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CONGRUENCES MODULO 2 FOR CERTAIN PARTITION FUNCTIONS

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Abstract

Let $b_{3,5}(n)$ denote the number of partitions of n into parts that are not multiples of 3 or 5. We establish several infinite families of congruences modulo 2 for $b_{3,5}(n)$. In the process, we also prove numerous parity results for broken 7-diamond partitions.

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1. Introduction

For any positive integer $\ell \geq 2$, a partition is said to be ℓ-regular if none of its parts is a multiple of ℓ. The number of ℓ-regular partitions of n is denoted by $b_\ell(n)$ and by convention we define $b_\ell(0) = 1$. The generating function of $b_\ell(n)$ satisfies the identity

$$\sum_{n=0}^{\infty} b_\ell(n) q^n = \frac{f_\ell}{f_1}.$$

Here and throughout this paper, we use the notation

$$f_k := (q^k; q^k)_\infty \quad (k = 1, 2, \ldots), \quad \text{where } (a; q)_\infty = \prod_{m=0}^{\infty} (1 - aq^m).$$

Let $f(a, b)$ be Ramanujan's general theta function [4, page 34] given by

$$f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}.$$

Jacobi's triple product identity states that

$$f(a, b) = (-a; ab)_\infty (-b; ab)_\infty (ab; ab)_\infty.$$

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In particular,

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty.$$

Ramanujan [13, page 212] stated the following identity without proof:

$$f_1 = f_{25}(R(q) - q - q^2 R^{-1}(q)), \quad \text{where } R(q) = \frac{f(-q^{15},-q^{10})}{f(-q^{20},-q^5)}. \tag{1.1}$$

Watson [14] gave a proof of (1.1) utilising the quintuple product identity

$$\sum_{n=-\infty}^{\infty} q^{(3n^2+n)/2}(z^{3n} - z^{-3n-1}) = (q;q)_\infty (zq;q)_\infty (1/z;q)_\infty (z^2q;q^2)_\infty (q/z^2;q^2)_\infty.$$

For any prime $p \geq 5$, Cui and Gu [6] obtained the p-dissection formula

$$\begin{aligned} f(-q) = &\sum_{\substack{k=-(p-1)/2 \\ k\neq(\pm p-1)/6}}^{(p-1)/2} (-1)^k q^{(3k^2+k)/2} f(-q^{(3p^2+(6k+1)p)/2}, -q^{(3p^2-(6k+1)p)/2}) \\ &+ (-1)^{(\pm p-1)/6} q^{(p^2-1)/24} f(-q^{p^2}), \end{aligned} \tag{1.2}$$

where the choice of the $\pm$ sign is made so that $(\pm p-1)/6$ is an integer. Note that $(3k^2+k)/2 \not\equiv (p^2-1)/24 \pmod p$ as k runs through the range of the summation.

Recently, numerous infinite families of congruences modulo 2 and 3 for $b_\ell(n)$ have been established. For example, Xia and Yao [15] proved that

$$b_9\Big(2^{6j+4}n + \frac{5\times 2^{6j+3}-1}{3}\Big) \equiv 0 \pmod 2$$

for all nonnegative integers j and n, while Cui and Gu [6] showed that

$$b_5\Big(4\times 5^{2j+1}n + \frac{31\times 5^{2j}-1}{6}\Big) \equiv 0 \pmod 2.$$

For more examples, see [5, 7–9].

MacMahon's partition analysis guided Andrews and Paule [2] to introduce broken k-diamond partitions. For any positive integer k, let $\Delta_k(n)$ denote the number of broken k-diamond partitions of n. The generating function of $\Delta_k(n)$ satisfies the identity

$$\sum_{n=0}^{\infty} \Delta_k(n) q^n = \frac{f_2 f_{2k+1}}{f_1^3 f_{4k+2}}. \tag{1.3}$$

Various authors have obtained parity results for broken k-diamond partitions (see, for example, [10, 12, 16]). In particular, Ahmed and Baruah [1] established

$$\Delta_7(8n+2) \equiv \Delta_7(64n+54) \equiv 0 \pmod 2 \tag{1.4}$$

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and

$$\Delta_7\Big(8\times 5^{2j+1}n+\frac{(24r+16)\times 5^{2j}+2}{3}\Big)\equiv 0 \pmod 2$$

for all $j, n \geq 0$ and $r \in \{3, 4\}$.

With this motivation, we obtain several infinite families of congruences for the broken 7-diamond partitions and for the partition function $b_{3,5}(n)$ which counts the number of partitions of n none of whose parts are multiples of 3 or 5. For example, we have the following results.

THEOREM 1.1. *If p is an odd prime with $(-15/p) = -1$, $1 \leq i \leq p-1$ and $j, n \geq 0$,*

$$b_{3,5}\Big(2\times p^{2j+2}n+\frac{(6i+2p)\times p^{2j+1}+1}{3}\Big)\equiv 0 \pmod 2. \tag{1.5}$$

THEOREM 1.2. *For all integers $n, j \geq 0$,*

$$\Delta_7\Big(2^{6j+3}n+\frac{2^{6j+2}+2}{3}\Big)\equiv 0 \pmod 2 \tag{1.6}$$

and

$$\Delta_7\Big(2^{6j+6}n+\frac{5\times 2^{6j+5}+2}{3}\Big)\equiv 0 \pmod 2. \tag{1.7}$$

Note that congruences (1.4) are special cases of (1.6) and (1.7). We prove Theorem 1.1 and more families of congruences for $b_{3,5}(n)$ in Section 3 and Theorem 1.2 and further infinite families of congruences for broken 7-diamond partitions in Section 4.

2. Preliminaries

In this section we present two lemmas which play a vital role in proving our main results. Note that the generating function for $b_{3,5}(n)$ satisfies

$$\sum_{n=0}^{\infty} b_{3,5}(n)q^n = \frac{f_3 f_5}{f_1 f_{15}}. \tag{2.1}$$

LEMMA 2.1. *We have*

$$\frac{f_3 f_5}{f_1 f_{15}} \equiv \sum_{n=0}^{\infty} b_{3,5}(2n)q^{2n} + q f_6 f_{10} \pmod 2 \tag{2.2}$$

and

$$\frac{f_1 f_{15}}{f_3 f_5} \equiv \sum_{n=0}^{\infty} d(2n)q^{2n} - q f_2 f_{30} \pmod 2, \tag{2.3}$$

where $d(n)$ is defined by

$$\sum_{n=0}^{\infty} d(n)q^n = \frac{f_1 f_{15}}{f_3 f_5}.$$

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PROOF. From (2.1),

$$\sum_{n=0}^{\infty} b_{3,5}(n)q^n = \frac{f_3 f_5}{f_1 f_{15}} = \frac{f_6 f_{10}}{f_2 f_{30}} \frac{(q^3;q^6)_\infty (q^5;q^{10})_\infty}{(q;q^2)_\infty (q^{15};q^{30})_\infty},$$

which implies that

$$2\sum_{n=0}^{\infty} b_{3,5}(2n+1)q^{2n+1} = \frac{f_6^2 f_{10}^2 f_4 f_{60}}{f_2^2 f_{30}^2 f_{12} f_{20}} \left\{ \frac{(-q;q^2)_\infty (-q^{15};q^{30})_\infty}{(-q^3;q^6)_\infty (-q^5;q^{10})_\infty} - \frac{(q;q^2)_\infty (q^{15};q^{30})_\infty}{(q^3;q^6)_\infty (q^5;q^{10})_\infty} \right\}. \tag{2.4}$$

Using the modular equation of degree 15 [4, Entry 11(iv), page 383], Baruah and Berndt [3] derived the identity

$$\left\{ \frac{(-q;q^2)_\infty (-q^{15};q^{30})_\infty}{(-q^3;q^6)_\infty (-q^5;q^{10})_\infty} - \frac{(q;q^2)_\infty (q^{15};q^{30})_\infty}{(q^3;q^6)_\infty (q^5;q^{10})_\infty} \right\} = 2q\frac{f_{12} f_{20}}{f_6 f_{10}}. \tag{2.5}$$

In view of (2.4) and (2.5), we can easily see that

$$\sum_{n=0}^{\infty} b_{3,5}(2n+1)q^{2n+1} = q\frac{f_4 f_6 f_{10} f_{60}}{f_2^2 f_{30}^2}. \tag{2.6}$$

From (2.1) and (2.6),

$$\frac{f_3 f_5}{f_1 f_{15}} = \sum_{n=0}^{\infty} b_{3,5}(2n)q^{2n} + q\frac{f_4 f_6 f_{10} f_{60}}{f_2^2 f_{30}^2}. \tag{2.7}$$

By the binomial theorem, we can see that for all positive integers k and m,

$$f_k^{2m} \equiv f_{2k}^m \pmod 2. \tag{2.8}$$

Congruence (2.2) follows from (2.7) and (2.8). Replacing q by $-q$ in (2.7) and using the relation

$$(-q;-q)_\infty = \frac{f_2^3}{f_1 f_4},$$

$$\frac{f_1 f_{15}}{f_3 f_5} = \sum_{n=0}^{\infty} d(2n)q^{2n} - q\frac{f_2 f_{30} f_{12} f_{20}}{f_6^2 f_{10}^2}. \tag{2.9}$$

Congruence (2.3) readily follows from (2.9) and (2.8). □

LEMMA 2.2. *Set* $\sum_{n=0}^{\infty} p_{[l^{-1}m^{-1}]}(n)q^n = f_l f_m$. *Then*

$$f_3 f_5 = \sum_{n=0}^{\infty} p_{[3^{-1}5^{-1}]}(2n)q^{2n} - q f_{12} f_{20} + q f_2 f_{30} \tag{2.10}$$

and

$$f_1 f_{15} = f_6 f_{10} - q^2 f_4 f_{60} + \sum_{n=0}^{\infty} p_{[1^{-1}15^{-1}]}(2n+1)q^{2n+1}. \tag{2.11}$$

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Proof. In [11], we used Ramanujan's theta function identities to derive the identities

$$(-q^3;q^6)_\infty(-q^5;q^{10})_\infty-(q^3;q^6)_\infty(q^5;q^{10})_\infty=2q\frac{f_{12}f_{20}}{f_6f_{10}}-2q\frac{f_2f_{30}}{f_6f_{10}} \tag{2.12}$$

and

$$(-q;q^2)_\infty(-q^{15};q^{30})_\infty+(q;q^2)_\infty(q^{15};q^{30})_\infty=2\frac{f_6f_{10}}{f_2f_{30}}-2q^2\frac{f_4f_{60}}{f_2f_{30}}. \tag{2.13}$$

Equations (2.10) and (2.11) follow easily from (2.12) and (2.13), respectively. □

3. Congruences for $b_{3,5}(n)$ modulo 2

In this section, we prove infinite families of congruences modulo 2 for $b_{3,5}(n)$.

Theorem 3.1. *Let $j\geq 0$. Then*

$$\sum_{n=0}^{\infty}b_{3,5}\Big(2\times 5^{2j}n+\frac{2\times 5^{2j}+1}{3}\Big)q^n\equiv f_3f_5 \pmod 2 \tag{3.1}$$

and, for all $n\geq 0$, $r\in\{14,26\}$ and $s\in\{22,28\}$,

$$b_{3,5}\Big(2\times 5^{2j+1}n+\frac{r\times 5^{2j}+1}{3}\Big)\equiv 0 \pmod 2 \tag{3.2}$$

and

$$b_{3,5}\Big(2\times 5^{2j+2}n+\frac{s\times 5^{2j+1}+1}{3}\Big)\equiv 0 \pmod 2. \tag{3.3}$$

Proof. From the equations (2.1) and (2.2),

$$\sum_{n=0}^{\infty}b_{3,5}(2n+1)\,q^n\equiv f_3f_5 \pmod 2, \tag{3.4}$$

which is the $j=0$ case of (3.1).

Suppose that the congruence (3.1) holds for some integer $j\geq 0$. Using (1.1),

$$\sum_{n=0}^{\infty}b_{3,5}\Big(2\times 5^{2j}n+\frac{2\times 5^{2j}+1}{3}\Big)q^n\equiv f_5f_{75}(R(q^3)-q^3-q^6R^{-1}(q^3)) \pmod 2. \tag{3.5}$$

Extracting the terms in q^n with $n\equiv 3 \pmod 5$ and using (1.1) again yields

$$\begin{aligned}\sum_{n=0}^{\infty}&b_{3,5}\Big(2\times 5^{2j}(5n+3)+\frac{2\times 5^{2j}+1}{3}\Big)q^n\\&=\sum_{n=0}^{\infty}b_{3,5}\Big(2\times 5^{2j+1}n+\frac{4\times 5^{2j+1}+1}{3}\Big)q^n\\&\equiv f_1f_{15}\equiv f_{15}f_{25}(R(q)-q-q^2R^{-1}(q)) \pmod 2.\end{aligned} \tag{3.6}$$

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In the same way, using (3.6),

$$\sum_{n=0}^{\infty} b_{3,5}\Big(2\times 5^{2j+1}(5n+1)+\frac{4\times 5^{2j+1}+1}{3}\Big)q^n$$
$$=\sum_{n=0}^{\infty} b_{3,5}\Big(2\times 5^{2j+2}n+\frac{2\times 5^{2j+2}+1}{3}\Big)q^n \equiv f_3 f_5 \pmod 2.$$

Thus, (3.1) is true for $j+1$. Hence, by induction, (3.1) is true for any nonnegative integer j. Congruences (3.2) and (3.3) follow from (3.5) and (3.6), respectively. □

Theorem 3.2. *For any prime $p \geq 5$ with $(-15/p) = -1$ and any integers $j, n \geq 0$,*

$$\sum_{n=0}^{\infty} b_{3,5}\Big(2\times p^{2j}n+\frac{2\times p^{2j}+1}{3}\Big)q^n \equiv f_3 f_5 \pmod 2. \tag{3.7}$$

Proof. Note that (3.4) is the $j=0$ case of (3.7). Consider the congruence

$$3\cdot\frac{3k^2+k}{2}+5\cdot\frac{3m^2+m}{2}\equiv\frac{p^2-1}{3} \pmod p, \tag{3.8}$$

where $-(p-1)/2 \leq k, m \leq (p-1)/2$. We can rewrite (3.8) as

$$(18k+3)^2+15(6m+1)^2 \equiv 0 \pmod p.$$

Since $(-15/p) = -1$, the only solution of this congruence is $k = m = (\pm p - 1)/6$. This fact along with (1.2) and (3.4) yields

$$\sum_{n=0}^{\infty} b_{3,5}\Big(2\Big(p^2 n+\frac{p^2-1}{3}\Big)+1\Big)\,q^n \equiv f_3 f_5 \pmod 2,$$

which is the $j=1$ case of (3.7). Iterating this procedure yields Theorem (3.2). □

Proof of Theorem (1.1). From the proof of Theorem (3.2) it is clear that the congruence (3.8) holds only when $k = m = (\pm p - 1)/6$. Thus, employing (1.2) in (3.7),

$$\sum_{n=0}^{\infty} b_{3,5}\Big(2\times p^{2j}\Big(pn+\frac{p^2-1}{3}\Big)+\frac{2\times p^{2j}+1}{3}\Big)q^n$$
$$=\sum_{n=0}^{\infty} b_{3,5}\Big(2\times p^{2j+1}n+\frac{2\times p^{2j+2}+1}{3}\Big)q^n \equiv f_{3p} f_{5p} \pmod 2,$$

which implies that, for $i = 1, 2, \ldots, p-1$,

$$b_{3,5}\Big(2\times p^{2j+1}(pn+i)+\frac{2\times p^{2j+2}+1}{3}\Big) \equiv 0 \pmod 2,$$

from which the congruence (1.5) follows immediately. □

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THEOREM 3.3. *For all integers $n \geq 0$ and $j \geq 0$,*

$$b_{3,5}\Big(2^{6j+1}n + \frac{2^{6j+1}+1}{3}\Big) \equiv b_{3,5}(2n+1) \pmod 2, \tag{3.9}$$

$$b_{3,5}\Big(2^{6j+4}n + \frac{2^{6j+3}+1}{3}\Big) \equiv 0 \pmod 2, \tag{3.10}$$

$$b_{3,5}\Big(2^{6j+7}n + \frac{5\times 2^{6j+6}+1}{3}\Big) \equiv 0 \pmod 2, \tag{3.11}$$

$$b_{3,5}\Big(2^{6j+6}n + \frac{2^{6j+5}+1}{3}\Big) \equiv b_{3,5}\Big(2^{6j+2}n + \frac{2^{6j+1}+1}{3}\Big) \pmod 2, \tag{3.12}$$

$$b_{3,5}\Big(2^{6j+5}n + \frac{5\times 2^{6j+4}+1}{3}\Big) \equiv b_{3,5}\Big(2^{6j+3}n + \frac{5\times 2^{6j+2}+1}{3}\Big) \pmod 2. \tag{3.13}$$

PROOF. By using (2.10) in (3.4) and extracting the terms involving odd powers of q,

$$\sum_{n=0}^{\infty} b_{3,5}(4n+3)q^{2n+1} \equiv qf_2f_{30} - qf_{12}f_{20} \pmod 2,$$

which yields

$$\sum_{n=0}^{\infty} b_{3,5}(4n+3)q^{n} \equiv f_1f_{15} - f_6f_{10} \pmod 2. \tag{3.14}$$

By substituting (2.11) in (3.14) and extracting the terms involving even powers of q,

$$\sum_{n=0}^{\infty} b_{3,5}(8n+3)q^{n} \equiv qf_2f_{30} \pmod 2. \tag{3.15}$$

It follows from (3.15) that

$$\sum_{n=0}^{\infty} b_{3,5}(16n+11)q^{n} \equiv f_1f_{15} \pmod 2 \tag{3.16}$$

and

$$b_{3,5}(16n+3) \equiv 0 \pmod 2. \tag{3.17}$$

In view of (3.14) and (3.16),

$$b_{3,5}(32n+27) \equiv b_{3,5}(8n+7) \pmod 2. \tag{3.18}$$

Using (2.11) in (3.16) and extracting the terms involving even powers of q,

$$\sum_{n=0}^{\infty} b_{3,5}(32n+11)q^{n} \equiv f_3f_5 - qf_2f_{30} \pmod 2. \tag{3.19}$$

It follows from (3.4) and (3.19) that

$$b_{3,5}(64n+11) \equiv b_{3,5}(4n+1) \pmod 2. \tag{3.20}$$

By substituting (2.10) in (3.19) and extracting the terms containing odd powers of q,

$$\sum_{n=0}^{\infty} b_{3,5}(64n+43)q^n \equiv f_6 f_{10} \pmod{2}. \tag{3.21}$$

From (3.21),

$$b_{3,5}(128n+107) \equiv 0 \pmod{2} \tag{3.22}$$

for all $n \geq 0$ and

$$\sum_{n=0}^{\infty} b_{3,5}(128n+43)q^n \equiv f_3 f_5 \pmod{2}. \tag{3.23}$$

By (3.4) and (3.23),

$$b_{3,5}(128n+43) \equiv b_{3,5}(2n+1) \pmod{2}. \tag{3.24}$$

Congruence (3.9) follows from (3.24) and mathematical induction. Using (3.17) and (3.22) in (3.9), we deduce (3.10) and (3.11), respectively. Congruence (3.12) follows from (3.9) and (3.20). Congruence (3.13) follows from (3.9) and (3.18). □

4. Parity results for broken 7-diamond partitions

In this section, we prove infinite families of congruences modulo 2 for $\Delta_7(n)$, using our results for $b_{3,5}(n)$.

THEOREM 4.1. *For any integers $n \geq 0$ and $j \geq 0$,*

$$\Delta_7\Big(2^{6j+2}n + \frac{2^{6j+2}+2}{3}\Big) \equiv b_{3,5}(8n+3) \pmod{2}. \tag{4.1}$$

PROOF. Setting $k = 7$ in (1.3),

$$\sum_{n=0}^{\infty} \Delta_7(n)q^n = \frac{f_2 f_{15}}{f_1^3 f_{30}} \equiv \frac{1}{f_1 f_{15}} \pmod{2},$$

which implies that

$$\sum_{n=0}^{\infty} \Delta_7(2n)q^{2n} \equiv \frac{1}{2 f_2 f_{30}}\Big\{\frac{(-q;q^2)_\infty(-q^{15};q^{30})_\infty + (q;q^2)_\infty(q^{15};q^{30})_\infty}{(q^2;q^4)_\infty(q^{30};q^{60})_\infty}\Big\} \pmod{2}. \tag{4.2}$$

Using equation (2.13) in (4.2) and replacing q^2 by q,

$$\sum_{n=0}^{\infty} \Delta_7(2n)q^n \equiv \Big(\frac{f_2 f_3 f_5 f_{30}}{f_1^3 f_{15}^3} - q\frac{f_2^2 f_{30}^2}{f_1^3 f_{15}^3}\Big) \equiv \Big(\frac{f_3 f_5}{f_1 f_{15}} - q f_1 f_{15}\Big) \pmod{2}. \tag{4.3}$$

Using equation (2.2) and (2.11) in (4.3),

$$\sum_{n=0}^{\infty} \Delta_7(4n+2)q^{2n+1} \equiv q^3 f_4 f_{60} \pmod{2}. \tag{4.4}$$

From (3.15) and (4.4),

$$\Delta_7(4n+2) \equiv b_{3,5}(8n+3) \pmod{2}. \tag{4.5}$$

Congruence (4.1) follows from (3.9) and (4.5). □

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Proof of Theorem 1.2. From (4.1),

$$\Delta_7\left(2^{6j+2}(2n) + \frac{2^{6j+2}+2}{3}\right) \equiv b_{3,5}(8(2n)+3) \pmod 2 \tag{4.6}$$

and

$$\Delta_7\left(2^{6j+2}(16n+13) + \frac{2^{6j+2}+2}{3}\right) \equiv b_{3,5}(8(16n+13)+3) \pmod 2. \tag{4.7}$$

Congruence (1.6) follows readily from (3.17) and (4.6). Congruence (1.7) follows from (3.22) and (4.7). □

Theorem 4.2. *For any prime $p \geq 5$ with $(-15/p) = -1$ and any integers $j, n \geq 0$,*

$$\Delta_7\left(4 \times p^{2j}n + \frac{4 \times p^{2j}+2}{3}\right) \equiv b_{3,5}(8n+3) \pmod 2. \tag{4.8}$$

Proof. Combining equations (3.4) and (3.7) with (4.5) yields (4.8). □

Theorem 4.3. *If p is an odd prime with $(-15/p) = -1$ and $1 \leq i \leq p-1$, then, for all $j, n \geq 0$,*

$$\Delta_7\left(8 \times p^{2j+2}n + \frac{(24i+16p) \times p^{2j+1}+2}{3}\right) \equiv 0 \pmod 2.$$

Proof. We omit the proof, since it is similar to the proof of Theorem 1.1. □

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References

[1] Z. Ahmed and N. D. Baruah, 'Parity results for broken 5-diamond, 7-diamond and 11-diamond partitions', *Int. J. Number Theory* **11** (2015), 527–542.
[2] G. E. Andrews and P. Paule, 'MacMahon's partition analysis XI: broken diamonds and modular forms', *Acta Arith.* **126** (2007), 281–294.
[3] N. D. Baruah and B. C. Berndt, 'Partition identities and Ramanujan's modular equations', *J. Combin. Theory Ser. A* **114** (2007), 1024–1045.
[4] B. C. Berndt, *Ramanujan's Notebooks, Part III* (Springer, New York, 1991).
[5] N. Calkin, N. Drake, K. James, S. Law, P. Lee, D. Penniston and J. Radder, 'Divisibility properties of the 5-regular and 13-regular partition functions', *Integers* **8** (2008), A60.
[6] S. P. Cui and N. S. S. Gu, 'Arithmetic properties of l-regular partitions', *Adv. Appl. Math.* **51** (2013), 507–523.
[7] D. Furcy and D. Penniston, 'Congruences for ℓ-regular partition functions modulo 3', *Ramanujan J.* **27** (2012), 101–108.
[8] M. D. Hirschhorn and J. A. Sellers, 'Elementary proofs of parity results for 5-regular partitions', *Bull. Aust. Math. Soc.* **81** (2010), 58–63.
[9] W. J. Keith, 'Congruences for 9-regular partitions modulo 3', *Ramanujan J.* **35** (2014), 157–164.
[10] B. L. S. Lin, 'Elementary proofs of parity results for broken 3-diamond partitions', *J. Number Theory* **135** (2014), 1–7.

[11] M. S. Mahadeva Naika, B. Hemanthkumar and H. S. Sumanth Bharadwaj, 'Color partition identities arising from Ramanujan's theta functions', *Acta Math. Vietnam.*, to appear.

[12] S. Radu and J. A. Sellers, 'Parity results for broken k-diamond partitions and $(2k + 1)$-cores', *Acta Arith.* **146** (2011), 43–52.

[13] S. Ramanujan, *Collected Papers* (Cambridge University Press, Cambridge, 1927).

[14] G. N. Watson, 'Ramanujan's Vermutung über Zerfällungsanzahlem', *J. reine angew. Math.* **179** (1938), 97–128.

[15] E. X. W. Xia and O. X. M. Yao, 'Parity results for 9-regular partitions', *Ramanujan J.* **34** (2014), 109–117.

[16] O. X. M. Yao, 'New parity results for broken 11-diamond partitions', *J. Number Theory* **140** (2014), 267–276.

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