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Homological Aspects of Semigroup Gradings on Rings and Algebras

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Abstract. This article studies algebras R over a simple artinian ring A, presented by a quiver and relations and graded by a semigroup Σ. Suitable semigroups often arise from a presentation of R. Throughout, the algebras need not be finite dimensional. The graded K_0, along with the Σ-graded Cartan endomorphisms and Cartan matrices, is examined. It is used to study homological properties.

A test is found for finiteness of the global dimension of a monomial algebra in terms of the invertibility of the Hilbert Σ-series in the associated path incidence ring.

The rationality of the Σ-Euler characteristic, the Hilbert Σ-series and the Poincaré-Betti Σ-series is studied when Σ is torsion-free commutative and A is a division ring. These results are then applied to the classical series. Finally, we find new finite dimensional algebras for which the strong no loops conjecture holds.

Introduction

There is a vast literature on the use of gradings (usually over **Z** or **N**) to extract ring theoretic information. Our purpose is to examine semigroup gradings on algebras which are not necessarily finite dimensional over a base ring A, where A can be a simple artinian ring; and to use them to obtain (mostly) homological information about the algebras. All rings below are unitary and modules are unitary left modules. We shall see that the gradings used arise quite naturally in many contexts. Before proceeding with a summary of the contents, here are the basic definitions.

All semigroups Σ considered in this work are written multiplicatively and have a zero element, 0, so that $0\sigma = \sigma 0 = 0$ for all $\sigma \in \Sigma$. There is also a distinguished family of non-zero orthogonal idempotents $\{\epsilon_1, \ldots, \epsilon_n\}$, so that $\Sigma = \bigcup_{ij} \epsilon_i \Sigma \epsilon_j$. That family is uniquely determined by Σ, up to reordering. When $\sigma \in \epsilon_i \Sigma$ or, equivalently, $\sigma = \epsilon_i \sigma$, we say that ϵ_i is the *origin* σ and write $o(\sigma) = \epsilon_i$. We put $\Sigma^* = \Sigma - \{\epsilon_1, \ldots, \epsilon_n\}$. Every Σ-grading on a ring R is assumed to have $R_0 = 0$. Finally, we set $R_1 = R_{\epsilon_1} \oplus \cdots \oplus R_{\epsilon_n}$.

We now specify the type of graded ring which will be the subject of the paper.

Definition 0.1 Let Σ be a semigroup with distinguished family of orthogonal idempotents $\{\epsilon_1, \ldots, \epsilon_n\}$. A Σ-grading $R = \bigoplus_{\sigma \in \Sigma} R_\sigma$ is called *(left) admissible* if the following hold:

a) $I = \bigoplus_{\sigma \in \Sigma^*} R_\sigma$ is a two-sided ideal of R.
b) For every $\sigma \in \Sigma$, there is a positive integer n_σ such that whenever $n \geq n_\sigma$ and ν is a left divisor of σ, $(I^n)_\nu = 0$.

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c) R_1 is a semisimple artinian ring.
d) Each R_σ is finitely generated as a left R_1-module.

The following identifies a class of semigroups which is important in what follows.

Definition 0.2 A semigroup Σ with 0 is called a *Möbius semigroup* (adapted from [8]) if, in addition to the conventions above,

(i) each $0 \neq p \in \Sigma^*$ has only finitely many factorizations $p = p_1 p_2$ in Σ^*; and
(ii) the only elements $p \in \Sigma$ with $p^m = p^n \neq 0$ for distinct m, n are the ε_i.

Given a semigroup Σ, two related rings will be especially important for us. The additive abelian group $\mathbf{Z}[[\Sigma]] = \{f\colon \Sigma \to \mathbf{Z} \mid f(0) = 0\}$ contains $\mathbf{Z}\Sigma = \{f \in \mathbf{Z}[[\Sigma]] \mid f(\sigma) = 0$ for almost all $\sigma \in \Sigma\}$ as a subgroup. The latter admits a convolution product $(f * g)(\sigma) = \sum_{\nu\tau=\sigma} f(\nu)g(\tau)$, which makes it into a ring, called the *(truncated) semigroup ring* of Σ. When, in addition, Σ is Möbius, the product is also valid in $\mathbf{Z}[[\Sigma]]$ and $\mathbf{Z}[[\Sigma]]$ becomes a ring of which $\mathbf{Z}\Sigma$ is a subring. The ring $\mathbf{Z}[[\Sigma]]$ is called the *incidence ring* of Σ. For convenience, we write an element $f \in \mathbf{Z}[[\Sigma]]$ as $f = \sum_{\sigma\in\Sigma} m_\sigma \sigma$, where $m_\sigma = f(\sigma)$ for all $\sigma \in \Sigma$.

For the rest of this introduction we shall simplify matters by assuming that our Σ-graded ring R is an algebra over a field, Σ is Möbius and each simple direct summand of R/I occurs exactly once. We often need that our Σ-graded modules M have resolutions (as ungraded modules) by finitely generated free modules. More general situations are studied in the body of the article.

We consider the subcategories $(R,\Sigma)_{\mathrm{fd}}$-gr and $(R,\Sigma)_{\mathrm{lfd}}$-gr of finite dimensional and locally finite dimensional Σ-graded modules, respectively, with maps which are degree preserving homomorphisms. Their classical Grothendieck groups, $K_0[R,\Sigma]_{\mathrm{fd}}$ and $K_0[R,\Sigma]_{lfg}$ are studied. When $M \in (R,\Sigma)_{\mathrm{lfd}}$-gr one can consider the Hilbert Σ-series of M, $\Delta_M = \sum_{\sigma\in\Sigma} c(M_\sigma)\sigma \in \mathbf{Z}[[\Sigma]]$, where $c(*)$ denotes the R_1-composition length. We then look at the group $K_0^*[R,\Sigma]_{\mathrm{lfd}} = K_0[R,\Sigma]_{\mathrm{lfd}}/B$, where B is the subgroup generated by the differences $M - N$, which $M, N \in (R,\Sigma)_{\mathrm{lfd}}$-gr and $\Delta_M = \Delta_N$. The key tools developed in the paper (Theorems 1.8 and 1.9) are that $K_0[R,\Sigma]_{\mathrm{fd}}$ and $K_0^*[R,\Sigma]_{\mathrm{lfd}}$ have canonical structures as right $\mathbf{Z}\Sigma$- and $\mathbf{Z}[[\Sigma]]$-modules, which are isomorphic to $\mathbf{Z}\Sigma_{\mathbf{Z}\Sigma}$ and $\mathbf{Z}[[\Sigma]]_{\mathbf{Z}[[\Sigma]]}$, respectively. The assignment to every simple graded module of a projective module which is a cover as a graded module is shown to yield a $\mathbf{Z}[[\Sigma]]$-module Cartan endomorphism of $K_0^*[R,\Sigma]_{\mathrm{lfd}} \cong \mathbf{Z}[[\Sigma]]$, which is multiplication by Δ_R; there is a corresponding Cartan matrix (Proposition 2.2 and Corollary 2.3).

We apply these results to homological questions. Section 3 is about computing the projective dimensions of simple graded modules over (not necessarily finite dimensional) monomial algebras (Theorem 3.1). This yields a new test for the global dimension of the algebra (Corollary 3.3). Section 4 proves the rationality of the so-called Anick-Poincaré-Betti Σ-series of an algebra where the ideal of relations has a finite Gröbner basis. In that case, the rationality of the Hilbert and Euler characteristic Σ-series follows and, when the Anick resolution is minimal, so does that of the Poincaré-Betti Σ-series (Theorem 4.4). In the last section we improve on one of the few known results on the "strong no loops conjecture" for finite dimensional algebras.

1 Admissible Gradings on Rings

The terminology and notation presented in the introduction are used.

Lemma 1.1 *Let $R = \bigoplus_{\sigma\in\Sigma} R_\sigma$ be a Σ-grading on the ring R such that $I = \bigoplus_{\sigma\in\Sigma^*} R_\sigma$ is a two-sided ideal of R with $\bigcap_{n\geq 1} I^n = 0$. Then $1 \in R_1$ and we have a R_1-bimodule decomposition $R = R_1 \oplus I$.*

Proof Clearly $R = R_1 \oplus I$ as abelian groups. If now $1 = e + f$, with $e \in R_1$ and $f \in I$, one gets $ef = e - e^2 = fe$ and so $ef = fe = 0$. But then $1 = (e + f)^2 = e^2 + f^2$, from which it follows that $e = e^2$ and $f = f^2$. The latter implies that $f \in \bigcap_{n\geq 1} I^n = 0$. So $1 = e \in R_1$. The rest is obvious. ■

Remark 1.2 Let M be a Σ-graded module (*i.e.*, $M = \bigoplus_{\sigma\in\Sigma} M_\sigma$ as an abelian group and $R_\sigma M_\tau \subseteq M_{\sigma\tau}$, for all σ and τ). Then $\bigcap_{n\geq 1} I^n M = 0$. Indeed, since $N = \bigcap_{n\geq 1} I^n M$ is a graded submodule of M, we need to check that $N_\sigma = 0$, for every $\sigma \in \Sigma$. From $I^{2n}M = I^n I^n M$, $[I^{2n}M]_\sigma = \sum_{\nu\tau=\sigma} (I^n)_\nu (I^n M)_\tau$. Now Definition 0.1 b) guarantees that, for $n \geq n_\sigma$, $[I^{2n}M]_\sigma = 0$. Then $N_\sigma = 0$, as desired.

Proposition 1.3 **(Nakayama Lemma for Graded Modules)** *Let R be a ring with an admissible Σ-grading. If M is a graded module such that $M = IM$ then $M = 0$. Moreover, if M is a graded module and N a graded submodule such that $M = IM + N$, then $N = M$ (i.e., IM is a superfluous as a graded submodule).*

Examples 1.4

1) An adequate Σ-grading on a left artinian ring in the sense of [19] is admissible.
2) Suppose that Σ is a Möbius semigroup. If A is a semisimple ring, then the canonical Σ-grading on the (truncated) semigroup ring $A\Sigma$ is admissible. An example of a Möbius semigroup is the path semigroup $\Sigma = \mathbf{P}(\Gamma)$ of a quiver Γ with only a finite number of vertices, in which case $A\Sigma$ is the *path A-algebra*, $A\Gamma$.
3) (Factors of path algebras) Let us consider $R = A\Gamma/H$, where A is a simple Artinian ring, Γ is as in 2) and H is an ideal of $A\Gamma$ generated by a set ρ of A-linear combinations of paths (called *relations*) of length ≥ 2, assuming that all paths appearing with nonzero coefficient in an element $r \in \rho$ share origin and terminus. We take the *associated semigroup* [19] $\Sigma = \Sigma(\Gamma, \rho)$ which is the factor of $\mathbf{P}(\Gamma)$ by the congruences: $p \equiv q$ if and only if the paths p and q both appear with nonzero coefficients in a relation in ρ. Then R inherits a canonical Σ-grading as in [19, Proposition 2.1]. When R is left artinian (the grading is finite) or when Σ is a Möbius semigroup, the grading is left admissible. When ρ consists of paths, we say that R is a *monomial A-algebra*. Then the associated semigroup is $\Sigma = \mathbf{P}(\Gamma)$. When ρ can be chosen to be finite, the algebra R is called *finitely presented*.
4) If, as in [20, Proposition 1.1(iii)], one considers $K\langle X, Y\rangle/(f_1, f_2)$, where $f_1 = Y^2X - XY^2$ and $f_2 = X^2Y + YX^2 + Y^2$, then the associated monoid Σ is Möbius. That can be seen by taking $\mathbf{N}$-degrees $\deg(Y) = 2\deg(X) = 2$ and noting that any way of writing a word in X and Y has bounded X and Y-degrees.

Remark 1.5 If $R = \bigoplus_{\sigma\in\Sigma} R_\sigma$ is a left admissible grading and $I = \bigoplus_{\sigma\in\Sigma^*} R_\sigma$, then $R_1 = R_{\epsilon_1} \oplus \cdots \oplus R_{\epsilon_n}$ is also a decomposition of R_1 as a ring, so that each R_{ϵ_i} is a semisimple ring and, for each simple left R-module S with $IS = 0$, there is a unique $i \in \{1, \dots, n\}$ such that $R_{\epsilon_i} S \neq 0$. In that case we shall say that the *origin* $o(S)$ of S is ϵ_i. Any such a simple module admits a canonical (trivial) grading by putting $S_{\epsilon_i} = S$ and $S_\sigma = 0$, for every $\sigma \in \Sigma - \{\epsilon_i\}$.

We fix a left admissible Σ-grading on a ring R and consider the Grothendieck category (R, Σ)-gr of Σ-graded R-modules. We observe that if $M = \bigoplus_{\sigma\in\Sigma} M_\sigma$ is a graded module, then $M[\epsilon_i] = \bigoplus_{\sigma\in\Sigma\epsilon_i} M_\sigma$ is a graded submodule of M, for every $i = 1, \dots, n$ (of course, by putting $\mathrm{M}_\sigma = 0$, for all $\sigma \notin \Sigma\epsilon_i$). More generally, for every $\sigma \in \Sigma$ we can form the σ-*shifting* of M, denoted $M[\sigma]$: as an ungraded R-module $M[\sigma] = M[\epsilon_i]$, where $\epsilon_i = o(\sigma)$, but the grading is given by $M[\sigma]_\tau = \oplus\{M_\nu \mid \nu\sigma = \tau\}$. In particular, the support of $M[\sigma]$ is always contained in $\Sigma\sigma$.

Definition 1.6 A Σ-graded left R-module $M = \bigoplus_{\sigma\in\Sigma} M_\sigma$ will be called *finite-dimensional* (*locally finite-dimensional*) whenever we have $\sum_{\sigma\in\Sigma} c_{R_1}(M_\sigma) < \infty$ ($c_{R_1}(M_\sigma) < \infty$, for every $\sigma \in \Sigma$), where $c_{R_1}(M_\sigma)$ is the composition length of M_σ as a left R_1-module.

Terminology In the sequel $(R, \Sigma)_{\mathrm{fd}}$-gr and $(R, \Sigma)_{\mathrm{lfd}}$-gr will denote the full subcategories of (R, Σ)-gr whose objects are the finite-dimensional and locally finite-dimensional graded R-modules, respectively. Maps are *degree preserving homomorphisms*. Moreover, $\mathbf{S}_I$ stands for a set of representatives, up to isomorphism, of the simple left R-modules S such that $IS = 0$.

Proposition 1.7 *The categories* $(R, \Sigma)_{\mathrm{fd}}$-gr *and* $(R, \Sigma)_{\mathrm{lfd}}$-gr *are skeletally small abelian categories in which* $\{S[\sigma] \mid S \in \mathbf{S}_I \text{ and } o(S) = o(\sigma)\}$ *is a set of representatives, up to isomorphism, of the simple objects.*

Proof It is immediately seen that, as full subcategories of (R, Σ)-gr, the two subcategories are closed under finite direct sums, subobjects, factors and extensions. It then follows that both are abelian categories. That $(R, \Sigma)_{\mathrm{fd}}$-gr is skeletally small is clear. For $(R, \Sigma)_{\mathrm{lfd}}$-gr, we abbreviate $c_{R_1}(M_\sigma)$ by $c(M_\sigma)$ and see that M is a factor in (R, Σ)-gr of $\bigoplus_{\sigma\in\Sigma} R[\sigma]^{c(M_\sigma)}$, where $R[\sigma]$ denotes the σ-shifting of R. As a consequence, if we take the direct sum Q of countably infinitely many copies of $P = \bigoplus_{\sigma\in\Sigma} R[\sigma]$, we see that every locally finite-dimensional graded module is a factor of Q in (R, Σ)-gr, so that $(R, \Sigma)_{\mathrm{lfd}}$-gr is skeletally small.

If $S \in \mathbf{S}_I$ then, by definition, $S[\sigma] \neq 0$ if and only if $o(S) = o(\sigma)$. The last assertion follows as in Lemma 1.3 of [19], bearing in mind that a simple object T of either of the subcategories satisfies $IT = 0$, by Remark 1.2. ■

In what follows, we shall denote the Grothendieck groups by $K_0[R, \Sigma]_{\mathrm{fd}}$ and $K_0[R, \Sigma]_{\mathrm{lfd}}$, respectively. The notation is abused throughout by using of the same symbol for a graded module *and* its image in the corresponding Grothendieck group.

Theorem 1.8 *The operation* $M \cdot \sigma = M[\sigma]$ *yields a right* $\mathbf{Z}\Sigma$*-module structure on* $K_0[R, \Sigma]_{\mathrm{fd}}$*, for which the following assertions hold:*

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a) *If* $S \in \mathbf{S}_I$, *then* $\operatorname{ann}_{\mathbf{Z}\Sigma} S = (1-\epsilon_i)\mathbf{Z}\Sigma$, *where* $\epsilon_i = o(S)$.
b) $K_0[R,\Sigma]_{\rm fd} = \bigoplus\{S\cdot\mathbf{Z}\Sigma \mid S \in \mathbf{S}_I\}$.
c) $K_0[R,\Sigma]_{\rm fd}$ *is isomorphic to* $(\epsilon_1\mathbf{Z}\Sigma)^{m_1} \oplus \cdots \oplus (\epsilon_n\mathbf{Z}\Sigma)^{m_n}$, *where* m_i *is the cardinality of* $\{S \in \mathbf{S}_I \mid o(S) = \epsilon_i\}$.

Proof Since all objects of $(R,\Sigma)_{\rm fd}$-gr have finite length, it is well-known that, as an abelian group, $K_0[R,\Sigma]_{\rm fd}$ is free and generated by the simple objects, *i.e.*, by $\{S[\sigma] \mid S \in \mathbf{S}_I$ and $o(S) = o(\sigma)\}$. The fact that $K_0[R,\Sigma]_{\rm fd}$ becomes canonically a right $\mathbf{Z}\Sigma$ module with the given multiplication is straightforward. If now $f = \sum_{\sigma\in\Sigma} f(\sigma)\sigma \in \mathbf{Z}\Sigma$, where $f(\sigma) \in \mathbf{Z}$ for every $\sigma \in \Sigma$, satisfies $S\cdot f = 0$ and $\epsilon_i = o(S)$, then from the fact that $S[\sigma] = 0$ if and only if $\sigma \notin \epsilon_i\Sigma$ one gets that the support of f is contained in $\bigcup_{j\neq i}\epsilon_j\Sigma$, from which a) follows.

The abelian group generating set given above yields that $\{S \mid S \in \mathbf{S}_I\}$ is a generating set of $K_0[R,\Sigma]_{\rm fd}$ as a right $\mathbf{Z}\Sigma$-module. If $x \in K_0[R,\Sigma]_{\rm fd}$ belongs to the $\mathbf{Z}\Sigma$-submodule generated by S and, also, to the $\mathbf{Z}\Sigma$ submodule generated by the remaining elements of $\mathbf{S}_I$, then x can be expressed as a $\mathbf{Z}$-linear combination of $\{S[\sigma] \mid o(\sigma) = o(S)\}$ and as a $\mathbf{Z}$-linear combination of $\{S'[\sigma] \mid S' \in \mathbf{S}_I,\ S' \neq S$ and $o(\sigma) = o(S')\}$. The fact that $K_0[R,\Sigma]_{\rm fd}$ is free as abelian group implies that $x = 0$, giving b). Finally, if $\epsilon_i = o(S)$ then, from a), one gets that the assignment $S\cdot f \mapsto \epsilon_i\cdot f$ yields an isomorphism of right $\mathbf{Z}\Sigma$-modules between $S\cdot\mathbf{Z}\Sigma$ and $\epsilon_i\cdot\mathbf{Z}\Sigma$. From this and b), assertion c) follows. ■

To every M in $(R,\Sigma)_{\rm lfd}$-gr we can associate the element $f^M = (f^M_S)_{S\in\mathbf{S}_I}$ of $\mathbf{Z}[[\Sigma]]^{\mathbf{S}_I}$ given by $f^M_S = \sum_{\sigma\in\Sigma, o(\sigma)=o(S)} c^S(M_\sigma)\sigma$, where $c^S(M_\sigma)$ denotes the multiplicity of S as a R_1-composition factor of M_σ. Notice that $f^M \in (\mathbf{Z}\Sigma)^{\mathbf{S}_I}$ when M is finite-dimensional. While f^M characterizes a finite-dimensional module M as an element of $K_0[R,\Sigma]_{\rm fd}$, the same is not true for a locally finite-dimensional graded module as an element of $K_0[R,\Sigma]_{\rm lfd}$.

Terminology To avoid problems caused by this, *we work with a factor group,* $K_0^*[R,\Sigma]_{\rm lfd} := K_0[R,\Sigma]_{\rm lfd}/B$, *where* B *is the subgroup generated by all differences* $M - N$, *with* $M, N \in (R,\Sigma)_{\rm lfd}$-gr *and* $f^M = f^N$. The canonical composition $K_0[R,\Sigma]_{\rm fd} \to K_0[R,\Sigma]_{\rm lfd} \to K_0^*[R,\Sigma]_{\rm lfd}$ is an injective homomorphism of abelian groups, thus allowing us to view $K_0[R,\Sigma]_{\rm fd}$ as a subgroup of $K_0^*[R,\Sigma]_{\rm lfd}$.

Theorem 1.9 *Let* Σ *be a Möbius semigroup and* $R = \bigoplus_{\sigma\in\Sigma} R_\sigma$ *be an admissible grading on the ring* R. *The operation* $M\cdot\sigma = M[\sigma]$ *yields a right* $\mathbf{Z}[[\Sigma]]$*-module structure on* $K_0^*[R,\Sigma]_{\rm lfd}$ *with the following properties:*

a) *If* $S \in \mathbf{S}_I$, *then* $\operatorname{ann}_{\mathbf{Z}[[\Sigma]]} S = (1-\epsilon_i)\mathbf{Z}[[\Sigma]]$.
b) $K_0^*[R,\Sigma]_{\rm lfd} = \bigoplus\{S\cdot\mathbf{Z}[[\Sigma]] \mid S \in \mathbf{S}_I\}$.
c) $K_0^*[R,\Sigma]_{\rm lfd}$ *is isomorphic to* $(\epsilon_1\mathbf{Z}[[\Sigma]])^{m_1}\oplus\cdots\oplus(\epsilon_n\mathbf{Z}[[\Sigma]])^{m_n}$, *where* m_i *is the cardinality of* $\{S \in \mathbf{S}_I \mid o(S) = \epsilon_i\}$.
d) *There is a commutative diagram:*

$$\begin{array}{ccc} K_0[R,\Sigma]_{\rm fd} & \longrightarrow & (\epsilon_1\mathbf{Z}\Sigma)^{m_1}\oplus\cdots\oplus(\epsilon_1\mathbf{Z}\Sigma)^{m_n} \\ \downarrow & & \downarrow \\ K_0^*[R,\Sigma]_{\rm lfd} & \longrightarrow & (\epsilon_1\mathbf{Z}[[\Sigma]])^{m_1}\oplus\cdots\oplus(\epsilon_1\mathbf{Z}[[\Sigma]])^{m_n} \end{array}$$

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where the vertical arrows are the canonical inclusions and the horizontal ones are the given isomorphisms of right $\mathbf{Z}\Sigma$*- and* $\mathbf{Z}[[\Sigma]]$*-modules respectively.*

Proof The assignment $M \to f^M$ is additive on short exact sequences, from which we get an abelian group homomorphism $\phi\colon K_0^*[R,\Sigma]_{\mathrm{lfd}} \to \mathbf{Z}[[\Sigma]]^{\mathbf{S}_I}$. On the other hand, an element of $K_0^*[R,\Sigma]_{\mathrm{lfd}}$ can be written in the form $(M-N)+B$, where $M, N \in (R,\Sigma)_{\mathrm{lfd}}$-gr. If it is in $\ker\phi$, then $f^M = f^N$ and so $M + B = N + B$. Hence ϕ is injective.

Moreover, if we put $f^M = (f_S^M)_{S\in\mathbf{S}_I}$ as above, then $f_S^M \in \epsilon_i \mathbf{Z}[[\Sigma]]$, where $o(S) = \epsilon_i$. By using direct sums of graded simples, for example, one derives that $\operatorname{Im}\phi = \bigoplus_{S\in\mathbf{S}_I} o(S)\mathbf{Z}[[\Sigma]]$, which is clearly isomorphic to $(\epsilon_1\mathbf{Z}[[\Sigma]])^{m_1} \oplus \cdots \oplus (\epsilon_n\mathbf{Z}[[\Sigma]])^{m_n}$ as an abelian group. So ϕ can be viewed as an isomorphism of abelian groups between $K_0^*[R,\Sigma]_{\mathrm{lfd}}$ and $(\epsilon_1\mathbf{Z}[[\Sigma]])^{m_1} \oplus \cdots \oplus (\epsilon_n\mathbf{Z}[[\Sigma]])^{m_n}$. By transporting the right $\mathbf{Z}[[\Sigma]]$-module structure from the latter to the former via ϕ, we get $K_0^*[R,\Sigma]_{\mathrm{lfd}}$ as a $\mathbf{Z}[[\Sigma]]$-module satisfying all the requirements. ∎

2 Graded Cartan Matrices and Graded Cartan Endomorphisms

When $R = \bigoplus_{\sigma\in\Sigma} R_\sigma$ is an admissible grading on the ring R, we fix a numbering of the elements of $\mathbf{S}_I$, $\{S_{11}, \ldots, S_{1m_1}, \ldots, S_{n1}, \ldots, S_{nm_n}\}$, with $o(S_{ij}) = \epsilon_i$, for $i = 1, \ldots, n$ and $j = 1, \ldots, m_i$. There is a family $\{e_{11}, \ldots, e_{1m_1}, \ldots, e_{n1}, \ldots, e_{nm_n}\}$ of primitive orthogonal idempotents, with each e_{ij} homogeneous of degree ϵ_i and $\mathrm{R}e_{ij}/Ie_{ij} \cong S_{ij}$. In the sequel, we put $P_{ij} = \mathrm{R}e_{ij}$, viewed as a graded module. We have seen in the proof of Theorem 1.9 that the assignment $M \to f^M$ can be thought of as an abelian group isomorphism $\phi\colon K_0^*[R,\Sigma]_{\mathrm{lfd}} \cong (\epsilon_1\mathbf{Z}[[\Sigma]])^{m_1} \oplus \cdots \oplus (\epsilon_n\mathbf{Z}[[\Sigma]])^{m_n}$, so that it makes sense to talk about the (i,j) component of f^M. Before giving our key definition, we notice that, even when $\mathbf{Z}[[\Sigma]]$ does not have a ring structure, the abelian group $\epsilon_i\mathbf{Z}[[\Sigma]]\epsilon_j = \{f \in \mathbf{Z}[[\Sigma]] \mid f(\sigma) = 0$, for all $\sigma \notin \epsilon_i\Sigma\epsilon_j\}$ makes sense.

Definition 2.1 Given an admissible grading $R = \bigoplus_{\sigma\in\Sigma} R_\sigma$, we define the associated *graded Cartan matrix*, denoted C_R, as a $n \times n$ block matrix $C_R = (C_{ij})$, where the (i,j) block C_{ij} is the $m_i \times m_j$-matrix with entries in $\epsilon_i\mathbf{Z}[[\Sigma]]\epsilon_j$ whose (k,l) entry $C_{ij}(k,l)$ is the (i,k) component of $f^{P_{jl}}$.

The l-th column within the j-th block column of C_R,

$$\big(C_{1j}(1,l), \ldots, C_{1j}(m_1,l), \ldots, C_{nj}(1,l), \ldots, C_{nj}(m_n,l)\big)^T,$$

is $f^{P_{jl}}$. In particular, when R is finitely graded (and, hence, artinian), the graded Cartan matrix has entries in $\mathbf{Z}\Sigma$. From now on we shall think of the elements of $(\epsilon_1\mathbf{Z}[[\Sigma]])^{m_1} \oplus \cdots \oplus (\epsilon_n\mathbf{Z}[[\Sigma]])^{m_n}$ as column-vectors

$$(f_{11}, \ldots, f_{1m_1}, \ldots, f_{n1}, \ldots, f_{nm_n})^T, \quad \text{where } f_{ij} \in \epsilon_i\mathbf{Z}[[\Sigma]].$$

Proposition 2.2 *Let* $R = \bigoplus_{\sigma\in\Sigma} R_\sigma$ *be an admissible grading. The following assertions hold:*

a) If the grading is finite, then the assignment $S_{ij} \to P_{ij}$ *extends to a homomorphism of right* $\mathbf{Z}\Sigma$*-modules* $K_0[R,\Sigma]_{\mathrm{fd}} \to K_0[R,\Sigma]_{\mathrm{fd}}$*, that, when we identify* $K_0[R,\Sigma]_{\mathrm{fd}}$ *with* $(\epsilon_1\mathbf{Z}\Sigma)^{m_1} \oplus \cdots \oplus (\epsilon_n\mathbf{Z}\Sigma)^{m_n}$*, is left multiplication by* C_R*.*

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b) *If Σ is a Möbius semigroup, assertion a) holds with $K_0[R,\Sigma]_{\mathrm{fd}}$ replaced by $K_0^*[R,\Sigma]_{\mathrm{lfd}}$ and $\mathbf{Z}\Sigma$ replaced by $\mathbf{Z}[[\Sigma]]$.*

Proof We prove the finite-dimensional case. The other one is similar. From Theorem 1.8 it follows that a typical element of $K_0[R,\Sigma]_{\mathrm{fd}}$ can be expressed in the form $\sum_{i,j} S_{ij} f_{ij}$, where f_{ij} is a uniquely determined element of $\epsilon_i\mathbf{Z}\Sigma$, for all i,j. Since $\mathrm{ann}_{\mathbf{Z}\Sigma} P_{ij} = (1-\epsilon_i)\mathbf{Z}\Sigma = \mathrm{ann}_{\mathbf{Z}\Sigma} S_{ij}$, we see that the assignment $\sum_{i,j} S_{ij} f_{ij} \mapsto \sum_{i,j} P_{ij} f_{ij}$ is a well-defined homomorphism of right $\mathbf{Z}\Sigma$-modules $\psi\colon K_0[R,\Sigma]_{\mathrm{fd}} \to K_0[R,\Sigma]_{\mathrm{fd}}$ that maps S_{ij} onto P_{ij}.

On the other hand, left multiplication by C_R yields a $\mathbf{Z}\Sigma$-homomorphism

$$(\epsilon_1\mathbf{Z}\Sigma)^{m_1} \oplus \cdots \oplus (\epsilon_n\mathbf{Z}\Sigma)^{m_n} \to (\epsilon_1\mathbf{Z}\Sigma)^{m_1} \oplus \cdots \oplus (\epsilon_n\mathbf{Z}\Sigma)^{m_n},$$

which in turn gives $\psi'\colon K_0[R,\Sigma]_{\mathrm{fd}} \to K_0[R,\Sigma]_{\mathrm{fd}}$, by Theorem 1.8 (c). Since the simple S_{jl} corresponds to the column vector of $(\epsilon_1\mathbf{Z}\Sigma)^{m_1} \oplus \cdots \oplus (\epsilon_n\mathbf{Z}\Sigma)^{m_n}$ having ϵ_j in its (j,l) component and zero elsewhere, we see that $\psi'(S_{jl})$ is the element of $K_0[R,\Sigma]_{\mathrm{fd}}$ corresponding to the (j,l) column of the graded Cartan matrix, *i.e.*, $\psi'(S_{jl}) = P_{jl}$, for $j = 1,\dots,n$ and $l = 1,\dots,m_j$, and so $\psi' = \psi$. ■

Terminology The endomorphisms of $K_0[R,\Sigma]_{\mathrm{fd}}$ and $K_0^*[R,\Sigma]_{\mathrm{lfd}}$ from Proposition 2.2 a) and b) are called the *(graded) Cartan endomorphisms*. There are two cases of particular interest here. The first is when Σ is a monoid and $\{1\}$ is the distinguished set, where C_R has a unique block whose size is $m \times m$, with m the number of non-isomorphic simple R-modules S such that $IS = 0$. In such a situation, $K_0[R,\Sigma]_{\mathrm{fd}} \cong (\mathbf{Z}\Sigma)^m$ and, in the Möbius semigroup case, $K_0^*[R,\Sigma]_{\mathrm{lfd}} \cong \mathbf{Z}[[\Sigma]]^m$. When, instead, $m_i = 1$, for $i = 1,\dots,n$, we shall say, recalling terminology from finite dimensional algebras, that the grading is *basic*. In this latter situation the (i,j)-block of C_R is an element of $\epsilon_i\mathbf{Z}[[\Sigma]]\epsilon_j$, for every pair (i,j), $K_0^*[R,\Sigma]_{\mathrm{lfd}} \cong \mathbf{Z}[[\Sigma]]_{\mathbf{Z}[[\Sigma]]}$ and $K_0[R,\Sigma]_{\mathrm{fd}} \cong \mathbf{Z}\Sigma_{\mathbf{Z}\Sigma}$. This has the following consequence.

Corollary 2.3 *Let $R = \bigoplus_{\sigma\in\Sigma} R_\sigma$ be a basic admissible grading on the ring R, $C_R = (C_{ij})$ the associated graded Cartan matrix and $\Delta_R = \sum_{i,j} C_{ij}$. Then:*

a) *If the grading is finite, when we identify $K_0[R,\Sigma]_{\mathrm{fd}}$ with $\mathbf{Z}\Sigma_{\mathbf{Z}\Sigma}$, the Cartan endomorphism is left multiplication by Δ_R.*
b) *If Σ is a Möbius semigroup, when we identify $K_0^*[R,\Sigma]_{\mathrm{lfd}}$ with $\mathbf{Z}[[\Sigma]]_{\mathbf{Z}[[\Sigma]]}$, the Cartan endomorphism is left multiplication by Δ_R.*

Proof Since in both cases left multiplication by Δ_R defines an endomorphism φ of the given module ($\mathbf{Z}\Sigma_{\mathbf{Z}\Sigma}$ or $\mathbf{Z}[[\Sigma]]_{Z[[\Sigma]]}$), we only have to check $\varphi(S_j) = \phi(S_j)$ for all $j = 1,\dots,n$, where ϕ is the Cartan endomorphism and S_j is the unique simple, up to isomorphism, such that $IS = 0$ and $o(S) = \epsilon_j$. One should first notice that, under the canonical isomorphisms $K_0[R,\Sigma]_{\mathrm{fd}} \cong \mathbf{Z}\Sigma_{\mathbf{Z}\Sigma}$ and $K_0^*[R,\Sigma]_{\mathrm{lfd}} \cong \mathbf{Z}[[\Sigma]]_{\mathbf{Z}[[\Sigma]]}$, S_j corresponds to ϵ_j. Now, viewed as a column-vector of $\epsilon_1\mathbf{Z}\Sigma \oplus \cdots \oplus \epsilon_n\mathbf{Z}\Sigma$, $\phi(S_j)$ is the j-th column of C_R. But, when we identify $\epsilon_1\mathbf{Z}\Sigma \oplus \cdots \oplus \epsilon_n\mathbf{Z}\Sigma$ with $\mathbf{Z}\Sigma$ in the obvious way, that element is $\sum_{i=1}^n C_{ij} = \Delta_R \cdot \epsilon_j$. ■

Remark 2.4 When the grading is basic, if we consider f^R as in the paragraph after Theorem 1.8, we immediately see that Δ_R is the sum of all components of f^R. To generalize this to a locally finite-dimensional graded R-module M, we denote by Δ_M the sum of all components of f^M, *i.e.*, $\Delta_M = \sum_{\sigma\in\Sigma} c_{R_1}(M_\sigma)\sigma$: Δ_M will be called the *Hilbert* Σ*-series* of M.

We shall look at resolutions using the $P_{ij}[\sigma]$, with $o(\sigma) = \epsilon_i$. A direct sum $P \in (R,\Sigma)$-gr of such modules will be called an *isoprojective module.* Then P is projective as an ungraded module and it behaves *like* a projective in (R,Σ)-gr (Proposition 2.6 and Remark 2.8). We adapt the standard definition to our situation.

Definition 2.5 Let $M \in (R,\Sigma)$-gr. An *isoprojective cover* of M is an epimorphism $p\colon P \to M$ in (R,Σ)-gr, where P is a isoprojective and $\ker p \subseteq IP$.

The following is fundamental.

Proposition 2.6 *Let* $R = \bigoplus_{\sigma\in\Sigma} R_\sigma$ *be an admissible grading on the ring* R*. Every* $M \in (R,\Sigma)$-gr *has an isoprojective cover and, hence, a minimal isoprojective resolution. Moreover:*

a) If the grading is finite, then the isoprojective cover of every finite-dimensional graded module is finite-dimensional.
b) If Σ *is a Möbius semigroup, then the isoprojective cover of every locally finite-dimensional graded module is locally finite-dimensional.*

Proof The proof is similar to the proof for left perfect rings, bearing in mind that IM is a superfluous subobject of M in (R,Σ)-gr (Proposition 1.3). One notes also that when Σ is Möbius, a direct sum, P, of the $P_{ij}[\sigma]$, where there are only finitely many summands of each graded isomorphism type, is locally finite dimensional. ∎

Definition 2.7 Let M be a graded left R-module. We say that the *isoprojective dimension* of M is $n \in \mathbf{N}$, written $\operatorname{isopd}(M) = n$, if there exists an exact sequence in (R,Σ)-gr, $0 \to Q_n \to \cdots \to Q_0 \to M \to 0$, with all Q_i isoprojective modules and n is minimal with this property. If there is no such n, $\operatorname{isopd}(M) = \infty$.

Remark 2.8 Suppose in Proposition 2.6 that (i) the grading is finite, or (ii) Σ is left cancellative. If P is a graded module which is projective as an ungraded module then it is isoprojective. In particular, the projective and isoprojective dimensions of a graded module coincide.

Proof In case (i), R is left artinian and $I = J(R)$. Then an isoprojective cover is projective. In case (ii), the proof of [17, Lemma I.2.1] applies here. ∎

Recall [21, Definition 8.10] that the λ*-dimension* of a module M, written $\lambda(M)$, is the supremum (possibly ∞, the most useful case) of the natural numbers $n \in \mathbf{N}$ such that there is an exact sequence $F_n \to F_{n-1} \to \cdots \to F_0 \to 0$, with all the F_k free (or projective) and finitely generated.

Proposition 2.9 *Let $R = \bigoplus_{\sigma\in\Sigma} R_\sigma$ be an admissible grading on R.*

a) If Σ is a Möbius semigroup, the Cartan map $K_0^*[R,\Sigma]_{\mathrm{lfd}} \to K_0^*[R,\Sigma]_{\mathrm{lfd}}$ is surjective. In addition, if $M \in (R,\Sigma)_{\mathrm{lfd}}\text{-gr}$ satisfies $\lambda(M) = \infty$ and $\operatorname{isopd}(M) < \infty$, then a preimage of M exists in $K_0[R,\Sigma]_{\mathrm{fd}}$.

b) If the grading is finite, then every $M \in (R,\Sigma)_{\mathrm{fd}}\text{-gr}$ with $\operatorname{isopd}(M) < \infty$ is in the image of the graded Cartan map $K_0[R,\Sigma]_{\mathrm{fd}} \to K_0[R,\Sigma]_{\mathrm{fd}}$.

Proof Take $M \in (R,\Sigma)_{\mathrm{lfd}}\text{-gr}$ and consider its minimal isoprojective resolution $\cdots \to Q_n \to \cdots \to Q_0 \to M \to 0$. Then, Proposition 2.6 tells us that all its terms are in $(R,\Sigma)_{\mathrm{lfd}}\text{-gr}$. We claim that $\sum_{n\geq 0}(-1)^n Q_n$ is a well-defined element of $K_0^*[R,\Sigma]_{\mathrm{lfd}}$. Indeed, each Q_m is a direct sum of modules of the form $P_{ij}[\sigma]$, with only finitely many copies of each $P_{ij}[\sigma]$ appearing in a given Q_m. Moreover, if $P_{ij}[\sigma]$ appears as a direct summand of Q_m, then $\sigma = \nu\tau$, for some $\tau \in \Sigma$ such that $P_{kl}[\tau]$ appears as a direct summand of Q_{m-1}, where $\epsilon_k = o(\tau)$ and $l \in \{1,\dots,m_k\}$. Consequently, for $\sigma \in \Sigma$ fixed, the appearance of the graded modules $\{P_{ij}[\sigma] \mid j = 1,\dots,m_i\}$ as direct summands of infinitely many Q_m would imply the existence of infinitely many right divisors for σ in Σ, which contradicts our assumption on Σ. So, if we denote by $c_{jm\sigma}$ the number of times that $P_{ij}[\sigma]$ appears as direct summand of Q_m, then $c_\sigma = \sum_{1\leq j\leq m_i}\sum_{m\geq 0}(-1)^m c_{jm\sigma}$ is a well-defined integer and we can identify $\sum_{\sigma\in\Sigma} c_\sigma\sigma$ with $\sum_{n\geq 0}(-1)^n Q_n$. Then one has $M = \sum_{n\geq 0}(-1)^n Q_n$ in $K_0^*[R,\Sigma]_{\mathrm{lfd}}$.

By expanding that equality we get $M = \sum_{1\leq i\leq n}\sum_{1\leq j\leq m_i} P_{ij}\cdot g_{ij}$, where $g_{ij} = \sum_{\sigma\in\epsilon_i\Sigma}\bigl(\sum_{m\geq 0}(-1)^m c_{jm\sigma}\bigr)\sigma$. Since $P_{ij} = \phi(S_{ij})$, where ϕ is the Cartan map, we conclude that $M = \phi(\sum_{i,j} S_{ij}\cdot g_{ij}) \in \operatorname{Im}\phi$. When $\lambda(M) = \infty$, all the Q_m are finitely generated. When in addition $\operatorname{isopd}(M) < \infty$, the g_{ij} are elements of $\mathbf{Z}\Sigma$. Therefore $\sum_{i,j} S_{ij}\cdot g_{ij}$ is an element of $K_0[R,\Sigma]_{\mathrm{fd}}$. This latter argument also works for part b) of the proposition, bearing in mind that, since R is left artinian in this case, every finitely generated left R-module M has $\lambda(M) = \infty$. ■

Some important but straightforward consequences follow.

Corollary 2.10 *Let $R = \bigoplus_{\sigma\in\Sigma} R_\sigma$ be an admissible grading on R.*

a) If Σ is Möbius then, for every $M \in (R,\Sigma)_{\mathrm{lfd}}\text{-gr}$, there is a column vector $g^M \in (\epsilon_1\mathbf{Z}[[\Sigma]])^{m_1}\oplus\cdots\oplus(\epsilon_n\mathbf{Z}[[\Sigma]])^{m_n}$ such that $C_R\cdot g^M = f^M$. When, moreover, $\operatorname{isopd}(M) < \infty$ and $\lambda(M) = \infty$, g^M can be chosen from $(\epsilon_1\mathbf{Z}\Sigma)^{m_1}\oplus\cdots\oplus(\epsilon_n\mathbf{Z}\Sigma)^{m_n}$.

b) If the grading is finite, then, for every $M \in (R,\Sigma)_{\mathrm{fd}}\text{-gr}$ with $\operatorname{isopd}(M) < \infty$, there is a g^M in $(\epsilon_1\mathbf{Z}\Sigma)^{m_1}\oplus\cdots\oplus(\epsilon_n\mathbf{Z}\Sigma)^{m_n}$ such that $C_R\cdot g^M = f^M$.

Corollary 2.11 *Suppose that the admissible grading on R is basic.*

a) If Σ is Möbius then, for every $M \in (R,\Sigma)_{\mathrm{lfd}}\text{-gr}$, there is a $\chi^M \in \mathbf{Z}[[\Sigma]]$ such that $\Delta_R\cdot\chi^M = \Delta_M$. When, moreover, $\operatorname{isopd}(M) < \infty$ and $\lambda(M) = \infty$, χ^M can be chosen from $\mathbf{Z}\Sigma$.

b) If the grading is finite, then, for every $M \in (R,\Sigma)_{\mathrm{fd}}\text{-gr}$ with $\operatorname{isopd}(M) < \infty$, there is a $\chi^M \in \mathbf{Z}\Sigma$ such that $\Delta_R\cdot\chi^M = \Delta_M$.

Corollary 2.12 *Let $R = \bigoplus_{\sigma\in\Sigma} R_\sigma$ be an admissible grading.*

a) If Σ is Möbius, there is a block matrix $D = (D_{ij}) \in \mathbf{M}_n(\mathbf{Z}[[\Sigma]])$, with $D_{ij} \in M_{m_i\times m_j}(\epsilon_i\mathbf{Z}[[\Sigma]]\epsilon_j)$ for each pair (i, j), such that $C_R \cdot D = \epsilon_1 I_{m_1} \oplus \cdots \oplus \epsilon_n I_{m_n}$ (diagonal sum). If, moreover, $\sup\{\operatorname{isopd}(S) \mid S \in \mathbf{S}_I\} < \infty$ and $\lambda(R/I) = \infty$, then D can be chosen so that $D_{ij} \in M_{m_i\times m_j}(\epsilon_i\mathbf{Z}\Sigma\epsilon_j)$.

b) If the grading is finite and $\operatorname{gl.dim.}(R) < \infty$, then there is a D as above, with $D_{ij} \in M_{m_i\times m_j}(\epsilon_i\mathbf{Z}\Sigma\epsilon_j)$ for all i, j, such that $C_R \cdot D = \epsilon_1 I_{m_1} \oplus \cdots \oplus \epsilon_n I_{m_n}$.

c) When the grading is basic, there is a $\chi^{R/I} \in \mathbf{Z}[[\Sigma]]$ such that $\Delta_R \cdot \chi^{R/I} = 1$. If, moreover, $\sup\{\operatorname{isopd}(S) \mid S \in \mathbf{S}_I\} < \infty$ and $\lambda(R/I) = \infty$, then $\chi^{R/I}$ can be chosen in $\mathbf{Z}\Sigma$, this latter fact is also true when the grading is finite, for any Σ.

Proof a) By Corollary 2.10, for each $i = 1, \dots, n$ and $j = 1, \dots, m_i$, there is a column vector $g^{S_{ij}} \in (\epsilon_1\mathbf{Z}[[\Sigma]])^{m_1} \oplus \cdots \oplus (\epsilon_n\mathbf{Z}[[\Sigma]])^{m_n}$ such that $C_R \cdot g^{S_{ij}} = f^{S_{ij}}$. If we now consider the matrix D with entries in $\mathbf{Z}[[\Sigma]]$ whose columns are

$$g^{S_{11}}, \dots, g^{S_{1m_1}}, \dots, g^{S_{n1}}, \dots, g^{S_{nm_n}},$$

we see that $C_R \cdot D = \epsilon_1 I_{m_1} \oplus \cdots \oplus \epsilon_n I_{m_n}$ as desired. Since $g^{S_{ij}}$ has entries in $\mathbf{Z}\Sigma$ when $\operatorname{isopd}(S_{ij}) < \infty$ and $\lambda(S_{ij}) = \infty$, the last statement of a) follows from the corresponding statement in Corollary 2.10.

Statement b) is clear and, for c), we only have to apply Corollary 2.11 (b) with $M = R/I$ and realize that, when the grading is basic, $\Delta_{R/I} = \epsilon_1 + \cdots + \epsilon_n = 1$. ■

Observation 2.13.1 If, under the assumptions of Corollary 2.12, Σ is a monoid and m is the number of elements in $\mathbf{S}_I$, assertion a) says that the Cartan matrix C_R always has a right inverse D in $M_{m\times m}(\mathbf{Z}[[\Sigma]])$, with entries in $\mathbf{Z}\Sigma$ whenever $\sup\{\operatorname{isopd}(S) \mid S \in \mathbf{S}_I\} < \infty$ and $\lambda(R/I) = \infty$. When the grading is finite, the latter fact is true for arbitrary Σ; see [19, Theorem 1.7], where the fact was used to confirm the Cartan determinant conjecture for finite dimensional algebras in some new cases.

Observation 2.13.2 The proof of Proposition 2.9 yields a special element

$$g^M \in (\epsilon_1\mathbf{Z}[[\Sigma]])^{m_1} \oplus \cdots \oplus (\epsilon_n\mathbf{Z}[[\Sigma]])^{m_n}$$

such that $C_R \cdot g^M = f^M$, namely, $g^M = (g_{11}, \dots, g_{1m_1}, \dots, g_{n1}, \dots, g_{nm_n})^T$, where $g_{ij} = \sum_{\sigma\in\epsilon_i\Sigma}\bigl(\sum_{k\geq 0}(-1)^k c_{jk\sigma}\bigr)\sigma$, and $c_{jk\sigma}$ is the number of times that $P_{ij}[\sigma]$ appears as a direct summand of the k-th term Q_k of the minimal isoprojective resolution of M. Since $\operatorname{Tor}_k^R(R/I, M)$ is isomorphic to Q_k/IQ_k there is a canonical Σ-grading on each Tor group. It is not hard to see that

$$g_{ij} = \sum_{\sigma\in\epsilon_i\Sigma}\Bigl(\sum_{k\geq 0}(-1)^k c_{R_1}\bigl(\operatorname{Tor}_k^R(R/I, M)_\sigma^j\bigr)\Bigr)\sigma,$$

where $\operatorname{Tor}_k^R(R/I, M)_\sigma^j$ is the trace of $S_{ij}[\sigma]$ in $\operatorname{Tor}_k^R(R/I, M)$. In particular, when the grading is basic, the index j may be omitted above and the element $\chi^M \in \mathbf{Z}[[\Sigma]]$ such that

$\Delta_R \cdot \chi^M = \Delta_M$ is

$$\chi^M = \sum_{1\leq i\leq n} g_i = \sum_{\sigma\in\Sigma}\Big(\sum_{k\geq 0}(-1)^k c_{R_1}\big(\mathrm{Tor}_k^R(R/I, M)_\sigma\big)\Big)\sigma.$$

We call χ^M the *(graded) Euler characteristic* of M. Govorov uses the name "multiplicity series" for χ^M and [12, Theorem 3] proves the equality $\Delta_R \cdot \chi^M = \Delta_M$ for classically graded algebras, *i.e.*, positively graded algebras, $R = \bigoplus_{n\geq 0} R_n$, over a field K such that $R_0 = K$ and R_1 is a finite-dimensional K-vector space that generates R as a K-algebra. Equations like $\Delta_R \cdot \chi^M = \Delta_M$ appear in many contexts. See, for example [2, p. 654] or [4, equation (2)] and Section 4, below.

Example As a simple illustration, take R to be the path algebra $K\Gamma$ of the quiver $\Gamma\colon 1 \underset{\beta}{\overset{\alpha}{\leftrightarrows}} 2$. There are several possible gradings. Here we take $\Sigma = \mathbf{P}(\Gamma)$. Put $\Omega_{11} = \epsilon_1 + \sum_k(\beta\alpha)^k$, $\Omega_{22} = \epsilon_2 + \sum_k(\alpha\beta)^k$. Then $\Delta_R = \Omega_{11} + \Omega_{22}\alpha + \Omega_{11}\beta + \Omega_{22}$ and $\chi^{R/I} = \epsilon_1 + \epsilon_2 - \alpha - \beta$. The equation $\Delta_R \cdot \chi^{R/I} = \epsilon_1 + \epsilon_2 = 1$ is readily verified.

Observation 2.13.3 If $R = A\Gamma/H$, with Γ finite, the associated semigroup is Möbius and H has a finite Gröbner basis for some ordering of the paths in Γ, then a suitable adaptation of the arguments in [10] proves that R is a not necessarily Noetherian Σ-graded ring for which $\lambda(R/I) = \infty$.

3 Applications to Monomial Algebras

The term "monomial A-algebra" will be as in Examples 1.4(3). We consider the canonical $\mathbf{P}(\Gamma)$-grading on it, except as noted. This grading is basic. As seen in (1.4 (3)), the associated semigroup ring is $\mathbf{Z}\Gamma$, while the associated incidence ring is the usual incidence ring $\mathbf{Z}[[\Gamma]]$ of a quiver (see, *e.g.*, [16]). Notice also that, by [8, Theorem 1.1], Δ_R is always invertible in $\mathbf{Z}[[\Gamma]]$, implying, in particular, that the Cartan endomorphism of $K_0^*[R,\Sigma]_{\mathrm{lfd}}$ is bijective. By Corollary 2.12(c), $\Delta_R^{-1} = \chi^{R/I}$. If $A = M_{n\times n}(D)$, where D is a division ring, then $R = A\Gamma/H = M_{n\times n}(D\Gamma/H)$ and so we may assume in our proofs that A is a division ring. Clearly, Δ_R does not depend on the ground (division) ring A and, hence, neither does $\Delta_R^{-1} = \chi^{R/I}$. To emphasize this, we shall use Δ_R^{-1} instead of $\chi^{R/I}$ in the statements, but, whenever necessary, will use the properties of $\chi^{R/I}$.

Let the simple $S \in \mathbf{S}_I$ be with $o(S) = \epsilon_i$. Then, by Corollary 2.10(a), $\chi^S = \Delta_R^{-1}\cdot\epsilon_i$. For a path $p \in \mathrm{supp}(\Delta_R^{-1}\cdot\epsilon_i)$, we shall denote by $\xi(p)$ the number of right divisors q of p, with length ≥ 1 in $\mathbf{P}(\Gamma)$, such that $q \in \mathrm{supp}(\Delta_R^{-1}\cdot\epsilon_i)$. Moreover, an element $f \in \mathbf{Z}[[\Gamma]]$ will be said to be of *bounded support* in case there is a $k \geq 0$, such that $l(p) \leq k$, for every path $p \in \mathrm{supp}(f)$.

Theorem 3.1 **(*cf.* [5, Proposition 1.4])** *Let $R = A\Gamma/H$ be a monomial A-algebra, where A is a simple artinian ring. Suppose that S is a simple left R-module such that $IS = 0$ and $o(S) = \epsilon_i$. Then $\mathrm{pd}(S) = \sup\{\xi(p) \mid p \in \mathrm{supp}(\Delta_R^{-1}\cdot\epsilon_i), l(p) \geq 1\}$ and it is independent of the ring A. If there is an upper bound on the lengths of paths in a generating set of H, the following assertions are equivalent:*

a) $\mathrm{pd}(S) < \infty$.
b) $\Delta_R^{-1} \cdot \epsilon_i$ *has bounded support.*

When, moreover, R is finitely presented, the above conditions are also equivalent to:

c) $\Delta_R^{-1} \cdot \epsilon_i \in \mathbf{Z}\Gamma$.

Proof We again assume that A is a division ring. The second member of the proposed equality is independent of A, so that we just have to prove that equality. On the other hand, the last part of Observation 2.13.1 tells us that $\mathrm{pd}(S) = \mathrm{isopd}(S)$. The method of Anick and Green [3] applies in our setting and, as in [3, Lemma 2.8], one finds a graded projective resolution $\cdots Q_k \to Q_{k-1} \to \cdots \to Q_0 \to S \to 0$, with $Q_k = \bigoplus_{p\in\Gamma_k^i} P_{o(p)}[p]$ for every $k \geq 0$, where Γ_k^i denotes the set of k-chains with terminus i. As in [3, Lemma 2.8], the monomial condition of R implies that the resolution is minimal. Since the Γ_k^i, $k \geq 0$, are disjoint we cannot have a $P_j[p]$ ($j = o(p)$) appearing as direct summand in Q_k and Q_l, with $k \neq l$. It follows that $p \in \mathrm{supp}(\chi^S) = \mathrm{supp}(\Delta_R^{-1} \cdot \epsilon_i)$ if and only if $P_j[p]$ appears as direct summand of a (unique) Q_k if and only if $p \in \Gamma_k^i$, for a (unique) $k = 0, 1, \ldots$. But, for $k \geq 2$, $p \in \Gamma_k^i$ if and only if $p = p_k p_{k-1} \cdots p_1$, where the $p^{(r)} = p_r p_{r-1} \cdots p_1$, $r = 1, \ldots, k$, are the right divisors of p which appear in $\mathrm{supp}(\chi^S) = \mathrm{supp}(\Delta_R^{-1} \cdot \epsilon_i)$. The equality $\mathrm{pd}(S) = \sup\{\xi(p) \mid p \in \mathrm{supp}(\Delta_R^{-1} \cdot \epsilon_i), l(p) \geq 1\}$ now follows.

For the second part, using the equality even without our extra assumption on H, we have b) $\Rightarrow$ a). Now $t = \sup\{l(q) \mid q \in \Gamma_2\}$ is finite when the extra hypothesis is assumed. Since every k-chain is an overlapping of $k-1$ of the 2-chains; the length of a path in $\mathrm{supp}(\Delta_R^{-1} \cdot \epsilon_i)$ is $\leq t \cdot \mathrm{pd}(S)$, and so b) holds.

Finally, when R is finitely presented, the equivalence b) $\Leftrightarrow$ c) is obvious, because, when Γ is finite, an element $f \in \mathbf{Z}[[\Gamma]]$ has bounded support if and only if $f \in \mathbf{Z}\Gamma$. ■

In this context, the global dimension is independent of A. This is known in the finite dimensional case (see [14]). The more general version may be derived from the recent results in [6], but this fact also falls out from the method of Theorem 3.1.

Lemma 3.2 *Let $R = A\Gamma/H$ be a monomial algebra, where A is a simple artinian ring. Then* $\sup\{\mathrm{isopd}(S) \mid S \in \mathbf{S}_I\} = \text{gr-gl}\,.\dim.(R) = \mathrm{gl}\,.\dim.(R)$.

Proof Let us denote by a, b and c the three members of the equalities. We always have $a \leq b$ and, since $\mathrm{isopd}(M) = \mathrm{pd}(M)$ for every graded module M (see Observation 2.13.1), also $b \leq c$. Finally, by [18, Corollary, p. 424], we know that $c = \mathrm{pd}(R/I) = \sup\{\mathrm{pd}(S) \mid S \in \mathbf{S}_I\} = a$. ■

Corollary 3.3 *Let $R = A\Gamma/H$ be a monomial A-algebra, where A is a simple artinian ring. Then* $\mathrm{gl}\,.\dim.(R) = \sup\{\xi(p) \mid p \in \mathrm{supp}(\Delta_R^{-1}), l(p) \geq 1\}$ *and it is independent of A.*

If there is an upper bound on the lengths of paths in a generating set of H, the following assertions are equivalent:
a)$\mathrm{gl}\,.\dim.(R) < \infty$.
b)Δ_R^{-1} *has bounded support.*

When, moreover, R is finitely presented, the above conditions are also equivalent to:

c) $\Delta_R^{-1} \in \mathbf{Z}\Gamma$.

Example Consider the free monoid Σ on the countable set $\mathbf{X} = \{x_1, \dots, x_n, \dots\}$, so that $\mathbf{Z}[[\Sigma]]$ is the free incidence ring on $\mathbf{X}$. If $\Delta \in \mathbf{Z}[[\Sigma]]$ is the sum of all words not containing a subword of the form $x_i x_{i+1}$, unless $i \in 3\mathbf{N} = \{3, 6, 9, \dots\}$, one readily sees that $\Delta - \sum_{i\geq 1} x_i\Delta + \sum_{i\geq 1, i\notin 3\mathbf{N}} x_i x_{i+1}\Delta - \sum_{i\in 3\mathbf{N}+1} x_i x_{i+1} x_{i+2}\Delta = 1$. Now if, for a simple artinian ring A, $R = R(3) = A\langle \mathbf{X}\rangle/I(3)$, where $I(3)$ is the ideal of $A\langle \mathbf{X}\rangle$ generated by the monomials $\{x_i x_{i+1} : i \notin 3\mathbf{N}\}$, then its Hilbert Σ-series is $\Delta_R = \Delta$. Consequently,

$$\Delta_R^{-1} = 1 - \sum_{i\geq 1} x_i + \sum_{i\geq 1, i\notin 3\mathbf{N}} x_i x_{i+1} - \sum_{i\in 3\mathbf{N}+1} x_i x_{i+1} x_{i+2}.$$

The above corollary states that $\operatorname{gl.dim}(R) = 3$. This can be also checked by using the (minimal) Anick resolution of $A = R/I$. An analogous argument shows that $\operatorname{gl.dim.}\big(R(n)\big) = n$, for every $n \geq 2$.

In a classically graded K-algebra $R = \bigoplus_{n\geq 0} R_n$, K a field, the Hilbert series takes the form $\Delta_R = \sum_{n\geq 0} \dim_K(R_n)T^n \in \mathbf{Z}[[T]]$. In [12, Corollary 1] the author claims that when $\Delta_R^{-1} = \chi^K$ is a polynomial then $\operatorname{gl.dim.}(R) < \infty$. This seems not to be the case, even for a finitely presented monomial algebra.

Proposition 3.4 *There is a finitely presented monomial algebra R so that $\Delta_R^{-1} = \chi^K$ is a polynomial while* $\operatorname{gl.dim.}(R) = \infty$.

Proof Consider the monomial K-algebra $R = K\langle X, Y\rangle/H$, where H is the two-sided ideal of $K\langle X, Y\rangle$ generated by $\{XY^2, X^2\}$, that we view as a classically graded algebra using total degree. A K-basis of R is given by $\{1, x, xy, xyx, xyxy, \dots, y, y^2, y^3, \dots, yx, y^2x, y^3x, \dots, yxy, y^2xy, y^3xy, \dots, yxyx, y^2xyx, y^3xyx, \dots\}$. The associated semigroup, Σ, is the free monoid on x and y and $\Delta_R \in \mathbf{Z}[[\Sigma]]$. Putting $\Omega = 1 + x + xy + xyx + xyxy + \cdots$ and $\Phi = y + y^2 + y^3 + \cdots$, $\Delta_R = (1 + \Phi)\Omega$. The corresponding classical Hilbert series is $1/(1 - T)^2$. However, the left ideal Rx appears as direct summand of every syzygy of K, so that $\operatorname{gl.dim.}(R) = \infty$. ■

4 On the Rationality of the Hilbert Σ-Series

In this section, A is a division ring, Γ a finite quiver and Σ is a Möbius semigroup with distinguished family of idempotents $\{\epsilon_1, \dots, \epsilon_n\}$ ($n = |\Gamma_0|$), that grades $A\Gamma$ in such a way that the vertices and the arrows of Γ are homogeneous and the ϵ_i are the degrees of the vertices. In that case, we have a canonical semigroup homomorphism $P(\Gamma) \to \Sigma$, $p \mapsto \deg(p)$ and we shall put $\tilde{p} = \deg(p)$, for each path p. Finally, we will assume that H is a two-sided ideal of $A\Gamma$ generated by a finite number of Σ-homogeneous linear combinations of paths of length ≥ 2, so that $R = A\Gamma/H$ inherits a canonical admissible Σ-grading.

Definition 4.1 An element $h \in (\mathbf{Z}[[\Sigma]])[[T]]$ (resp. $\mathbf{Z}[[\Sigma]]$) will be called *rational* if there exist $f, g \in \mathbf{Z}\Sigma[T]$ (resp. $\mathbf{Z}\Sigma$) such that $h \cdot g = f$.

The equality $\Delta_R \cdot \chi^{R/I} = 1$ shows that the rationality of $\chi^{R/I}$ implies that of Δ_R. The latter will be established in some particular situations.

According to Anick and Green [3], there is a projective resolution (not necessarily minimal), which is also a Σ-graded resolution, of R/I given by sets of paths $\Gamma_0, \Gamma_1, \ldots$, where there is an explicit construction (based on a fixed well-ordering of the paths in Γ) using Γ_2, which, in turn, is built from a finite generating set of H by Gröbner methods. Then the resolution

$$\cdots \to P_m \to \cdots \to P_1 \to R \to R/I \to 0$$

has $P_m = \bigoplus_{p\in\Gamma_m} \mathrm{R}e_{o(p)}$ or $P_m = \bigoplus_{p\in\Gamma_m} \mathrm{R}e_{o(p)}[\tilde{p}]$, in Σ-graded form.

The gradablilty of the resolution must first be established.

Proposition 4.2 *Let $R = A\Gamma/H$ be a Σ-graded algebra as described above. Then the Anick-Green resolution of R/I gives rise to a Σ-graded graded projective resolution.*

Proof The approach of Farkas in [9] is useful here once the necessary modifications are in place to pass from quotients of free algebras to quotients of path algebras (as remarked in [13, p. 319]). (The notation used here is a hybrid of those of [3] and [9]. The two indexings differ by 1; that of [3] is used.) The maps, δ_n, in the resolution are first defined in terms of modules over the path algebra and then it is shown that they are compatible with the ideal H. The maps are presented using two operations. The first is defined on a set of paths, including those of Γ_n, which factor $p = \pi_1 c_1 \pi_2 c_2 \cdots$, where the c_i are paths and the $\pi_i \in \Gamma_2$. Then $S(p)$ is obtained by replacing the paths π_i by the corresponding relation in the reduced Gröbner generating set being used (π_i is the "tip" or leading term in the relation). Since these relations are Σ-homogeneous, this operation is degree preserving.

The next maps to be considered are called $D_\alpha\colon I[n] \to A\Gamma$, $\alpha \in \Gamma_n$, where $I[n]$ is a left ideal generated by a set of paths which includes Γ_n. These are not degree preserving since, in particular, $D_\alpha\big(S(\alpha)\big) = 1$ (while $D_\alpha\big(S(\beta)\big) = 0$ for $\alpha \neq \beta \in \Gamma_n$).

We can now look at the maps making up the resolution, $\delta_n\colon P_n \to P_{n-1}$, as described in [9]. For an indecomposable component P_n, say $\mathrm{R}u_\alpha \cong \mathrm{R}e_i[\tilde{\alpha}]$, coming from $\alpha \in \Gamma_n$, $\delta_n(u_\alpha)(\beta) = D_\beta\big(S(\alpha)\big)u_\beta$, for $\beta \in \Gamma_{n-1}$ and u_β in degree $\tilde{\beta}$. The result, if non-zero, will be a linear combination of paths q so that $q\beta$ is of the same degree as α. Now, this image will be in a graded projective in degree $\tilde{\beta}$ and, hence, the β-component of $\delta_n(u_\alpha)$ will be in degree $\widetilde{q\beta} = \tilde{\alpha}$. ■

Definition 4.3 Let $R = A\Gamma/H$ as described above, let Γ_0 (the vertices), Γ_1 (the arrows), $\Gamma_2, \ldots$ the sets of chains (with respect to a fixed suitable ordering of the paths in Γ). Then $\mathcal{G}(\Sigma, T) = \sum_{k\geq 0}(\sum_{p\in\Gamma_k} \tilde{p})T^k$ will be called the *Anick-Poincaré-Betti Σ-series* of R. Let $|*|$ denote A-dimension. The *Poincaré-Betti Σ-series* of R (*cf.* [4]) is $\mathcal{P} \in (\mathbf{Z}[[\Sigma]])[[T]]$ given by

$$\mathcal{P} = \mathcal{P}(\Sigma, T) = \sum_{k\geq 0}\Big(\sum_{\sigma\in\Sigma} |\operatorname{Tor}_k^R(R/I, R/I)_\sigma|\sigma\Big)T^k.$$

By the proof of Proposition 2.9 for $M = R/I$, we know that $\mathcal{G}(\Sigma, -1)$ and $\mathcal{P}(\Sigma, -1) = \chi^{R/I}$ are (right) inverses of Δ_R. Uniqueness of that inverse (*cf.* [8, Theorem 1.1]) implies $\mathcal{G}(\Sigma, -1) = \mathcal{P}(\Sigma, -1) = \chi^{R/I}$. Moreover, when the Anick resolution is minimal, $\mathcal{G}(\Sigma, T) = \mathcal{P}(\Sigma, T)$. We study the rationality of Δ_R by looking at that of $\mathcal{G}(\Sigma, T)$.

We want an algorithmic way of getting $\mathcal{G}(\Sigma, T)$. We require that, for a suitable well-ordering of the paths in Γ, the set of obstructions Γ_2 is finite, *i.e.*, that H has a finite Gröbner basis with respect to the well-ordering. This is assumed from now on in this section. One property of the construction of the sets Γ_m is essential here. Namely, that the elements of Γ_{m+1} are built from those of Γ_m by a process which can be expressed as being independent of m (for $m \geq 2$).

We use the notation of [3] but written using left modules. If we can write a path $p = p_2 p_1$, where both p_1 and p_2 have at least one arrow, p_2 is called a *proper left factor* of p (similarly for right factors). The method below recalls the construction of the *syzygy quiver* of Cibils [7] and the reasoning in [4, Theorem 1]. The set of all paths is denoted $\mathbf{B}$ and M is the set of paths which have no subpaths in Γ_2. The elements of Γ_m, $m > 2$ have unique factorizations $\gamma = \gamma_2\gamma_1$, where $\gamma_1 \in \Gamma_{m-1}$ and $\gamma_2 \in M$. Moreover, $\gamma_1 = \beta_2\beta_1$, where $\beta_1 \in \Gamma_{m-2}$, $\beta_2 \in M$ and $\gamma_2\beta_2 \notin M$. The uniqueness in the construction [3, Definition 2.1 and Lemma 2.3] says that $\gamma_2\beta_2$ has a left factor (necessarily unique) in Γ_2. Hence, γ_2 must be a proper left factor of an element of Γ_2.

The elements of Γ_2 are listed in some fixed order, $\Gamma_2 = \{\gamma_1, \dots, \gamma_r\}$. For each γ_i we list its proper left factors $\mu_{i,1}, \dots, \mu_{i,l_i}$, again with the order arbitrary but fixed. Notice that the same path can appear with two different pairs of indices.

If $p \in \Gamma_k$, $k \geq 3$, we have factorizations $p = \beta_2\beta_1$, $\beta_1 \in \Gamma_{k-1}$ and $\beta_2 \in M$, as well as $p = \gamma_i q$, for some $i = 1, \dots, r$. Hence, $\beta_2 = \mu_{i,j}$ for some j. When can we form $p' \in \Gamma_{k+1}$ where $p' = \mu_{i',j'} p$ with left factor $\gamma_{i'}$? Three conditions together are necessary and sufficient:

(i) the product $\mu_{i',j'}\mu_{i,j} \neq 0$;
(ii) $\mu_{i',j'}\mu_{i,j}$ has left factor $\gamma_{i'}$; and
(iii) the following does *not* happen: $\mu_{i',j'}$ has a proper right factor ν so that $\nu\gamma_i$ has a left factor in Γ_2.

This can be encapsulated in the following coefficients: $a(i', j'; i, j) = 1$ if the three conditions are satisfied, and $a(i', j'; i, j) = 0$ otherwise.

These coefficients allow one to predict the indecomposable direct summands of the (graded) projective module P_{m+1} knowing the summands of P_m. (The maps between them are *much* more complicated, see [3] and [9].) We now view the Anick-Poincaré-Betti series as a generating function $\mathcal{G} = \mathcal{G}(\Sigma, T) = \sum_{k\geq 0} g_k T^k$, where, for $k \geq 0$, $g_k = \sum_{p\in\Gamma_k} \tilde{p} \in \mathbf{Z}\Sigma$. The initial conditions are: $g_0 = \widetilde{v_1} + \cdots + \widetilde{v_n}$, $g_1 = \sum_{\alpha\in\Gamma_1} \tilde{\alpha}$ and $g_2 = \sum_{p\in\Gamma_2} \tilde{p}$. We further introduce auxiliary generating functions $\mathcal{G}_{i,j}$, $i = 1, \dots, r$ and $j = 1, \dots, l_i$. To simplify notation, let $\Gamma_k(i, j)$, $k \geq 3$, be the set of elements $\beta \in \Gamma_k$ which factor $\beta = \gamma_i\beta' = \mu_{i,j}\beta''$, $\beta'' \in \Gamma_{k-1}$. Then, $\mathcal{G}_{i,j} = \sum_{k\geq 3}\sum_{p\in\Gamma_k(i,j)} \tilde{p}T^k$. Denote the coefficient of T^k by $g(i, j)_k$. Then $\mathcal{G} = \sum_{i,j} \mathcal{G}_{i,j} + g_2T^2 + g_1T + g_0$.

The coefficients $a(i, j; i', j')$ connect the auxiliary generating functions by

$$g(i, j)_{k+1} = \sum_{i',j'} a(i, j; i'j')\widetilde{\mu_{i,j}}g(i', j')_k \quad \text{for } k \geq 3.$$

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This can be expressed in matrix form where $A = \big(a(i,j;i',j')\widetilde{\mu_{i,j}}\big)^t \in \mathbf{M}_s(\mathbf{Z}\Sigma)$, and s is the number of pairs i, j:

$$\begin{pmatrix} \mathcal{G}_{1,1} \\ \vdots \\ \mathcal{G}_{r,l_r} \end{pmatrix} = A \begin{pmatrix} \mathcal{G}_{1,1} \\ \vdots \\ \mathcal{G}_{r,l_r} \end{pmatrix} T + \begin{pmatrix} g(1,1)_3 \\ \vdots \\ g(r,l_r)_3 \end{pmatrix} T^3.$$

Let I_s be the $s \times s$ identity matrix; then the existence of a "nice" formula for $\mathcal{G}$ depends on being able to solve $B(I_s - AT) = I_s$ in some reasonable fashion. When that occurs, we get

$$\begin{pmatrix} \mathcal{G}_{1,1} \\ \vdots \\ \mathcal{G}_{r,l_r} \end{pmatrix} = B \begin{pmatrix} g(1,1)_3 \\ \vdots \\ g(r,l_r)_3 \end{pmatrix} T^3.$$

Although the rationality question makes sense in the above general setting, we only tackle it in the case when Γ only has one vertex, *i.e.*, in the case when $A\Gamma = A\langle X_1, \ldots, X_m\rangle$. By our initial assumptions, Σ is a monoid in this case. We shall refer to this case simply by saying that R is a *factor of a free algebra*. In order to give our main result in this section, we recall that a commutative monoid Σ is *torsionfree* in case $x^n \neq y^n$, whenever $x \neq y$ and $n > 0$.

Theorem 4.4 *Let $R = A\langle X_1, \ldots, X_m\rangle/H$ be a factor of a free algebra and let Σ be a torsion-free cancellative commutative monoid which grades R in the way described at the beginning of this section. If H has a finite Gröbner basis (for a suitable well-ordering of the monomials in $A\langle X_1, \ldots, X_m\rangle$), then the corresponding Anick-Poincaré-Betti Σ-series of R is rational. In such a situation, in particular:*

a) The Hilbert Σ-series and the Σ-graded Euler characteristic of R are rational.
b) When the Anick resolution is minimal, the Poincaré-Betti Σ-series of R is rational.

Proof By [11, Theorem 8.1], $\mathbf{Z}\Sigma$ is a commutative integral domain in this case, so that $\mathbf{Z}\Sigma[T]$ has a field of quotients, namely $\mathbf{Q}(\Sigma)(T)$, where $\mathbf{Q}(\Sigma)$ is the field of quotients of $\mathbf{Z}\Sigma$. Now, our previous considerations reduce the proof to checking that the matrix $I_s - AT \in M_{s\times s}(\mathbf{Z}\Sigma[T])$ is invertible in $M_{s\times s}\big(\mathbf{Q}(\Sigma)(T)\big)$. But that is clear, for, by construction of A, $\det(I_s - AT)$ is a polynomial in T (with coefficients in $\mathbf{Z}\Sigma$) whose constant term is 1. ■

Remark 4.5 The Anick-Green resolution is minimal for monomial algebras [3, Corollary 2.9]. In particular, the above theorem implies [4, Theorem 1]. The Anick-Green resolution is also minimal when Γ_2 consists of paths of length 2 (*cf.* [13, Theorem 3] or, more generally, the situation described in [19, Remark 2.4]). In general, if one can guarantee that whenever $p \in \Gamma_m$, $q \in \Gamma_{m'}$, with $\tilde{p} = \tilde{q}$, necessarily $m = m'$, then the resolution is automatically minimal. When $\dim_A(A\Gamma/H) < \infty$ or H is generated by a finite number of paths, H has a finite Gröbner basis [10, Theorem 15].

Example Consider the algebra $R = K\langle X, Y, Z\rangle/(f_1, f_2)$, for a field K with $f_1 = XYX - XZX$ and $f_2 = X^2Y - X^2Z$. Let Σ be the free commutative monoid on the variables x, s. Then R has an admissible Σ-grading, by putting $\deg(X) = x$ and $\deg(Y) = \deg(Z) = s$.

We take the order $X > Y > Z$. The only "overlap" between f_1 and f_2 is $Xf_1 - f_2X = 0$. Neither element reduces over the other. This means that $\{f_1, f_2\}$ is a Gröbner generating set for the ideal it generates. (See [10] for the methods.) Then $\Gamma_2 = \{XYX, X^2Y\}$ and in general Γ_n consists of words with x-degree n. It follows that the Anick resolution is minimal. The three variable Poincaré-Betti series of R, $\mathcal{P}(x, s, T)$, is thus rational, as is the classical Poincaré-Betti series.

5 A Remark on the "Strong No Loops Conjecture"

In this section A is a division ring and Γ a finite quiver, and we look at factors of path algebras $R = A\Gamma/H$ that are finite-dimensional as A-vector spaces. As remarked by Igusa in [15], not much is known about the "strong no loops conjecture": if ${}_RS$ is a simple module and $\operatorname{pd}(S) < \infty$ then $\operatorname{Ext}^1_R(S, S) = 0$ (equivalently, there are no loops at the vertex corresponding to S in the quiver). In this section the grading introduced in Example 1.43) will be used to confirm the conjecture in some special cases. The following generalizes [15, Corollary 6.2].

Proposition 5.1 *Let $R = A\Gamma/\langle\rho\rangle$ be a finite dimensional. Suppose there is a non-empty subset L of the set of loops at v_1 so that if some $\alpha \in L$ appears in a relation $\sum_{i=1}^{l} k_i p_i \in \rho$, $0 \neq k_i \in K$, $p_i = e_1 p_i e_1$ a path of length ≥ 2, which has more than one term, then either each p_i is a composition of loops from L or each p_i contains an arrow which is not in L. Then* $\operatorname{pd} S_1 = \infty$.

Proof Let Σ_{ab} be the abelianized version of $\Sigma = \Sigma(\Gamma, \rho)$. Any path which is not a composition of loops becomes 0 in Σ_{ab}, because of the orthogonal idempotents. If, now, each ε_i, $i > 1$, and each loop at ν_1 not from L are also sent to 0, we get a commutative monoid, denoted Σ^1_{ab}, where only ϵ_1 and abelianized compositions of loops from L remain non-zero. Let θ be the composition $\Sigma \to \Sigma_{\text{ab}} \to \Sigma^1_{\text{ab}}$, as well as the induced ring homomorphism $\mathbf{Z}\Sigma \to \mathbf{Z}\Sigma_{\text{ab}} \to \mathbf{Z}\Sigma^1_{\text{ab}}$. Note that $\theta(\varepsilon_1)$ may be denoted by 1.

If $\operatorname{pd} S_1 < \infty$ then we know, by Corollary 2.3, that $\Delta_R \cdot \chi^{S_1} = \varepsilon_1$. Apply θ to this equality to get $\theta(\Delta_R)\theta(\chi^{S_1}) = 1 \in \mathbf{Z}\Sigma^1_{\text{ab}}$. If now X denotes a maximal A-linearly independent set of compositions of loops from L, modulo $\langle\rho\rangle$, then our assumption implies that $\theta(\Delta_R) = 1 + \sum_{p\in X} \theta(p)$. Note that $|X| = m \geq 1$. Now apply the augmentation map $\alpha\colon \mathbf{Z}\Sigma^1_{\text{ab}} \to \mathbf{Z}$ (which sends each image of a path to 1) to get $(1 + m) \cdot \alpha\theta(\chi^{S_1}) = 1$, which is not possible in $\mathbf{Z}$. ■

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