{(m)}(v)-\psi_{\xi_{i,j-1}}^{(m)}(v)\right]dv\right]\\
&\quad \times\left[\gamma_i\mu\sum_{l=0}^{\min(j,d)} N_{i,j-l}^{report}p_l^*\right].
\end{aligned}
$$

□

It should be noted that the results in Equation (11) are an extension of Hess (2009), who under the assumption $P(N_{ij}^{paid} > r) \approx 1$ used $E(Y_{i(\xi_{i,j}-m_1+1)}Y_{i(\xi_{i,j-1}-m_2+1)}|\mathcal{D}_I) \approx E(Y_{i(\xi_{i,j}-m_1+1)}Y_{ij}^{(1)}|\mathcal{D}_I)$ found this equation under two reinsurance treaties.

Lemma 2. *Under the Model Assumption 1 and the ECOMOR(r) treaty, given the information $\mathcal{D}_I$, we have:*

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(1) *The conditional variance* $Var(X^{Re}_{i,j}(\text{ECOMOR(r)})|\mathcal{D}_I)$ *is*

$$Var\Big(X^{Re}_{i,j}(\text{ECOMOR(r)})|\mathcal{D}_I\Big) = \sigma^2_{\text{LCR(r)}} + r^2\frac{1}{\Gamma(r)}\int_0^1 F^{-1}(v)^2\,[1-v]^{r-1}\,\psi^{(r)}_{\xi_{i,j}}(v)dv$$
$$-r^2\left(\frac{1}{\Gamma(r)}\int_0^1 F^{-1}(v)\,[1-v]^{r-1}\,\psi^{(r)}_{\xi_{i,j}}(v)dv\right)^2$$
$$+r^2\frac{1}{\Gamma(r)}\int_0^1 F^{-1}(v)^2\,[1-v]^{r-1}\,\psi^{(r)}_{\xi_{i,j-1}}(v)dv$$
$$-r^2\left(\frac{1}{\Gamma(r)}\int_0^1 F^{-1}(v)\,[1-v]^{r-1}\,\psi^{(r)}_{\xi_{i,j-1}}(v)dv\right)^2$$
$$+2r^2\frac{1}{\Gamma(r)}\int_0^1 F^{-1}(v)\,[1-v]^{r-1}\,\psi^{(r)}_{\xi_{i,j}}(v)dv\frac{1}{\Gamma(r)}$$
$$\int_0^1 F^{-1}(v)\,[1-v]^{r-1}\,\psi^{(r)}_{\xi_{i,j-1}}(v)dv$$
$$+2r\sum_{m=1}^{r}\sum_{k=m+1}^{N_i}P\big(\xi_{i,j}=k\big)\sum_{h=1}^{3}T_h(k;m)$$
$$-2r\frac{1}{\Gamma(r)}\int_0^1 F^{-1}(v)^2\,[1-v]^{r-1}\,\psi^{(r)}_{\xi_{i,j}}(v)dv - 2r\frac{1}{\Gamma(r)}$$
$$\int_0^1 F^{-1}(v)^2\,[1-v]^{r-1}\,\psi^{(r)}_{\xi_{i,j-1}}(v)dv$$
$$+2rE\Big(X_{i,j}X^{Re}_{i,j}(\text{LCR(r)})|\mathcal{D}_I\Big)\,\frac{1}{\Gamma(r)}\int_0^1 F^{-1}(v)\,[1-v]^{r-1}\,\psi^{(r)}_{\xi_{i,j}}(v)dv$$
$$-2rE\Big(X_{i,j}X^{Re}_{i,j}(\text{LCR(r)})|\mathcal{D}_I\Big)\,\frac{1}{\Gamma(r)}\int_0^1 F^{-1}(v)\,[1-v]^{r-1}\,\psi^{(r)}_{\xi_{i,j-1}}(v)dv$$

(2) *The conditional variance* $Cov\Big(X_{i,j}, X^{Re}_{i,j}(\text{ECOMOR(r)})|\mathcal{D}_I\Big)$ *is*

$$\sum_{l=0}^{N^{paid}_{i,j}} N^{report}_{i,j-l}\,p^*_l\left(\sum_{m=1}^{r}\sum_{k_1=m+1}^{N_i}P(\xi_{i,j}=k_1)\sum_{h=1}^{3}T_h(k_1;m) - \sum_{k_2=r+1}^{N_i}P(\xi_{i,j-1}=k_2)\sum_{h=1}^{3}T_h(k_2;r)\right)$$
$$-\gamma_i\mu\sum_{l=0}^{N^{paid}_{i,j}} N^{report}_{i,j-l}\,p^*_l\sum_{m=1}^{r}\frac{1}{\Gamma(m)}\int_0^1 F^{-1}(v)\,[1-v]^{m-1}\,\psi^{(m)}_{\xi_{i,j}}(v)dv$$
$$-\gamma_i\mu\sum_{l=0}^{N^{paid}_{i,j}} N^{report}_{i,j-l}\,p^*_l\frac{r}{\Gamma(r)}\int_0^1 F^{-1}(v)\,[1-v]^{r-1}\,\psi^{(r)}_{\xi_{i,j}}(v)dv$$
$$-\gamma_i\mu\sum_{l=0}^{N^{paid}_{i,j}} N^{report}_{i,j-l}\,p^*_l\sum_{m=1}^{r}\frac{1}{\Gamma(m)}\int_0^1 F^{-1}(v)\,[1-v]^{m-1}\,\psi^{(m)}_{\xi_{i,j-1}}(v)dv$$
$$+\gamma_i\mu\sum_{l=0}^{N^{paid}_{i,j}} N^{report}_{i,j-l}\,p^*_l\frac{r}{\Gamma(r)}\int_0^1 F^{-1}(v)\,[1-v]^{r-1}\,\psi^{(r)}_{\xi_{i,j-1}}(v)dv.$$

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where $\xi_{i,j} = \sum_{h=0}^{j} N_{i,h}^{paid}$.

Proof. For Part (1) observe that

$$\begin{aligned}
Var\Big(X_{i,j}^{Re}(\text{ECOMOR(r)})|\mathcal{D}_I\Big) &= Var\Big(\sum_{m_1=1}^{r} Y_{i(\xi_{i,j}-m_1+1)} - \sum_{m_2=1}^{r} Y_{i(\xi_{i,j}-m_2+1)} \\
&\quad -r\left(Y_{i\left(\xi_{i,j}-r+1\right)} - Y_{i\left(\xi_{i,j-1}-r+1\right)}\right)|\mathcal{D}_I\Big) \\
&= \overbrace{Var\Big(\sum_{m_1=1}^{r} Y_{i(\xi_{i,j}-m_1+1)} - \sum_{m_2=1}^{r} Y_{i(\xi_{i,j}-m_2+1)}|\mathcal{D}_I\Big)}^{I} \\
&\quad +r^2\, \overbrace{Var\Big(Y_{i\left(\xi_{i,j}-r+1\right)} - Y_{i\left(\xi_{i,j-1}-r+1\right)}\Big)|\mathcal{D}_I)}^{II} \\
&\quad -2\, \overbrace{Cov\left(\sum_{m_1=1}^{r} Y_{i(\xi_{i,j}-m_1+1)} - \sum_{m_2=1}^{r} Y_{i(\xi_{i,j}-m_2+1)}, r\left(Y_{i\left(\xi_{i,j}-r+1\right)} - Y_{i\left(\xi_{i,j-1}-r+1\right)}\right)|\mathcal{D}_I\right)}^{III}
\end{aligned}$$

The first expression, indicated by I, is equal to $\sigma^2_{\text{LCR(r)}}$. The second expression, indicated by II, can be simplified as

$$\begin{aligned}
Var\Big(Y_{i\left(\xi_{i,j}-r+1\right)} - Y_{i\left(\xi_{i,j-1}-r+1\right)}|\mathcal{D}_I\Big) &= Var(Y_{i\left(\xi_{i,j}-r+1\right)}|\mathcal{D}_I) + Var\Big(Y_{i\left(\xi_{i,j-1}-r+1\right)}|\mathcal{D}_I\Big) \\
&\quad -2Cov\left(Y_{i\left(\xi_{i,j}-r+1\right)}, Y_{i\left(\xi_{i,j-1}-r+1\right)}|\mathcal{D}_I\right)
\end{aligned}$$

where

$$Var\Big(Y_{i\left(\xi_{i,j}-r+1\right)}|\mathcal{D}_I\Big) = E\Big(Y^2_{i\left(\xi_{i,j}-r+1\right)}|\mathcal{D}_I\Big) - E^2\left(Y_{i\left(\xi_{i,j}-r+1\right)}|\mathcal{D}_I\right)$$

This equation is obtained from Equation (1). Similarly, the second variance will be obtained from Equation (1). For covariance term, using the covariance definition, we may have

$$\begin{aligned}
Cov\left(Y_{i\left(\xi_{i,j}-r+1\right)}, Y_{i\left(\xi_{i,j-1}-r+1\right)}|\mathcal{D}_I\right) &= E\Big(Y_{i\left(\xi_{i,j}-r+1\right)} Y_{i\left(\xi_{i,j-1}-r+1\right)}|\mathcal{D}_I\Big) \\
&\quad -E\Big(Y_{i\left(\xi_{i,j}-r+1\right)}|\mathcal{D}_I\Big)\, E\Big(Y_{i\left(\xi_{i,j-1}-r+1\right)}|\mathcal{D}_I\Big)
\end{aligned}$$

This equation is obtained from Equations (1) and (3).

Finally, for Part (III) observe that,

$$\begin{aligned}
Part(III) &= rE\left(\left(\sum_{m_1=1}^{r} Y_{i(\xi_{i,j}-m_1+1)} - \sum_{m_2=1}^{r} Y_{i(\xi_{i,j}-m_2+1)}\right)\left(Y_{i\left(\xi_{i,j}-r+1\right)} - Y_{i\left(\xi_{i,j-1}-r+1\right)}\right)|\mathcal{D}_I\right) \\
&\quad -rE\left(\sum_{m_1=1}^{r} Y_{i(\xi_{i,j}-m_1+1)} - \sum_{m_2=1}^{r} Y_{i(\xi_{i,j}-m_2+1)}|\mathcal{D}_I\right) E\Big(Y_{i\left(\xi_{i,j}-r+1\right)} - Y_{i\left(\xi_{i,j-1}-r+1\right)}\Big)|\mathcal{D}_I\Big)
\end{aligned}$$

This equation is obtained from Equations (1) and (3) and the employed method in Proposition (3.1) Hess (2009).

For Part (2), using the covariance definition, we may have

$$\begin{aligned}
Cov\left(X_{i,j}, X_{i,j}^{Re}(\text{ECOMOR(r)})|\mathcal{D}_I\right) &= E\Big(X_{i,j}, X_{i,j}^{Re}(\text{ECOMOR(r)})|\mathcal{D}_I\Big) \\
&\quad -E\big(X_{i,j}|\mathcal{D}_I\big)\, E\Big(X_{i,j}^{Re}(\text{ECOMOR(r)})|\mathcal{D}_I\Big).
\end{aligned}$$

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Two conditional expectations in the second term are calculated in Equation (5) and Theorem (2), respectively. Therefore, we just focus on

$$
\begin{aligned}
E\left(X_{i,j}|\mathcal{D}_I\right) E\left(X_{i,j}^{Re}(\text{ECOMOR(r)})|\mathcal{D}_I\right) &= E\left(\sum_{k=1}^{N_{i,j}^{paid}} Y_{i,j}^{(k)} \left(\sum_{m=1}^{r} Y_{i(\xi_{i,j}-m+1)} - rY_{i(\xi_{i,j}-r+1)}\right) |\mathcal{D}_I\right) \\
&\quad -E\left(\sum_{k=1}^{N_{i,j}^{paid}} Y_{i,j}^{(k)} \left(\sum_{m=1}^{r} Y_{i(\xi_{i,j-1}-m+1)} - rY_{i(\xi_{i,j-1}-r+1)}\right) |\mathcal{D}_I\right) \\
&= \sum_{m=1}^{r} E\left(E\left(\left(Y_{i(\xi_{i,j}-m+1)} - rY_{i(\xi_{i,j}-r+1)}\right) \sum_{k=1}^{N_{i,j}^{paid}} Y_{i,j}^{(k)} |\xi_{i,j}, N_{i,j}^{paid}\right) |\mathcal{D}_I\right) \\
&\quad - \sum_{m=1}^{r} E\left(E\left(\left(Y_{i(\xi_{i,j-1}-m+1)} - rY_{i(\xi_{i,j-1}-r+1)}\right) \sum_{k=1}^{N_{i,j}^{paid}} Y_{i,j}^{(k)} |\xi_{i,j-1}, N_{i,j}^{paid}\right) |\mathcal{D}_I\right) \\
&= \sum_{l=0}^{\min(j,d)} N_{i,j-l}^{report} p_l^* \sum_{m=1}^{r} \sum_{k_1=m+1}^{N_i} P(\xi_{i,j}=k_1) \sum_{h=1}^{3} T_h(k_1;m) \\
&\quad - \sum_{l=0}^{\min(j,d)} N_{i,j-l}^{report} p_l^* \sum_{k_2=r+1}^{N_i} P(\xi_{i,j-1}=k_2) \sum_{h=1}^{r} T_h(k_2,r)
\end{aligned}
$$

The rest of the proof is similar to Lemma 1. □

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