Bull. Aust. Math. Soc. **109** (2024), 89–93
doi:10.1017/S000497272300014X

ON THE NUMBER OF REAL ZEROS OF POLYNOMIALS OF EVEN DEGREE

LANDE MA ✉ and ZHAOKUN MA

(Received 11 January 2023; accepted 17 January 2023; first published online 27 February 2023)

Abstract

For any real polynomial $p(x)$ of even degree k, Shapiro [‘Problems around polynomials: the good, the bad and the ugly…’, *Arnold Math. J.* **1**(1) (2015), 91–99] proposed the conjecture that the sum of the number of real zeros of the two polynomials $(k-1)(p'(x))^2 - kp(x)p''(x)$ and $p(x)$ is larger than 0. We prove that the conjecture is true except in one case: when the polynomial $p(x)$ has no real zeros, the derivative polynomial $p'(x)$ has one real simple zero, that is, $p'(x) = C(x)(x-w)$, where $C(x)$ is a polynomial with $C(w) \neq 0$, and the polynomial $(k-1)(C(x))^2(x-w)^2 - kp(x)C'(x)(x-w) - kC(x)p(x)$ has no real zeros.

2020 *Mathematics subject classification*: primary 26C10; secondary 12D10, 30C10, 30C15.

Keywords and phrases: real zeros, polynomials, Rolle’s theorem, conjecture of B. Shapiro.

1. Introduction

The assertion that if a real polynomial $p(x)$ has only simple real zeros, then the function $p(x)$ is (locally) strictly monotone was known to Gauss (see [3]). We can reformulate it in the form of the classical Laguerre inequality: if $p(x)$ has only simple real zeros, then the polynomial $p_1(x) = (p'(x))^2 - p(x)p''(x)$ is strictly positive. A refinement of the Laguerre inequality constitutes the Hawaiian conjecture (see [1]), where if $p(x)$ is a real polynomial, then the number of real zeros of $(p'(x)/p(x))'$ does not exceed the number of nonreal zeros of $p(x)$. The Hawaiian conjecture was settled in 2011 by Tyaglov [4]. Shapiro proposed three conjectures around the Hawaiian conjecture (see Conjectures 11, 12 and 13 in [2]). Conjecture 11 is discussed in [5].

We consider Conjecture 12 which states: for any real polynomial $p(x)$ of even degree k, we have $\Delta := \sharp_r[(k-1)(p'(x))^2 - kp(x)p''(x)] + \sharp_r p(x) > 0$. Here, $\sharp_r p(x)$ stands for the number of real zeros of a polynomial $p(x)$ with real coefficients.

Our first result shows that, in most cases, the conjecture is true.

THEOREM 1.1. *Let $p(x)$ be a real polynomial of even degree k. Then the quantity $\Delta = \sharp_r[(k-1)(p'(x))^2 - kp(x)p''(x)] + \sharp_r p(x) > 0$ if and only if one of the following four cases holds:*

(1) *the polynomial* $p(x)$ *has real zeros;*
(2) *the polynomial* $p(x)$ *has no real zeros and the polynomial* $p'(x)$ *has at least three distinct real zeros;*
(3) *the polynomial* $p(x)$ *has no real zeros and the polynomial* $p'(x)$ *has one real zero with exponent larger than 1;*
(4) *the polynomial* $p(x)$ *has no real zeros, the polynomial* $p'(x)$ *has one real zero which is simple, that is,* $p'(x) = C(x)(x-w)$*, where* $C(x)$ *is a polynomial with* $C(w) \neq 0$*, and the polynomial* $(k-1)(C(x))^2(x-w)^2 - kp(x)C'(x)(x-w) - kC(x)p(x)$ *has at least one real zero.*

The only case in which the conjecture is false is described in our second result.

THEOREM 1.2. *Let* $p(x)$ *be a real polynomial of even degree* k*. Then the quantity* $\Delta = \sharp_r[(k-1)(p'(x))^2 - kp(x)p''(x)] + \sharp_r p(x) = 0$ *if and only if the polynomial* $p(x)$ *has no real zeros, the polynomial* $p'(x)$ *has one real zero which is simple, that is,* $p'(x) = C(x)(x-w)$*, where* $C(x)$ *is a polynomial with* $C(w) \neq 0$*, and the polynomial* $(k-1)(C(x))^2(x-w)^2 - kp(x)C'(x)(x-w) - kC(x)p(x)$ *has no real zeros.*

At the end of the paper, we give some examples to show that the case described in Theorem 1.2 does occur.

2. Proofs of the theorems

We derive Theorem 1.1 from a series of lemmas.

LEMMA 2.1. *For a real polynomial* $p(x)$ *of even degree* k*, the real zeros of the polynomial* $kp''(x)p(x) - (k-1)(p'(x))^2$ *are all included in the critical points of the rational fraction* $P(x) = (p'(x))^k/(p(x))^{k-1}$.

PROOF. Observe that

$$P'(x) = \Big(\frac{(p'(x))^k}{(p(x))^{k-1}}\Big)' = \frac{k(p'(x))^{k-1}p''(x)(p(x))^{k-1} - (k-1)(p'(x))^k(p(x))^{k-2}p'(x)}{(p(x))^{2k-2}}$$
$$= \frac{k(p'(x))^{k-1}p''(x)(p(x))^{k-1} - (k-1)(p'(x))^{k+1}(p(x))^{k-2}}{(p(x))^{2k-2}}$$
$$= \frac{(p'(x))^{k-1}(kp''(x)p(x) - (k-1)(p'(x))^2)}{(p(x))^k}.$$ □

LEMMA 2.2. *When the real polynomial* $p(x)$ *of even degree has real zeros, we have* $\sharp_r[(k-1)(p'(x))^2 - kp(x)p''(x)] + \sharp_r p(x) > 0$.

Now suppose $p(x)$ is a real polynomial of even degree with no real zeros, so that $\sharp_r p(x) = 0$. The derivative polynomial $p'(x)$ has odd degree. A real polynomial of odd degree has an odd number of real zeros. In particular, it has at least one real zero.

LEMMA 2.3. *Let* $p(x)$ *be a real polynomial of even degree with no real zeros. If* $p'(x)$ *has at least three distinct real zeros, then* $\sharp_r[(k-1)(p'(x))^2 - kp(x)p''(x)] + \sharp_r p(x) > 0$.

https://doi.org/10.1017/S000497272300014X Published online by Cambridge University Press

PROOF. The rational function $P(x)$ is a real function. Since $p(x)$ has no real zeros and $p'(x)$ has no real poles, the rational function $P(x)$ has no real poles and so satisfies the conditions of Rolle's theorem. By the hypothesis, the polynomial $p'(x)$ has at least three real zeros. By Rolle's theorem, between two adjacent real zeros of $P(x)$, there is at least one real critical point. So, $P(x)$ has at least two real critical points. These two real critical points of $P(x)$ are not zeros of $p'(x)$. So, by Lemma 2.1, at least two real critical points of $P(x)$ are real zeros of the polynomial $(k-1)(p'(x))^2 - kp(x)p''(x)$. So, $\sharp_r[(k-1)(p'(x))^2 - kp(x)p''(x)] \geq 2 > 0$. □

EXAMPLE 2.4. Let $p_1(x) = x^4 - 2x^2 + 5 = (x^2-1)^2 + 1$, so $k = 4$.

Obviously, $p_1(x)$ has four distinct complex zeros and it has no real zeros. Further, $p_1'(x) = 4x^3 - 4x = 4x(x^2-1)$ has three real zeros. In each of the intervals $(-1, 0)$ and $(0, 1)$, there is one critical point of the rational fraction $P_1(x) = (p'(x))^k/p^{k-1}(x) = (4x^3-4x)^4/(x^4-2x^2+5)^3$ and $\sharp_r[(k-1)(p_1'(x))^2 - kp_1(x)p_1''(x)] = 2 > 0$. This is in accord with Lemma 2.3.

LEMMA 2.5. *Let $p(x)$ be a real polynomial of even degree with no real zeros. If $p'(x)$ has one real zero with exponent larger than 1, then $\sharp_r[(k-1)(p'(x))^2 - kp(x)p''(x)] + \sharp_r p(x) > 0$.*

PROOF. By hypothesis, $p'(x) = C(x)(x-w)^l$, where $C(x)$ is a polynomial, w is real, $C(w) \neq 0$ and $l > 1$. Then,

$$\begin{aligned}
&(k-1)(p'(x))^2 - kp(x)p''(x)\\
&\quad = (k-1)(C(x))^2(x-w)^{2l} - kp(x)C'(x)(x-w)^l - klC(x)p(x)(x-w)^{l-1}\\
&\quad = (x-w)^{l-1}((k-1)(C(x))^2(x-w)^{l+1} - kp(x)C'(x)(x-w) - klC(x)p(x))
\end{aligned}$$

and this polynomial has a zero at $z = w$ with exponent $l-1$. From this, it follows that $\sharp_r[(k-1)(p'(x))^2 - kp(x)p''(x)] + \sharp_r p(x) \geq l-1 > 0$. □

LEMMA 2.6. *Let $p(x)$ be a real polynomial of even degree with no real zeros. If $p'(x)$ has one real zero which is simple, that is, $p'(x) = C(x)(x-w)$, where $C(x)$ is a polynomial with $C(w) \neq 0$, and $(k-1)(C(x))^2(x-w)^2 - kp(x)C'(x)(x-w) - kC(x)p(x)$ has real zeros, then $\sharp_r[(k-1)(p'(x))^2 - kp(x)p''(x)] + \sharp_r p(x) > 0$.*

PROOF. By hypothesis, the polynomial

$$(k-1)(p'(x))^2 - kp(x)p''(x) = (k-1)(C(x))^2(x-w)^2 - kp(x)C'(x)(x-w) - kC(x)p(x)$$

has real zeros. Consequently, $\sharp_r[(k-1)(p'(x))^2 - kp(x)p''(x)] + \sharp_r p(x) > 0$. □

PROOF OF THEOREM 1.1. Let $\Delta = \sharp_r[(k-1)(p'(x))^2 - kp(x)p''(x)] + \sharp_r p(x)$. The four cases of Theorem 1.1 arise as follows.

(1) If $p(x)$ has real zeros, then $\Delta > 0$ by Lemma 2.2.
(2) If $p(x)$ has no real zeros and $p'(x)$ has at least three distinct real zeros, then $\Delta > 0$ by Lemma 2.3.

(3) Suppose $p'(x)$ has fewer than three distinct real zeros. Because $p'(x)$ is a polynomial of odd degree, it must have just one real zero. If $p(x)$ has no real zeros and $p'(x)$ has one real zero with exponent larger than 1, then $\Delta > 0$ by Lemma 2.5.
(4) If $p(x)$ has no real zeros, $p'(x) = C(x)(x - w)$ has one real zero which is simple, and the polynomial $(k - 1)(C(x))^2(x - w)^2 - kp(x)C'(x)(x - w) - kC(x)p(x)$ has real zeros, then $\Delta > 0$ by Lemma 2.6.

The only remaining case is when $p(x)$ has no real zeros, $p'(x) = C(x)(x - w)$ has one real zero which is simple, that is, $C(x)$ is a polynomial with $C(w) \neq 0$, and the polynomial $(k - 1)(C(x))^2(x - w)^2 - kp(x)C'(x)(x - w) - kC(x)p(x)$ has no real zeros. In this case, the calculation in Lemma 2.6 shows that $\Delta = 0$. This completes the proof of Theorem 1.1. □

PROOF OF THEOREM 1.2. Let $\Delta = \sharp_r[(k - 1)(p'(x))^2 - kp(x)p''(x)] + \sharp_r p(x)$. From the proof of Theorem 1.1, the hypotheses of Theorem 1.2 describe the only case in which $\Delta = 0$; in all other cases, $\Delta > 0$. □

EXAMPLE 2.7. Let $p_2(x) = x^2 + ax + b$ with a, b real, so $k = 2$.

For this example, $(k - 1)(p_2'(x))^2 - kp_2(x)p''(x) = (2x + a)^2 - 4(x^2 + ax + b) = a^2 - 4b$. If $a^2 - 4b < 0$, then the polynomials $(k - 1)(p_2'(x))^2 - kp_2(x)p_2''(x)$ and $p_2(x)$ have no real zeros, that is, $\sharp_r[(k - 1)(p_2'(x))^2 - kp_2(x)p_2''(x)] + \sharp_r p_2(x) = 0$, in contrast to Shapiro's conjecture.

EXAMPLE 2.8. Let $p_3(x) = x^4 + x^2 + 1$, so $k = 4$. For this example, $(k - 1)(p_3'(x))^2 - kp_3(x)p_3''(x) = 3(4x^3 + 2x)^2 - 4(x^4 + x^2 + 1)(12x^2 + 2) = -4(2x^4 + 11x^2 + 2)$. The zeros of the polynomial $2t^2 + 11t + 2$ are $\frac{1}{2}(-11 \pm \sqrt{105})$ which are both negative real zeros. So, the polynomial $2x^4 + 11x^2 + 2$ has four complex zeros and no real zeros. So, $\sharp_r[(k - 1)(p_3'(x))^2 - kp_3(x)p_3''(x)] + \sharp_r p_3(x) = 0$.

References

[1] G. Csordas, T. Craven and W. Smith, 'The zeros of derivatives of entire functions and the Wiman–Pólya conjecture', *Ann. of Math. (2)* **125**(2) (1987), 405–431.
[2] B. Shapiro, 'Problems around polynomials: the good, the bad and the ugly...', *Arnold Math. J.* **1**(1) (2015), 91–99.
[3] T. Sheil-Small, *Complex Polynomials*, Cambridge Studies in Advanced Mathematics, 75 (Cambridge University Press, Cambridge, 2002).
[4] M. Tyaglov, 'On the number of critical points of logarithmic derivatives and the Hawaii conjecture', *J. Anal. Math.* **114** (2011), 1–62.
[5] M. Tyaglov and M. J. Atia, 'On the number of non-real zeroes of a homogeneous differential polynomial and a generalisation of the Laguerre inequalities', *J. Math. Anal. Appl.* **494**(2) (2021), Article no. 124652.

LANDE MA, School of Mathematical Sciences,
Tongji University, Shanghai 200092, PR China
e-mail: dzy200408@sina.cn

https://doi.org/10.1017/S000497272300014X Published online by Cambridge University Press

ZHAOKUN MA, YanZhou College,
ShanDong Radio and TV University, YanZhou, ShanDong 272100, PR China
e-mail: dzy200408@126.com

https://doi.org/10.1017/S000497272300014X Published online by Cambridge University Press