



Quasi-elementary contractions of Fano manifolds

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ABSTRACT

Let X be a smooth complex Fano variety. We study ‘quasi-elementary’ contractions of fiber type of X , which are a natural generalization of elementary contractions of fiber type. If $f: X \rightarrow Y$ is such a contraction, then the Picard numbers satisfy $\rho_X \leq \rho_Y + \rho_F$, where F is a general fiber of f . We show that, if $\dim Y \leq 3$ and $\rho_Y \geq 4$, then Y is smooth and Fano; if moreover $\rho_Y \geq 6$, then X is a product. This yields sharp bounds on ρ_X when $\dim X = 4$ and X has a quasi-elementary contraction of fiber type, and other applications in higher dimensions.

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1. Introduction

A Fano manifold is a smooth complex projective variety X with ample anticanonical class. These varieties have a rich structure, and play an important role in higher dimensional geometry from the viewpoint of Mori theory, because they appear as fibers of Mori contractions of fiber type of smooth projective varieties.

After [Cam92, KMM92] we know that X is rationally connected and simply connected. Moreover, smooth Fano varieties of dimension n form a limited family, so they have only a finite number of possible topological types. However, in general, very little is known about their topological invariants.

In particular, we consider here the second Betti number b_2 of X , which coincides with the Picard number ρ_X .

Recall that a Del Pezzo surface S has $\rho_S \leq 9$. Fano 3-folds have been classified by Iskovskikh, Mori, and Mukai (see [IP99] and references therein). Thus we know that a Fano 3-fold X has $\rho_X \leq 10$. In fact, more is true: as soon as $\rho_X \geq 6$, X is a product of a Del Pezzo surface with \mathbb{P}^1 (see [MM81, Theorem 2]).

Starting from dimension 4, we do not have a bound on ρ_X . The known examples with largest Picard number are just products of Del Pezzo surfaces with Picard number 9, which gives $\rho_X = \frac{9}{2}n$.

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Optimistically, one could think that Fano varieties with large Picard number are simpler, maybe a product of lower dimensional varieties. This would yield a linear bound (in the dimension n) for ρ_X ; in fact one could expect precisely $\rho_X \leq \frac{9}{2}n$ (see [Deb03, p. 122]).

This is actually what happens in the toric case: if X is a smooth toric Fano variety of dimension n , then $\rho_X \leq 2n$, and equality holds if and only if n is even and X is $(S)^{n/2}$, S the blow-up of \mathbb{P}^2 in three non-collinear points (see [Cas06]).

Let us also recall that there is a class of Fano varieties for which a stronger linear bound on ρ_X is expected. These are Fano varieties that do not contain curves of anticanonical degree 1, e.g. Fano varieties of index at least 2. In this case it is expected that $\rho_X \leq n$, with equality only for $(\mathbb{P}^1)^n$. This is a generalization of a conjecture by Mukai, and has been proved in dimension $n \leq 5$ and in the toric case, see [BCDD03, ACO04, Cas06].

A strategy in this direction is to look for a contraction $f: X \rightarrow Y$ of fiber type on X , and try to bound ρ_X in terms of ρ_Y and ρ_F , where F is a general fiber. There are (at least) two difficulties in this approach. First, Y may not be Fano, so that we do not know how to bound ρ_Y . Second, surely F is Fano, but in general ρ_F is much smaller than $\rho_X - \rho_Y$ (for instance, any Del Pezzo surface S admits a contraction $S \rightarrow \mathbb{P}^1$ with general fiber \mathbb{P}^1). This problem could be avoided by considering only elementary contractions of fiber type, for which $\rho_X = 1 + \rho_Y$. However, this is not very satisfactory, because we do not necessarily expect a Fano variety with large Picard number to have an elementary contraction of fiber type (think of a product of Del Pezzo surfaces).

This was our motivation to introduce the notion of ‘quasi-elementary’ contraction of fiber type. This is a contraction of fiber type as above, such that, if $i: F \hookrightarrow X$ is a general fiber, then the image of $i_*: \mathcal{N}_1(F) \rightarrow \mathcal{N}_1(X)$ contains all numerical classes of curves contracted by f (see Definition 3.1). This implies that $\rho_X \leq \rho_Y + \rho_F$. In particular, any elementary contraction of fiber type is quasi-elementary. If a contraction is a smooth morphism (such as a projection $Y \times F \rightarrow Y$), then it is quasi-elementary (see Lemma 3.3).

We study several properties of quasi-elementary contractions of smooth Fano varieties, in particular when the target has small dimension. The following is our main result.

THEOREM 1.1. *Let X be a smooth complex Fano variety, $f: X \rightarrow Y$ a quasi-elementary contraction of fiber type, and F a general fiber.*

- (i) *Suppose that $\dim Y = 2$. Then Y is a smooth Del Pezzo surface, f is equidimensional, and $\rho_X \leq \rho_Y + \rho_F \leq 9 + \rho_F$. If moreover $\rho_Y \geq 3$, then $X \cong Y \times F$ and f is the projection on the first factor.*
- (ii) *Suppose that $\dim Y = 3$. Then $\rho_X \leq \rho_Y + \rho_F \leq 10 + \rho_F$. If moreover $\rho_Y \geq 4$, then Y is smooth and Fano. If $\rho_Y \geq 6$, then $X \cong S \times W$ and $Y \cong S \times \mathbb{P}^1$, where S is a Del Pezzo surface, and W a smooth Fano variety with a quasi-elementary contraction onto \mathbb{P}^1 .*

In the case $\dim X = 3$, part (i) has been shown in [MM81, Proposition 8] under the more general assumption that $f: X \rightarrow Y$ is an equidimensional contraction onto a surface. See the end of § 7 herein for examples concerning the sharpness of the statement.

The following are some applications to Fano 4-folds and 5-folds.

COROLLARY 1.2. *Let X be a smooth complex Fano 4-fold.*

- (i) *If X has a non-trivial quasi-elementary contraction of fiber type, then $\rho_X \leq 18$, with equality if and only if $X \cong S_1 \times S_2$, where S_i are Del Pezzo surfaces with $\rho_{S_i} = 9$.*
- (ii) *If X has an elementary contraction onto a surface S and $\rho_X \geq 4$, then $X \cong \mathbb{P}^2 \times S$.*
- (iii) *If X has an elementary contraction onto a 3-fold and $\rho_X \geq 7$, then either $X \cong \mathbb{P}^1 \times \mathbb{P}^1 \times S$, or $X \cong \mathbb{F}_1 \times S$, with S a Del Pezzo surface.*

COROLLARY 1.3. *Let X be a smooth complex Fano 5-fold.*

- (i) *If X has two distinct elementary contractions of fiber type, then $\rho_X \leq 12$.*
- (ii) *If X has an elementary contraction onto Y with $\dim Y \leq 3$, then $\rho_X \leq 11$.*
- (iii) *Suppose that X has an elementary contraction $f: X \rightarrow Y$ with $\dim Y = 4$. If X has another elementary contraction φ of type $(3,0)$, $(4,0)$, $(4,1)$, or such that $f(\text{Exc}(\varphi)) = Y$, then $\rho_X \leq 12$.*

Finally we give an application to Fano varieties with two quasi-elementary contractions.

COROLLARY 1.4. *Let X be a smooth complex Fano variety of dimension n . Let $f_1: X \rightarrow Y_1$ and $f_2: X \rightarrow Y_2$ be two quasi-elementary contractions of fiber type with $\text{NE}(f_1) \cap \text{NE}(f_2) = \{0\}$. Let F_i be a general fiber of f_i . Then $\dim F_1 + \dim F_2 \leq n$, and moreover:*

- (i) *$\dim F_1 + \dim F_2 = n$ implies that $\rho_X \leq \rho_{F_1} + \rho_{F_2}$;*
- (ii) *$\dim F_1 + \dim F_2 = n - 1$ implies that $\rho_X \leq \rho_{F_1} + \rho_{F_2} + 1$;*
- (iii) *$\dim F_1 + \dim F_2 = n - 2$ implies that $\rho_X \leq \rho_{F_1} + \rho_{F_2} + 9$; and*
- (iv) *if $\dim F_1 + \dim F_2 = n - 3$ and f_2 is elementary, then $\rho_X \leq \rho_{F_1} + 11$.*

If f_i is elementary one can replace ρ_{F_i} by 1 in the statement, see Remark 7.1. These corollaries should be compared to the following result by Wiśniewski, involving an arbitrary number of elementary contractions of fiber type.

THEOREM 1.5 [Wiś91a, Theorem 2.2]. *Let X be a smooth complex Fano variety of dimension n . Suppose that X has k distinct elementary contractions of fiber type, and let F_1, \dots, F_k be the general fibers. Then $\sum_i \dim F_i \leq n$, and moreover:*

- (i) *$\sum_i \dim F_i = n$ implies that $\rho_X \leq k$; and*
- (ii) *$\sum_i \dim F_i = n - 1$ implies that $\rho_X \leq k + 1$.*

In particular $k \leq n$, and if $k \geq n - 1$ then $\rho_X \leq n$.

Our main tool is Mori theory; in particular we use many properties of contractions of Fano varieties shown in [Wiś91a].

In § 2 we recall some basic notions and properties. In § 2.5 we show that, when a variety Y is the target of a contraction $f: X \rightarrow Y$ where X is Fano, Y shares with X a good behavior with respect to Mori theory; see Lemma 2.6.

In § 3 we define quasi-elementary contractions of fiber type $f: X \rightarrow Y$ and give some related properties and examples. We study the singularities of Y in Lemma 3.10, generalizing results known in the elementary case and using in particular results from [ABW92]. Then in § 3.12 we study elementary contractions of Y by means of their liftings to X . This is a key ingredient in the proof of Theorem 1.1. Finally in § 3.15 we show that if X is Fano, $\dim Y \geq 3$ and Y contains a prime divisor D with $\rho_D = 1$, then $\rho_Y \leq 3$, so $\rho_X \leq 3 + \rho_F$ where F is a general fiber of f . This is a generalization of results from [Tsu06] and [BCW02].

In § 4 we apply results from [BCD07] to deduce that some contractions of X always induce contractions of Y . This is needed in the proof of Theorem 1.1(i); moreover, it extends the applicability of Theorem 1.1 (see § 7).

Section 5 contains the proof of Theorem 1.1(i), which relies on the results of §§ 3 and 4.

In § 6 we show Theorem 1.1(ii). This is based on a detailed analysis of the possible elementary contractions of the target Y . We need the classification of smooth Fano 3-folds by Mori and Mukai, and we imitate the strategy for the classification of imprimitive smooth Fano 3-folds (see [MM81, IP99]) to get some results about the singular case. We also use the existence of a smoothing of a

terminal Fano 3-fold Z shown in [Nam97], and some relations among Z and its smoothing shown in [JR06], see Lemma 6.10.

Finally in §7 we prove Corollaries 1.2, 1.3, and 1.4, and we give some other applications and related examples.

2. Preliminaries

We work over the field of complex numbers.

Let X be a normal projective variety of dimension n . We denote by X_{reg} the smooth locus of X .

Let $\mathcal{N}_1(X)$ be the vector space of 1-cycles in X with real coefficients, modulo numerical equivalence. The dimension of $\mathcal{N}_1(X)$ is equal to the Picard number ρ_X of X . Inside $\mathcal{N}_1(X)$ we denote by $\text{NE}(X)$ the convex cone generated by classes of effective curves, and by $\overline{\text{NE}}(X)$ its closure. If $C \subset X$ is a curve, its numerical class is $[C] \in \mathcal{N}_1(X)$.

If Z is a closed subset of X , call $i: Z \hookrightarrow X$ the inclusion, and consider the linear map

$$i_*: \mathcal{N}_1(Z) \longrightarrow \mathcal{N}_1(X).$$

We denote by $\mathcal{N}_1(Z, X)$ the image of i_* in $\mathcal{N}_1(X)$. Thus we have

$$\dim \mathcal{N}_1(Z, X) \leq \rho_Z \quad \text{and} \quad \dim \mathcal{N}_1(Z, X) \leq \rho_X.$$

A *contraction* of X is a surjective morphism with connected fibers $f: X \rightarrow Y$ onto a projective and normal variety Y . The push-forward of 1-cycles defined by f gives a surjective linear map

$$f_*: \mathcal{N}_1(X) \longrightarrow \mathcal{N}_1(Y),$$

so that $\rho_X - \rho_Y = \dim \ker f_*$. We also consider the convex cone $\text{NE}(f)$ in $\mathcal{N}_1(X)$ generated by classes of curves contracted by f , that is

$$\text{NE}(f) = \text{NE}(X) \cap \ker f_*.$$

The contraction f is determined (up to isomorphism) by $\text{NE}(f)$, see [Deb01, Proposition 1.14]. We say that f is of *fiber type* if $\dim Y < \dim X$, otherwise f is birational. When f is of fiber type, we say that f is non-trivial if $\dim Y > 0$. We denote by $\text{Exc}(f)$ the exceptional locus of f , i.e. the locus where f is not an isomorphism. We say that f is *divisorial* if $\text{Exc}(f)$ is a divisor, *small* if $\text{Exc}(f)$ has codimension bigger than 1. More generally we say that f is of *type* (a, b) if $\dim \text{Exc}(f) = a$ and $\dim f(\text{Exc}(f)) = b$. Finally f is *elementary* if $\rho_X - \rho_Y = 1$.

We will need to work with singular varieties; we refer the reader to [Deb01, KM98] for the definitions and properties of terminal and canonical singularities. We say that X is \mathbb{Q} -factorial if every Weil divisor is \mathbb{Q} -Cartier.

Suppose that X has canonical singularities (in particular K_X is \mathbb{Q} -Cartier). We say that $f: X \rightarrow Y$ is a *Mori contraction* if it is a contraction and moreover $-K_X \cdot C > 0$ for every curve $C \subset X$ contracted by f . We recall two important properties of Mori contractions:

- (2.1) $\dim \text{NE}(f) = \dim \ker f_* = \rho_X - \rho_Y$, namely $\ker f_*$ is the linear subspace generated by $\text{NE}(f)$;
- (2.2) for any $L \in \text{Pic } X$ one has $L \in f^*(\text{Pic } Y)$ if and only if $L \cdot C = 0$ for every curve $C \subset X$ contracted by f .

(See [KM98, Theorem 3.7(4)] for the second statement, which implies the first one.)

Suppose that $\text{NE}(X)$ is closed and polyhedral. By a *face* of $\text{NE}(X)$ we just mean a face in the geometrical sense. For any contraction f of X , $\text{NE}(f)$ is a face of $\text{NE}(X)$. An *extremal ray* is a one-dimensional face, with no assumptions on the intersection of K_X with its elements. We will use greek letters α, β , etc. to denote faces of $\text{NE}(X)$. If α is an extremal ray and D a \mathbb{Q} -Cartier divisor

on X , we will say that $D \cdot \alpha > 0$, $D \cdot \alpha = 0$, or $D \cdot \alpha < 0$ if respectively $D \cdot v > 0$, $D \cdot v = 0$, or $D \cdot v < 0$ for a non-zero element $v \in \alpha$. We denote by $\text{Locus}(\alpha) \subseteq X$ the union of all curves in X whose numerical class is in α .

Let X be a projective variety with canonical singularities and K_X Cartier. We say that X is *Fano* if $-K_X$ is ample. If so, the cone $\text{NE}(X)$ is closed and polyhedral, and any contraction of X is a Mori contraction. Moreover, for any face α of $\text{NE}(X)$ there exists a contraction f of X such that $\alpha = \text{NE}(f)$. This follows from the contraction theorem, see [KM98, Theorem 3.7].

Remark 2.3. Let X and Y be factorial projective varieties, and $\sigma: X \rightarrow Y$ the blow-up of a smooth subvariety $A \subset Y_{\text{reg}}$. Suppose that X is Fano and let $C \subset Y$ be an irreducible curve such that $C \not\subseteq A$ and $C \cap A \neq \emptyset$. Let \tilde{C} be the proper transform of C in X . Then

$$-K_X \cdot \tilde{C} < -K_Y \cdot C,$$

in particular $-K_Y \cdot C \geq 2$.

In fact, if $E = \text{Exc}(\sigma)$, we have $\tilde{C} \not\subseteq E$ and $\tilde{C} \cap E \neq \emptyset$, so $\tilde{C} \cdot E > 0$. Moreover $K_X = \sigma^*(K_Y) + aE$ where $a = \text{codim } A - 1$, so

$$-K_X \cdot \tilde{C} = -K_Y \cdot C - aE \cdot \tilde{C} < -K_Y \cdot C.$$

Remark 2.4. Let X be a normal and \mathbb{Q} -factorial projective variety and $f: X \rightarrow Y$ an elementary divisorial contraction. Then $\text{Exc}(f)$ is an irreducible divisor and $\text{Exc}(f) \cdot \text{NE}(f) < 0$.

In fact, let E be an irreducible component of $\text{Exc}(f)$; then $E \cdot \text{NE}(f) < 0$ (see for instance [KM98, Lemma 3.39] applied to $B = -E$). Now if $C \subset X$ is an irreducible curve contracted by f , we have $E \cdot C < 0$, thus $C \subseteq E$. Hence $E = \text{Exc}(f)$.

2.5 Targets of contractions of smooth Fano varieties

Let X be a smooth Fano variety and $f: X \rightarrow Y$ a contraction. Consider the push-forward $f_*: \mathcal{N}_1(X) \rightarrow \mathcal{N}_1(Y)$. We observe that

$$f_*(\text{NE}(X)) = \text{NE}(Y),$$

namely $\text{NE}(Y)$ is the linear projection of $\text{NE}(X)$ from the face $\text{NE}(f)$. This simple remark implies many properties of $\text{NE}(Y)$. For instance, it is closed and polyhedral, since $\text{NE}(X)$ is. Moreover, faces of $\text{NE}(Y)$ are in bijection (via f_*) with faces of $\text{NE}(X)$ containing $\text{NE}(f)$. In fact this description is the same as the one involving the ‘star of a cone’ in toric geometry, see [Ful93, p. 52].

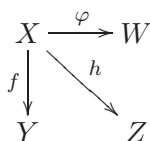
Let us consider a face α of $\text{NE}(Y)$, and let $\hat{\alpha}$ be the unique face of $\text{NE}(X)$ containing $\text{NE}(f)$ and such that $f_*(\hat{\alpha}) = \alpha$. Then $\dim \hat{\alpha} = \dim \alpha + \dim \text{NE}(f)$. Since $\text{NE}(f)$ is a face of $\hat{\alpha}$, we can choose another face $\tilde{\alpha}$ of $\hat{\alpha}$ with the properties:

$$\dim \tilde{\alpha} = \dim \alpha \quad \text{and} \quad \tilde{\alpha} \cap \text{NE}(f) = \{0\}.$$

Observe that the choice of $\tilde{\alpha}$ will not be unique in general, and that it can very well be $\tilde{\alpha} + \text{NE}(f) \subsetneq \hat{\alpha}$.

When α is an extremal ray, $\tilde{\alpha}$ is an extremal ray of $\text{NE}(X)$, and $f_*(\tilde{\alpha}) = \alpha$. There is a rational curve $C \subset X$ such that $[C] \in \tilde{\alpha}$, hence $f(C)$ is a rational curve in Y with numerical class in α .

Since X is Fano, there exist contractions $\varphi: X \rightarrow W$ and $h: X \rightarrow Z$ such that $\tilde{\alpha} = \text{NE}(\varphi)$ and $\hat{\alpha} = \text{NE}(h)$, as shown below.



Now by rigidity (see for instance [Deb01, Lemma 1.15]) there exist contractions $\psi: Y \rightarrow Z$ and $g: W \rightarrow Z$ that make the following diagram commute.

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & W \\ f \downarrow & \searrow h & \downarrow g \\ Y & \xrightarrow{\psi} & Z \end{array}$$

It is not difficult to check that $\text{NE}(\psi) = \alpha$ and that $\dim \ker \psi_* = \dim \alpha$. We will say that $\varphi: X \rightarrow W$ is a *lifting* of ψ . Summing up, we have proved the following result.

LEMMA 2.6. *Let X be a smooth Fano variety and $f: X \rightarrow Y$ a contraction. Then $\text{NE}(Y)$ is a closed polyhedral cone, and every extremal ray contains the class of some rational curve.*

Moreover for every face α of $\text{NE}(Y)$ there exists a contraction $\psi: Y \rightarrow Z$ such that $\text{NE}(\psi) = \alpha$ and $\rho_Y - \rho_Z = \dim \alpha$.

3. Quasi-elementary contractions

Let X be a smooth variety and $f: X \rightarrow Y$ a Mori contraction of fiber type. Recall that

$$\rho_X - \rho_Y = \dim \text{NE}(f) = \dim \ker f_*$$

Let F be a general fiber and consider $\mathcal{N}_1(F, X) \subseteq \mathcal{N}_1(X)$. We have

$$\mathcal{N}_1(F, X) \subseteq \ker f_*$$

hence $\dim \mathcal{N}_1(F, X) \leq \rho_X - \rho_Y$.

DEFINITION 3.1. We say that f is *quasi-elementary* if

$$\mathcal{N}_1(F, X) = \ker f_*$$

equivalently if $\dim \mathcal{N}_1(F, X) = \rho_X - \rho_Y$. Since $\dim \mathcal{N}_1(F, X) \leq \rho_F$, if f is quasi-elementary we get

$$\rho_X \leq \rho_Y + \rho_F.$$

Example 3.2. Let $f: X \rightarrow Y$ be a Mori contraction of fiber type.

- (a) If f is elementary, then it is also quasi-elementary. This is because $1 \leq \dim \mathcal{N}_1(F, X) \leq \dim \ker f_* = 1$.
- (b) Suppose that $\dim X - \dim Y = 1$. Then f is quasi-elementary if and only if it is elementary.
- (c) Suppose that X is Fano and f is quasi-elementary. If $\psi: Y \rightarrow Z$ is an elementary contraction of fiber type, then the composition $\psi \circ f: X \rightarrow Z$ is quasi-elementary. In fact, since ψ is elementary, we have $\dim \ker(\psi \circ f)_* = \dim \ker f_* + 1$. Let G be a general fiber of $\psi \circ f$, then $G \supset F$, hence $\ker(\psi \circ f)_* \supseteq \mathcal{N}_1(G, X) \supseteq \mathcal{N}_1(F, X) = \ker f_*$. Choosing a curve $C \subset G$ not contracted by f shows that $\mathcal{N}_1(G, X) \supsetneq \mathcal{N}_1(F, X)$, so $\mathcal{N}_1(G, X) = \ker(\psi \circ f)_*$ and $\psi \circ f$ is quasi-elementary.

Let us show that the notion of quasi-elementary is related to smoothness.

LEMMA 3.3. *Let X be a smooth variety and $f: X \rightarrow Y$ a Mori contraction of fiber type. Let $Y_0 \subseteq Y_{\text{reg}}$ be an open subset over which f is smooth, and set $X_0 := f^{-1}(Y_0)$. If $\text{codim}(X \setminus X_0) \geq 2$ and Y is \mathbb{Q} -factorial, then f is quasi-elementary.*

We remark that the converse to Lemma 3.3 does not hold (take, for instance, a smooth Fano 3-fold X with an elementary contraction $X \rightarrow \mathbb{P}^1$ that is not a smooth morphism). Moreover the hypothesis of \mathbb{Q} -factoriality on Y is necessary, see the contraction $f: V \rightarrow Y$ in Example 7.9.

Proof. Set $f_0 := f|_{X_0} : X_0 \rightarrow Y_0$. Following the notation of [KMM87, § 0-1], consider

$$\text{NE}(X_0/Y_0) \subset \mathcal{N}_1(X_0/Y_0).$$

Since f_0 is a Mori contraction, $\text{NE}(X_0/Y_0)$ is closed and polyhedral by the relative version of the cone theorem, see [KMM87, Theorem 4-2-1].

On the other hand $\mathcal{N}_1(X/Y) = \ker f_* \subseteq \mathcal{N}_1(X)$, and the inclusion $X_0 \hookrightarrow X$ induces a natural injective homomorphism

$$\mathcal{N}_1(X_0/Y_0) \hookrightarrow \ker f_*.$$

With a slight abuse of notation, we will consider $\mathcal{N}_1(X_0/Y_0)$ as a subspace of $\mathcal{N}_1(X)$.

For every $y \in Y_0$ we have $i_*(\text{NE}(F_y)) \subseteq \text{NE}(X_0/Y_0)$. Consider an extremal ray α of $\text{NE}(X_0/Y_0)$. Then [Wis91b, Proposition 1.3] says that $\text{Locus}(\alpha)$ dominates Y_0 via f , hence $\alpha \subseteq i_*(\text{NE}(F_y))$ for every $y \in Y_0$. Repeating this for every extremal ray of $\text{NE}(X_0/Y_0)$, we get

$$i_*(\text{NE}(F_y)) = \text{NE}(X_0/Y_0)$$

for every $y \in Y_0$, in particular

$$\mathcal{N}_1(F_y, X) = \mathcal{N}_1(X_0/Y_0) \subseteq \ker f_* \subseteq \mathcal{N}_1(X). \tag{3.4}$$

The homomorphism dual to the inclusion $\mathcal{N}_1(X_0/Y_0) \hookrightarrow \ker f_*$ is

$$r : \frac{(\text{Pic } X) \otimes \mathbb{R}}{f^*(\text{Pic } Y) \otimes \mathbb{R}} \longrightarrow \frac{(\text{Pic } X_0) \otimes \mathbb{R}}{f_0^*(\text{Pic } Y_0) \otimes \mathbb{R}}$$

induced by the restriction $\text{Pic } X \rightarrow \text{Pic } X_0$ (see [KMM87, § 0-1 and Lemma 3-2-5(2)]). Clearly r is surjective.

Let $L \in \text{Pic } X$ be such that $L|_{X_0} = f_0^*(M_0)$ for some $M_0 \in \text{Pic } Y_0$. Since Y is \mathbb{Q} -factorial, there exists $M \in \text{Pic } Y$ such that $M|_{Y_0} = M_0^{\otimes l}$, $l \in \mathbb{Z}_{\geq 1}$. Then $L^{\otimes l} \otimes f^*(M)^{\otimes (-1)}$ is trivial on X_0 , and by our hypothesis trivial on X . Thus r is an isomorphism. Dually, this gives $\mathcal{N}_1(F_y, X) = \mathcal{N}_1(X_0/Y_0) = \ker f_*$, so f is quasi-elementary. \square

Remark 3.5. Observe that (3.4) shows that $\mathcal{N}_1(F_y, X)$ does not depend on $y \in Y_0$, so that the condition $\mathcal{N}_1(F_y, X) = \ker f_*$ can be checked for an arbitrary $y \in Y_0$.

Remark 3.6. If $f : X \rightarrow Y$ is quasi-elementary, then $\mathcal{N}_1(F_0, X) = \ker f_*$ for every fiber F_0 of f (with the reduced structure).

In fact we have $\mathcal{N}_1(F_0, X) \subseteq \ker f_*$. Moreover if $f_0 : X_0 \rightarrow Y_0$ is as in the proof of Lemma 3.3, $\text{NE}(X_0/Y_0)$ has dimension $\rho_X - \rho_Y$. For any fixed extremal ray α of $\text{NE}(X_0/Y_0)$, we know by [Wis91b, Proposition 1.3] that $\text{Locus}(\alpha)$ dominates Y_0 . Then taking a family of curves whose class is in α and their degenerations, we see that $\alpha \subset \mathcal{N}_1(F_0, X)$. This implies that $\mathcal{N}_1(F_0, X) = \ker f_*$.

Remark 3.7. Suppose that X is smooth and Fano, $f : X \rightarrow Y$ a contraction of fiber type, and F a general fiber. In general the push-forward $i_* : \mathcal{N}_1(F) \rightarrow \mathcal{N}_1(X)$ does not need to be injective: for instance, there are smooth Fano 3-folds that have an elementary contraction onto \mathbb{P}^1 , with fibers Del Pezzo surfaces with $\rho > 1$. This is related to the monodromy of the fibration f .

Consider an open subset Y_0 as in Lemma 3.3 and let $y \in Y_0$. The dimension of $\mathcal{N}_1(F_y, X)$ is equal to the dimension of the image of the restriction

$$(\text{Pic } X) \otimes \mathbb{Q} = H^2(X, \mathbb{Q}) \longrightarrow H^2(F_y, \mathbb{Q}) = (\text{Pic } F_y) \otimes \mathbb{Q}.$$

In turn this is equal to the dimension of the linear subspace of $H^2(F_y, \mathbb{Q})$ which is invariant for the monodromy action of $\pi_1(Y_0, y)$ (see for instance [Voi02, ch. 15]). Hence $\dim \mathcal{N}_1(F_y, X) = \rho_F$ if and only if the monodromy action is trivial.

Example 3.8. Let X be a smooth Fano variety, $f: X \rightarrow Y$ a non-trivial contraction of fiber type with Y smooth, and F a general fiber. Suppose that f is smooth outside a finite number of points of Y , and that there are no fibers of codimension 1. Then Y is Fano, f is quasi-elementary, and $\rho_X = \rho_Y + \rho_F$.

In fact Y is Fano by [Miy93, Theorem 3]; in particular it is simply connected, and $\dim Y \geq 2$. Then $Y \setminus \{y_1, \dots, y_m\}$ stays simply connected, so the monodromy action on $H^2(F, \mathbb{Q})$ is trivial, and $\dim \mathcal{N}_1(F, X) = \rho_F$. On the other hand f is quasi-elementary by Lemma 3.3, so $\rho_X = \dim \mathcal{N}_1(F, X) + \rho_Y = \rho_F + \rho_Y$.

The following two lemmas give some basic properties of quasi-elementary contractions.

LEMMA 3.9. *Let X be a smooth variety and $f: X \rightarrow Y$ a quasi-elementary Mori contraction of fiber type.*

- (i) *If D is a prime divisor in X such that $f(D) \subsetneq Y$, then $f(D)$ is a Cartier divisor, and $D = f^*(f(D))$.*
- (ii) *The locus where f is not equidimensional has codimension at least 3 in Y .*

Proof. Since f is quasi-elementary and D is disjoint from the general fiber, we have $D \cdot C = 0$ for every curve $C \subset X$ contracted by f . By property (2.2) of Mori contractions, there exists an effective Cartier divisor D' on Y such that $D = f^*(D')$. Moreover D' is supported on $f(D)$, so there exists $m \in \mathbb{Z}_{>0}$ such that $D' = mf(D)$.

Set $U := f^{-1}(Y_{\text{reg}})$ and observe that $X \setminus U$ has codimension at least 2 in X . In Y_{reg} the intersection $f(D) \cap Y_{\text{reg}}$ is a (non-trivial) Cartier divisor and $D|_U = (f|_U)^*(mf(D))|_{Y_{\text{reg}}} = m(f|_U)^*(f(D) \cap Y_{\text{reg}})$. Since X is smooth, there exists an effective divisor D'' in X such that $D''|_U = (f|_U)^*(f(D) \cap Y_{\text{reg}})$. Then $D|_U = mD''|_U$, so $D = mD''$ and $m = 1$ because D is prime.

Let us show part (ii). Let $K \subset Y$ be an irreducible closed subset such that the general fiber of f over K has dimension at least $\dim X - \dim Y + 1$. Then

$$\dim X - 1 \geq \dim f^{-1}(K) \geq \dim X - \dim Y + 1 + \dim K,$$

so $\dim K \leq \dim Y - 2$. If by contradiction $\dim K = \dim Y - 2$, then $\dim f^{-1}(K) \geq \dim X - 1$. Consider a prime divisor D contained in $f^{-1}(K)$: we have $f(D) \subseteq K$, which contradicts (i). \square

We generalize to quasi-elementary contractions some known properties of elementary contractions of fiber type. In particular part (ii) below is shown in [ABW92] in the elementary case.

LEMMA 3.10. *Let $f: X \rightarrow Y$ be a Mori contraction of fiber type, with X smooth.*

- (i) *If f is quasi-elementary, then Y is factorial and has canonical singularities.*
- (ii) *If Y is a surface and f is equidimensional, then Y is smooth.*
- (iii) *If Y is a surface and f is quasi-elementary, then f is equidimensional and Y is smooth.*
- (iv) *If $\dim Y = 3$ and f is quasi-elementary, then Y has isolated singularities.*

Proof. (i) Let D be a prime Weil divisor in Y and let D' be a prime divisor in X such that $f(D') = D$. Then Lemma 3.9(i) yields that D is Cartier.

Thus Y is factorial, in particular K_Y is Cartier. It is known that Y has rational singularities, see [Kol86, Corollary 7.4]. Then Y also has canonical singularities, see [KM98, Corollary 5.24].

(ii) When f is elementary, this is [ABW92, Proposition 1.4.1]. In general, observe that Y is a normal surface with rational singularities; in particular it is \mathbb{Q} -factorial and has isolated singularities.

We want to show that in fact Y has quotient singularities, using results from [Wat80] and [FZ03]. More precisely, let $y_0 \in Y$ be a singular point. In [Wat80, Definition 1.4] and [FZ03, Definition 1.25]

one can find the definition of the plurigenera $\delta_m(Y, y_0)$ of Y in y_0 , for $m \in \mathbb{Z}_{>0}$. Since f is equidimensional, then it is ‘non-degenerate’ in the sense of [FZ03, Definition 1.14]. Then [FZ03, Corollary 1.27] gives $\delta_m(Y, y_0) = 0$ for every $m \in \mathbb{Z}_{>0}$. This is equivalent to saying that y_0 is a quotient singularity by [Wat80, Theorem 3.9].

Hence Y has quotient singularities, and we can apply [ABW92, Proposition 1.4] to deduce that Y is actually smooth. Observe that in [ABW92] the contraction f is assumed to be elementary, but the proof works word for word in the case of an equidimensional Mori contraction. Let us also remark that, by the definition of quotient singularities, we can cover Y by open subsets in the complex topology which are quotients of an open subset of \mathbb{C}^2 by a finite group. Hence in the proof of [ABW92, Proposition 1.4] one has actually to work in the analytic category; however everything works in the same way.

(iii) This follows immediately from Lemma 3.9 and part (ii).

(iv) By Lemma 3.9, f can have at most isolated fibers of dimension $n - 2$. Let $S \subset Y$ be a general hyperplane section and $D := f^{-1}(S)$. Then S is a normal surface (see [KM98, Lemma 5.30]) and $f|_D: D \rightarrow S$ is equidimensional. Moreover D is general in a base point free linear system, so it is smooth. Since D is disjoint from the general fiber of f and f is quasi-elementary, we have $D \cdot C = 0$ for every curve $C \subset X$ contracted by f . In particular, if $C \subset D$ we get

$$-K_D \cdot C = -K_X \cdot C > 0,$$

so $f|_D$ is an equidimensional Mori contraction. Then S is smooth by part (ii), and this yields $\dim \text{Sing } Y = 0$. □

Thus the target of a quasi-elementary contraction has reasonable singularities. The following simple remark will be very useful.

Remark 3.11. Let X be a smooth Fano variety and $f: X \rightarrow Y$ a quasi-elementary contraction, so that Y is factorial and has canonical singularities by Lemma 3.10. Then Y is Fano if and only if $-K_Y \cdot \alpha > 0$ for every extremal ray α of $\text{NE}(Y)$. Equivalently, Y is Fano if and only if every elementary contraction $\psi: Y \rightarrow Z$ is a Mori contraction.

This is a straightforward consequence of Lemma 2.6.

3.12 Elementary contractions of the target

Let X be a smooth projective variety and $f: X \rightarrow Y$ a quasi-elementary Mori contraction of fiber type. Recall that Y is factorial with canonical singularities by Lemma 3.10.

Let $\psi: Y \rightarrow Z$ be a contraction. We say that ψ has a *lifting* if there exist a Mori contraction $\varphi: X \rightarrow W$ such that $\text{NE}(\varphi) \cap \text{NE}(f) = \{0\}$, $\rho_X - \rho_W = \rho_Y - \rho_Z$, and a commutative diagram as below.

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & W \\ f \downarrow & & \downarrow g \\ Y & \xrightarrow{\psi} & Z \end{array} \tag{3.13}$$

We will also say that φ is a *lifting* of ψ .

Notice that ψ is elementary if and only if φ is elementary. We are interested in comparing properties of φ and ψ .

When X is Fano, any contraction $\psi: Y \rightarrow Z$ has a lifting, as explained in § 2.5. Conversely, in Theorems 4.1 and 4.4 we will give conditions on a Mori contraction $\varphi: X \rightarrow W$ to be a lifting of some ψ .

THEOREM 3.14. *Let X be a smooth projective variety and $f: X \rightarrow Y$ a quasi-elementary Mori contraction of fiber type.*

Let $\psi: Y \rightarrow Z$ be a birational elementary contraction with fibers of dimension at most 1. Assume that ψ has a lifting $\varphi: X \rightarrow W$ as in (3.13). Then the following hold:

- (i) W is smooth and φ is the blow-up of a smooth subvariety of codimension 2;
- (ii) g is a quasi-elementary Mori contraction and Z is factorial with canonical singularities;
- (iii) ψ is divisorial, $\text{Exc}(\varphi) = f^*(\text{Exc}(\psi))$, and $-K_Y \cdot \text{NE}(\psi) \geq 0$;
- (iv) if $\dim Y = 3$, then ψ is the blow-up of a smooth curve $C \subset Z_{\text{reg}}$; and
- (v) if $\dim Y = 2$, then
 - (v.a) Y and Z are smooth, ψ is the blow-up of $z_0 \in Z$, and g has smooth fiber F_0 over z_0 ;
 - (v.b) φ is the blow-up of F_0 , $\text{Exc}(\varphi) \cong \mathbb{P}^1 \times F_0$, and $f|_{\text{Exc}(\varphi)}, \varphi|_{\text{Exc}(\varphi)}$ are the two projections;
 - (v.c) if X is Fano, then W is Fano.

Proof. Let F be a non-trivial fiber of φ . Then f is finite on F and $f(F)$ is contained in a non-trivial fiber of ψ , hence $f(F) \subseteq \text{Exc}(\psi)$ and $\dim F = 1$. Since $\text{Exc}(\varphi)$ is covered by non-trivial fibers of φ , we get $f(\text{Exc}(\varphi)) \subseteq \text{Exc}(\psi)$. In particular, φ is birational with fibers of dimension at most 1, so [Wis91a, Theorem 1.2] yields that W is smooth and φ is the blow-up of a smooth subvariety of codimension 2.

Let $E \subset X$ be the exceptional divisor of φ , then $f(E)$ is contained in $\text{Exc}(\psi)$ and it is a divisor by Lemma 3.9(i). Hence ψ is elementary and divisorial, and $\text{Exc}(\psi)$ is irreducible by Remark 2.4, that is $\text{Exc}(\psi) = f(E)$. Then again Lemma 3.9(i) gives $E = f^*(\text{Exc}(\psi))$.

We have

$$K_X = \varphi^*(K_W) + E.$$

Let C be an irreducible curve contracted by g and \tilde{C} an irreducible curve in X such that $\varphi(\tilde{C}) = C$. Then $\varphi_*(\tilde{C}) = mC$ with $m \in \mathbb{Z}_{>0}$. Moreover $f(\tilde{C})$ is a point, so $\tilde{C} \cdot E = 0$. Then

$$m(-K_W \cdot C) = -K_W \cdot \varphi_*(\tilde{C}) = (-K_X - E) \cdot \tilde{C} = -K_X \cdot \tilde{C} > 0,$$

so g is a Mori contraction.

We have $\dim \ker f_* = \rho_X - \rho_Y = \rho_W - \rho_Z = \dim \ker g_*$. Moreover $\text{NE}(\varphi)$ is an extremal ray of $\text{NE}(X)$ not contained in $\text{NE}(f)$, hence $\ker \varphi_* \cap \ker f_* = \{0\}$. Then $\dim \varphi_*(\ker f_*) = \dim \ker g_*$; on the other hand, $\varphi_*(\ker f_*) \subseteq \ker g_*$, so equality holds.

Observe that φ is an isomorphism over the general fibers of f and g . Let F_1 be a general fiber of f , so that $\mathcal{N}_1(F_1, X) = \ker f_*$ because f is quasi-elementary. Then $\varphi(F_1)$ is a general fiber of g , and $\mathcal{N}_1(\varphi(F_1), W) = \varphi_*(\mathcal{N}_1(F_1, X)) = \varphi_*(\ker f_*) = \ker g_*$. Thus g is quasi-elementary.

Then Z is factorial with canonical singularities by Lemma 3.10(i). Let us show that $-K_Y \cdot \text{NE}(\psi) \geq 0$. Write $K_Y = \psi^*(K_Z) + r \text{Exc}(\psi)$ with $r \in \mathbb{Z}$, and let $C_1 \subset Y$ be a curve contracted by ψ . Then $\text{Exc}(\psi) \cdot C_1 < 0$ by Remark 2.4, and $-K_Y \cdot C_1 = r(-\text{Exc}(\psi) \cdot C_1)$. Hence we have to show that $r \geq 0$. Let $h: Y' \rightarrow Y$ be a resolution of singularities of Y , and consider the composition $\psi \circ h: Y' \rightarrow Z$. Call E' the proper transform of $\text{Exc}(\psi)$ in Y' . Then r is the coefficient of E' in $K_{Y'} - (\psi \circ h)^*(K_Z)$, thus $r \geq 0$ because Z has canonical singularities.

Assume now that $\dim Y = 3$ and set $S := \text{Exc}(\psi)$.

Let us first notice that $S_{\text{reg}} = S \cap Y_{\text{reg}}$. This is because, if $y_0 \in S_{\text{reg}}$, then y_0 must be smooth for Y too, because S is a Cartier divisor in Y . On the other hand, let $y_1 \in Y_{\text{reg}}$ and let $h \in \mathcal{O}_{Y, y_1}$ be a local equation for S in y_1 . Since $E = f^*(S)$, $f^*(h)$ is a local equation for E near the fiber over y_1 . Take x_1 in this fiber. Since E is smooth, the differential $d_{x_1}(f^*(h))$ is non-zero, and since $d_{x_1}(f^*(h)) = d_{y_1}h \circ d_{x_1}f$, the differential $d_{y_1}h$ must be non-zero too. Thus S is smooth at y_1 .

Recall that Y has rational singularities, in particular it is Cohen–Macaulay (see [Kol86, Corollary 7.4] and [KM98, Theorem 5.10]). Since S is a Cartier divisor in Y , it is Cohen–Macaulay too. On the other hand, Lemma 3.10(iv) implies that Y and S have isolated singularities. Then S is normal by Serre’s criterion.

By Lemma 3.9 f can have at most isolated fibers of dimension $n - 2$. Let us show that f is equidimensional over S . If $F_0 \subset E$ is an irreducible component of a fiber of f with $\dim F_0 = n - 2$, then $\varphi(F_0) \subseteq \varphi(E)$ and $\dim \varphi(F_0) = n - 2 = \dim \varphi(E)$, hence $\varphi(F_0) = \varphi(E)$. This gives

$$\psi(S) = \psi(f(E)) = g(\varphi(E)) = g(\varphi(F_0)) = \psi(f(F_0)) = pt,$$

a contradiction.

Now E is smooth, S is normal, and $f|_E: E \rightarrow S$ has connected fibers. As in the proof of Lemma 3.10(iv) we see that $f|_E$ is a Mori contraction, and by Lemma 3.10(ii) the surface S is smooth, so that $S \cap \text{Sing}(Y) = \emptyset$.

The general fiber l of $\psi|_S: S \rightarrow \psi(S)$ is a rational curve, and it is smooth because S is, hence $-K_S \cdot l = 2$.

Let \tilde{l} be a fiber of φ such that $f(\tilde{l}) = l$. Then

$$-1 = \tilde{l} \cdot E = \tilde{l} \cdot f^*(S) = f_*(\tilde{l}) \cdot S,$$

which gives $l \cdot S = -1$ and $-K_Y \cdot l = -K_S \cdot l + S \cdot l = 1$.

This shows also that $\psi|_S: S \rightarrow \psi(S)$ is a \mathbb{P}^1 -bundle. Therefore ψ is the blow-up of a smooth curve contained in the smooth locus of Z , and Z has the same singularities as Y .

Finally, let us suppose that $\dim Y = 2$. Then Y and Z are smooth by Lemma 3.10(iii), and f and g are equidimensional. Hence ψ is the blow-up of a point $z_0 \in Z$. We have

$$E = f^{-1}(\text{Exc}(\psi)) = \varphi^{-1}(g^{-1}(z_0)),$$

so the center of φ is $\varphi(E) = g^{-1}(z_0) = F_0$.

Since $E = f^*(\text{Exc}(\psi))$, E is the (schematic) fiber of $\psi \circ f$ over z_0 , and it is reduced. Thus $g \circ \varphi$ has reduced fiber over z_0 , and the same must hold for g . Now g is an equidimensional morphism between smooth varieties, whose fiber F_0 over z_0 is reduced and smooth. This implies that g is smooth over z_0 , and hence the normal bundle of F_0 in W is trivial. Since E is the exceptional divisor of the blow-up of F_0 , we deduce that $E \cong \mathbb{P}^1 \times F_0$, and $f|_E$ and $\varphi|_E$ are the two projections.

Suppose that X is Fano and W is not. Then [Wiś91a, Proposition 3.4] says that there exists an extremal ray α of $\text{NE}(X)$, different from $\text{NE}(\varphi)$, such that $\alpha \cdot E < 0$. This implies that α is not contained in $\text{NE}(f)$, because $C \cdot E = 0$ for every curve C contracted by f . However $E \cong \mathbb{P}^1 \times F_0$, so every curve in E is numerically equivalent to a linear combination with coefficients in $\mathbb{Q}_{\geq 0}$ of a curve in $\text{NE}(f)$ and a curve in $\text{NE}(\varphi)$. Thus no curve in E can have numerical class in an extremal ray α not contained in $\text{NE}(f) \cup \text{NE}(\varphi)$, and we have a contradiction. This completes the proof of Theorem 3.14. □

3.15 Divisors D with small $\dim \mathcal{N}_1(D, X)$

Let X be a smooth Fano variety of dimension n , and D a prime divisor in X . If D is simple, e.g. if ρ_D is very small, then one can hope to deduce information on X itself. For instance, in [BCW02] the authors classify the possible pairs (X, D) when $D \cong \mathbb{P}^{n-1}$ and $\mathcal{N}_{D/X} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(-1)$. This is equivalent to asking that X is obtained by blowing up a smooth variety in a point.

This classification has been generalized in [Tsu06] to the case $D \cong \mathbb{P}^{n-1}$ and $\mathcal{N}_{D/X} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(-a)$ with $a \in \mathbb{Z}_{\geq 1}$.

With the same techniques, [Tsu06, Proposition 5] shows that, if X contains a prime divisor D with $\rho_D = 1$ and $\dim X \geq 3$, then $\rho_X \leq 3$. The proof of [Tsu06, Proposition 5] can be generalized in order to obtain the following result.

PROPOSITION 3.16. *Let X be a smooth Fano variety, and let $f: X \rightarrow Y$ be either a quasi-elementary contraction of fiber type or an isomorphism.*

Suppose that $\dim Y \geq 3$ and that there exists a prime divisor $D \subset Y$ such that $\dim \mathcal{N}_1(D, Y) = 1$. Then $\rho_Y \leq 3$.

In the case where f is an isomorphism, this says that if $\dim X \geq 3$ and X contains a prime divisor D with $\dim \mathcal{N}_1(D, X) = 1$, then $\rho_X \leq 3$. In particular, if $\dim X = n \geq 3$ and X has an elementary contraction of type $(n - 1, 0)$, then $\rho_X \leq 3$.

Proof. Recall that Y is factorial by Lemma 3.10(i). Moreover $\text{NE}(Y)$ is closed and polyhedral, and every extremal ray can be contracted, by Lemma 2.6.

Let us assume that $\rho_Y > 1$. There exists at least one extremal ray α_1 of $\text{NE}(Y)$ such that $\alpha_1 \cdot D > 0$; let $\psi_1: Y \rightarrow Z_1$ be its contraction. Then $\dim Z_1 > 0$ and D must intersect every non-trivial fiber of ψ_1 .

If $\alpha_1 \subset \mathcal{N}_1(D, Y)$, then every curve in D has numerical class contained in α_1 , hence $\psi_1(D)$ is a point. Then ψ_1 cannot be of fiber type (otherwise Z_1 is a point), so ψ_1 is birational and $D \subseteq \text{Exc}(\psi_1)$. This implies that $D \cdot \alpha_1 < 0$ by Remark 2.4, a contradiction.

Hence $\alpha_1 \notin \mathcal{N}_1(D, Y)$. Consider a non-trivial fiber F of ψ_1 . Then $F \cap D \neq \emptyset$; on the other hand $\dim(F \cap D) = 0$ otherwise we would get a curve in D with numerical class in α_1 . Since Y is factorial, we get

$$0 = \dim(F \cap D) \geq \dim F - 1,$$

so F has dimension 1.

If ψ_1 is of fiber type, then

$$(\psi_1)_*|_{\mathcal{N}_1(D, Y)}: \mathcal{N}_1(D, Y) \longrightarrow \mathcal{N}_1(Z_1)$$

is surjective, so $\rho_{Z_1} = 1$ and $\rho_Y = 2$.

Assume that ψ_1 is birational and consider a lifting of ψ_1 as in § 2.5.

$$\begin{array}{ccc} X & \xrightarrow{\varphi_1} & W_1 \\ f \downarrow & \searrow h & \downarrow g \\ Y & \xrightarrow{\psi_1} & Z_1 \end{array}$$

Since X is Fano, φ_1 is a Mori contraction, and Theorem 3.14 applies. Thus g is a quasi-elementary Mori contraction, W_1 is smooth, φ_1 is a blow-up, ψ_1 is divisorial, Z_1 is factorial, and $\text{Exc}(\varphi_1) = f^{-1}(\text{Exc}(\psi_1))$.

Since ψ_1 is finite on D , $D_2 := \psi_1(D)$ is a Cartier divisor in Z_1 . Moreover

$$(\psi_1)_*|_{\mathcal{N}_1(D, Y)}: \mathcal{N}_1(D, Y) \longrightarrow \mathcal{N}_1(D_2, Z_1)$$

is surjective, hence $\dim \mathcal{N}_1(D_2, Z_1) = 1$. Recall also that D intersects every non-trivial fiber of ψ_1 , hence $D_2 \supset \psi_1(\text{Exc}(\psi_1))$.

Again by Lemma 2.6, $\text{NE}(Z_1)$ is closed and polyhedral, and every extremal ray can be contracted.

Thus consider an extremal ray α_2 of $\text{NE}(Z_1)$ such that $\alpha_2 \cdot D_2 > 0$, and let $\psi_2: Z_1 \rightarrow Z_2$ be the associated contraction.

As before, we see that if ψ_2 is of fiber type then $\rho_{Z_1} \leq 2$ and $\rho_Y \leq 3$.

If ψ_2 is birational, then $\alpha_2 \notin \mathcal{N}_1(D_2, Z_1)$ and again every non-trivial fiber of ψ_2 has dimension 1. We show that this case leads to a contradiction.

Let $\widehat{\alpha}_2$ be the unique extremal face of $\text{NE}(X)$ such that $\widehat{\alpha}_2$ contains $\text{NE}(h)$ and $h_*(\widehat{\alpha}_2) = \alpha_2$ (see § 2.5). Notice that $\text{NE}(\varphi_1)$ is an extremal ray of $\text{NE}(h)$.

Let us make an easy remark on convex polyhedral cones. If $\sigma \subset \mathbb{R}^d$ is a convex polyhedral cone, γ is a proper face of σ , and ρ is an extremal ray of γ , then there exists an extremal ray ρ_2 of σ , not contained in γ , and such that $\rho + \rho_2$ is a face of σ . The reader who is not familiar with convex geometry may easily prove this by induction on $\dim \sigma$.

Applied to our situation, this says that there is an extremal ray $\widetilde{\beta}$ of $\widehat{\alpha}_2$ such that $\widetilde{\beta} \not\subset \text{NE}(h)$ and $\widetilde{\beta} + \text{NE}(\varphi_1)$ is a face of $\widehat{\alpha}_2$.

Then $\beta = (\varphi_1)_*(\widetilde{\beta})$ is an extremal ray of $\text{NE}(W_1)$, whose contraction $\varphi_2: W_1 \rightarrow W_2$ yields the following commutative diagram.

$$\begin{array}{ccccc}
 X & \xrightarrow{\varphi_1} & W_1 & \xrightarrow{\varphi_2} & W_2 \\
 \downarrow f & \searrow h & \downarrow g & & \downarrow \\
 Y & \xrightarrow{\psi_1} & Z_1 & \xrightarrow{\psi_2} & Z_2
 \end{array}$$

Observe that $\text{NE}(\psi_2 \circ h) = \widehat{\alpha}_2$ and $\text{NE}(\varphi_2 \circ \varphi_1) = \widetilde{\beta} + \text{NE}(\varphi_1) \subseteq \text{NE}(\psi_2 \circ h)$, so the morphism $W_2 \rightarrow Z_2$ exists by rigidity (see § 2.5). Since ψ_2 is birational and has fibers of dimension at most 1, it is easy to see that the same holds for φ_2 .

Suppose that there exists a non-trivial fiber C of φ_2 contained in $\varphi_1(\text{Exc}(\varphi_1))$. Let \widetilde{C} be an irreducible curve in $\text{Exc}(\varphi_1)$ such that $\varphi_1(\widetilde{C}) = C$. Then $f(\widetilde{C}) \subset f(\text{Exc}(\varphi_1)) = \text{Exc}(\psi_1)$ and $h(\widetilde{C}) \subseteq \psi_1(\text{Exc}(\psi_1)) \subset D_2$.

On the other hand, since $\varphi_2(C)$ is a point, $\psi_2(h(\widetilde{C}))$ must also be a point, namely the numerical class of $h(\widetilde{C})$ must be in $\alpha_2 = \text{NE}(\psi_2)$. But this contradicts $\alpha_2 \notin \mathcal{N}_1(D_2, Z_1)$.

Therefore if C is a non-trivial fiber of φ_2 , we have $C \not\subseteq \varphi_1(\text{Exc}(\varphi_1))$, and this implies that $-K_{W_1} \cdot C > 0$.

Thus φ_2 is a Mori contraction, and Theorem 3.14 yields that W_2 is smooth, φ_2 is the blow-up of a smooth subvariety of codimension 2 in W_2 , and $\text{Exc}(\varphi_2) = g^{-1}(\text{Exc}(\psi_2))$. In particular, $-K_{W_1} \cdot C = 1$.

Now using Remark 2.3 we see that any non-trivial fiber C of φ_2 cannot intersect $\varphi_1(\text{Exc}(\varphi_1))$, hence

$$\text{Exc}(\varphi_2) \cap \varphi_1(\text{Exc}(\varphi_1)) = \emptyset.$$

Recall that $\text{Exc}(\varphi_1) = f^{-1}(\text{Exc}(\psi_1))$, which gives $\varphi_1(\text{Exc}(\varphi_1)) = g^{-1}(\psi_1(\text{Exc}(\psi_1)))$. Therefore

$$\emptyset = g^{-1}(\text{Exc}(\psi_2)) \cap g^{-1}(\psi_1(\text{Exc}(\psi_1))) = g^{-1}(\text{Exc}(\psi_2) \cap \psi_1(\text{Exc}(\psi_1))),$$

namely $\text{Exc}(\psi_2) \cap \psi_1(\text{Exc}(\psi_1)) = \emptyset$.

Notice that $\dim \psi_1(\text{Exc}(\psi_1)) = \dim Y - 2 \geq 1$, so there exists a curve $C_2 \subseteq \psi_1(\text{Exc}(\psi_1))$, and $C_2 \cdot \text{Exc}(\psi_2) = 0$.

On the other hand $\psi_1(\text{Exc}(\psi_1)) \subset D_2$ and $\dim \mathcal{N}_1(D_2, Z_1) = 1$. This implies that $C_2' \cdot \text{Exc}(\psi_2) = 0$ for every curve $C_2' \subset D_2$. But this is impossible, because D_2 and $\text{Exc}(\psi_2)$ are distinct prime divisors with non-empty intersection. This completes the proof of Proposition 3.16. \square

Remark 3.17. More precisely we have shown that in the hypotheses of Proposition 3.16, either $\rho_Y = 1$, or one of the following occurs:

- (a) $\rho_Y = 2$ and Y has an elementary contraction of fiber type with one-dimensional fibers; or
- (b) $\rho_Y = 3$ and Y has a divisorial elementary contraction such that every fiber has dimension at most 1.

4. Existence of contractions after [BCD07]

Let X be any normal projective variety. Consider an irreducible closed subset V of $\text{Chow}(X)$ such that:

- (a) for $v \in V$ general, the corresponding cycle $C_v \subset X$ is an irreducible, reduced, and connected rational curve; and
- (b) every point of X is contained in C_v for some $v \in V$.

We call V a *covering family of rational curves* in X . Such a family induces an equivalence relation on X (as a set), called V -equivalence, as follows. Two points $x, y \in X$ are V -equivalent if there exist $v_1, \dots, v_m \in V$ such that $C_{v_1} \cup \dots \cup C_{v_m}$ is connected and contains x and y . This notion was originally introduced by Campana [Cam81] and is by now well known; see [Kol96, §IV.4], [Deb01, §5], and [Cam04]. We refer specifically to [BCD07] for the set-up and for precise references.

In particular it is known that there exist an open subset X_0 that is closed under V -equivalence, a normal quasi-projective variety Y_0 , and a proper, equidimensional morphism $q: X_0 \rightarrow Y_0$, such that every fiber of q is a V -equivalence class. In general, there are no morphisms defined on the whole X that extend q .

If $f: X \rightarrow Y$ is a Mori contraction of fiber type, then one can find a family V as above such that $q = f|_{X_0}$ (see the proof of Theorem 4.1). Using the properties of this family, we can apply the results of [BCD07] to deduce the following result.

THEOREM 4.1. *Let X be a smooth variety of dimension n and consider two Mori contractions*

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & W \\ f \downarrow & & \\ Y & & \end{array}$$

where f is elementary of fiber type and $\text{NE}(f) \cap \text{NE}(\varphi) = \{0\}$. Let k_f and k_φ be the dimensions of the general fibers of f and $\varphi|_{\text{Exc}(\varphi)}$ respectively. Assume that we are in one of the following situations:

- (i) φ is quasi-elementary of fiber type, and $k_f + k_\varphi \geq n - 3$;
- (ii) φ is elementary and divisorial, and $\dim Y \leq 3$; or
- (iii) φ is elementary and divisorial, $f(\text{Exc}(\varphi)) = Y$, and $k_\varphi \geq \dim Y - 3$.

Then there exists a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & W \\ f \downarrow & \searrow h & \downarrow g \\ Y & \longrightarrow & Z \end{array} \tag{4.2}$$

where $g: W \rightarrow Z$ is an elementary Mori contraction and $\dim Z \leq 3$.

Let us point out that we do not know whether the hypotheses on $k_f + k_\varphi$, $\dim Y$, and k_φ respectively are really necessary for the statement to hold.

Proof. In the first part of the proof we will just assume that f is quasi-elementary and that $\text{NE}(f) \cap \text{NE}(\varphi) = \{0\}$.

We first construct a suitable covering family V of rational curves in X , such that f is the quotient for V -equivalence over an open subset of X .

Let F be a general fiber of f , then F is a smooth Fano variety of dimension k_f . In particular F is rationally connected, and there exists a smooth rational curve $C_0 \subset F$ that is very free, namely with ample normal bundle in F (see [Deb01, §§ 4.3 and 5.6]):

$$\mathcal{N}_{C_0/F} \cong \bigoplus_i \mathcal{O}_{\mathbb{P}^1}(a_i), \quad a_i \in \mathbb{Z}_{>0} \text{ for every } i.$$

Since $\mathcal{N}_{F/X}$ is trivial, we have

$$\mathcal{N}_{C_0/X} \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus(n-k_f)} \oplus \bigoplus_i \mathcal{O}_{\mathbb{P}^1}(a_i),$$

hence C_0 is a free curve in X . This means that the deformations of C_0 cover the whole X : by [Deb01, Proposition 4.8] there exists a covering family V of rational curves in X such that $C_0 = C_{v_0}$ for some $v_0 \in V$.

Clearly all curves parametrized by V are numerically equivalent in X , and their numerical class is contained in $\text{NE}(f)$, because $f(C_0)$ is a point. Since $\text{NE}(f)$ is a face of $\text{NE}(X)$, we deduce that every irreducible component of every curve parametrized by V is contained in some fiber of f . This implies that every V -equivalence class is contained in some fiber of f .

Consider the quotient for V -equivalence $q: X_0 \rightarrow Y_0$; let F_0 be a general fiber, and $C_v \subset F_0$ a general curve of the family. Then $C_v \cong \mathbb{P}^1$ and we have

$$\mathcal{N}_{C_v/X} \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus(n-\dim F_0)} \oplus \mathcal{N}_{C_v/F_0}.$$

On the other hand, by [Kol96, II.3.9.2] the number of trivial summands in $\mathcal{N}_{C_v/X}$ is at most the number of trivial summands in $\mathcal{N}_{C_0/X}$, so that $n - \dim F_0 \leq n - k_f$ and $\dim F_0 \geq k_f$.

This implies that $\dim F_0 = k_f$ and hence all fibers of f of dimension k_f are V -equivalence classes.

We observe that V induces in a natural way a covering family of rational curves on W . First of all, the hypothesis $\text{NE}(f) \cap \text{NE}(\varphi) = \{0\}$ implies that φ does not contract any irreducible component of any curve of V .

Consider now the incidence diagram associated to V , shown below.

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & X \\ \downarrow & & \\ V & & \end{array}$$

We proceed as in [BCD07, proof of Lemma 2]. Let $\tilde{\mathcal{C}}$ be the normalization of \mathcal{C} and $\tilde{\mathcal{C}} \rightarrow \tilde{V}$ be the Stein factorization of the composite map $\tilde{\mathcal{C}} \rightarrow \mathcal{C} \rightarrow V$. Then \tilde{V} is normal, the general fiber of $\tilde{\mathcal{C}} \rightarrow \tilde{V}$ is \mathbb{P}^1 , and the composite map

$$\tilde{\mathcal{C}} \longrightarrow \mathcal{C} \longrightarrow X \xrightarrow{\varphi} W$$

yields a family of 1-cycles in W , thus a morphism $\tilde{V} \rightarrow \text{Chow}(W)$. We call V' the image of this morphism. By the construction, for every irreducible component C of a curve parametrized by V , the image $\varphi(C)$ is a component of some curve parametrized by V' , and conversely. In particular, if $x, y \in X$ are V -equivalent, then $\varphi(x), \varphi(y) \in W$ are V' -equivalent.

CLAIM. Let $T \subseteq W$ be a general V' -equivalence class. Then $\text{codim } T \leq n - k_f$, and $\text{codim } T \leq n - (k_f + k_\varphi)$ if $f(\text{Exc}(\varphi)) = Y$.

Observe that the second case always holds if φ is of fiber type.

Proof of the claim. The inverse image $\varphi^{-1}(T)$ is closed for V -equivalence. Let F be a general V -equivalence class contained in $\varphi^{-1}(T)$, so that F is a fiber of f . Since φ is finite on F and $\varphi(F) \subseteq T$, we have $\dim T \geq \dim F = k_f$, while $\dim W \leq n$, which gives the first statement.

If φ is of fiber type, then $\dim W = n - k_\varphi$, so the same argument gives $\text{codim } T \leq n - (k_f + k_\varphi)$.

Let us assume that φ is birational and that $\text{Exc}(\varphi)$ dominates Y via f . Then $F \cap \text{Exc}(\varphi) \neq \emptyset$, so $T \cap \varphi(\text{Exc}(\varphi)) \neq \emptyset$, and since T is general, every V' -equivalence class intersects $\varphi(\text{Exc}(\varphi))$. This means that $\varphi^{-1}(T)$ contains a general fiber N of $\varphi|_{\text{Exc}(\varphi)}$. Let N_0 be an irreducible component of N with $\dim N_0 = k_\varphi$.

There exists a non-empty open subset U_0 of $f(N_0)$ such that every fiber of f over U_0 is an irreducible V -equivalence class. Then $f^{-1}(U_0)$ is irreducible of dimension $k_f + k_\varphi$, and it must be contained in $\varphi^{-1}(T)$ because $\varphi^{-1}(T)$ is closed for V -equivalence. Moreover $\varphi|_{f^{-1}(U_0)}$ is birational, so $\dim T \geq k_f + k_\varphi$. This completes the proof of the claim. \square

Consider now the linear subspaces H_V of $\mathcal{N}_1(X)$ and $H_{V'}$ of $\mathcal{N}_1(W)$ generated by the numerical classes of all irreducible components of all curves in V and V' respectively. Then we have $\varphi_*(H_V) = H_{V'}$; and moreover $\mathcal{N}_1(F, X) \subseteq H_V$ and $\mathcal{N}_1(T, W) \subseteq H_{V'}$ by [Kol96, Proposition IV.3.13.3] (see also [BCD07, Remark 1]).

Since f is quasi-elementary, we get $\ker f_* = \mathcal{N}_1(F, X) \subseteq H_V \subseteq \ker f_*$, hence $\mathcal{N}_1(F, X) = H_V = \ker f_*$. Then we have

$$\mathcal{N}_1(T, W) \subseteq H_{V'} = \varphi_*(H_V) = \varphi_*(\mathcal{N}_1(F, X)) = \mathcal{N}_1(\varphi(F), W) \subseteq \mathcal{N}_1(T, W),$$

hence

$$\mathcal{N}_1(T, W) = H_{V'} = \varphi_*(\mathcal{N}_1(F, X)) = \varphi_*(\ker f_*). \tag{4.3}$$

We observe that φ is either elementary and divisorial, or quasi-elementary of fiber type. Then in any case W is \mathbb{Q} -factorial and has canonical singularities, by Lemma 3.10 and [Deb01, Proposition 7.44].

Let us assume now that f is an elementary contraction. Then $\dim H_V = \dim H_{V'} = 1$, and this means that V and V' are quasi-unsplit in the sense of [BCD07]. Moreover, using the claim we see that in any case the general V' -equivalence class has codimension at most 3. Hence [BCD07, Theorem 2] yields the existence of an elementary Mori contraction $g: W \rightarrow Z$ such that every fiber of g is a V' -equivalence class. Then $\dim Z \leq 3$, and by rigidity (see for instance [Deb01, Lemma 1.15]) there exists a morphism $Y \rightarrow Z$ as in the statement. This completes the proof of Theorem 4.1. \square

We observe that, with a slight modification of the argument, we can prove a different version of the previous theorem. Namely we can allow f to be quasi-elementary instead of elementary, if we impose a stronger condition on k_f and k_φ .

THEOREM 4.4. *Let X be a smooth variety of dimension n and consider two Mori contractions $f: X \rightarrow Y$ and $\varphi: X \rightarrow W$ such that $\text{NE}(f) \cap \text{NE}(\varphi) = \{0\}$ and f is quasi-elementary of fiber type. Let k_f and k_φ be the dimensions of the general fibers of f and $\varphi|_{\text{Exc}(\varphi)}$ respectively. Assume that we are in one of the following situations:*

- (i) φ is quasi-elementary and $k_f + k_\varphi \geq n - 2$;
- (ii) φ is elementary and divisorial, and $\dim Y \leq 2$; or
- (iii) φ is elementary and divisorial, $f(\text{Exc}(\varphi)) = Y$, and $k_\varphi \geq \dim Y - 2$.

Then there exists a commutative diagram as (4.2) where $h: X \rightarrow Z$ is a contraction, $\dim Z \leq 2$, and $\rho_X - \rho_Z \leq (\rho_X - \rho_Y) + (\rho_X - \rho_W)$ (equality holds except possibly in (i)).

Proof. We perform the same construction as in the previous proof, so that W is a normal and \mathbb{Q} -factorial projective variety, and V' is a covering family of rational curves in W . Moreover by the claim and the assumptions, we see that now the general V' -equivalence class has codimension at most 2 in W . On the other hand, V' is not quasi-unsplit if f is not elementary. However (4.3) implies the following property.

Let D be a Weil divisor in W whose support is disjoint from the general V' -equivalence class. Then $D \cdot C = 0$ for every irreducible component of every curve in V' .

We claim that we can apply [BCD07, Proposition 1] to the family V' on W , even if V' is not quasi-unsplit. Indeed the quasi-unsplit assumption is used in the proof of this proposition uniquely to deduce the property above. One can think that the property above generalizes the quasi-unsplit property, in the same way as quasi-elementary contractions generalize elementary contractions of fiber type.

Now as in the proof of [BCD07, Theorem 1], we get the existence of a normal projective variety Z and a surjective morphism $g: W \rightarrow Z$ such that every fiber of g is a V' -equivalence class (even if g is not necessarily a Mori contraction).

As before this implies the existence of the diagram (4.2) by rigidity, and $h := g \circ \varphi$ is a contraction. Moreover, it is clear that $\ker g_* = H_{V'}$ and

$$\ker h_* = (\varphi_*)^{-1}(H_{V'}) = (\varphi_*)^{-1}(\varphi_*(\ker f_*)) = \ker f_* + \ker \varphi_*.$$

Observe that the hypothesis $\text{NE}(f) \cap \text{NE}(\varphi) = \{0\}$ does not imply that $\ker f_* \cap \ker \varphi_* = \{0\}$, unless we know that one of f or φ is elementary, as in (ii) or (iii). Thus $\dim \ker h_* \leq \dim \ker f_* + \dim \ker \varphi_*$, and equality holds except possibly in (i). □

Remark 4.5. In both Theorems 4.1 and 4.4, we have in fact proved that $\dim Z \leq \dim Y$ in case (ii), and $\dim Z \leq n - (k_f + k_\varphi) = \dim Y - k_\varphi$ in cases (i) and (iii). Thus the contraction $Y \rightarrow Z$ is of fiber type in case (i), and elementary of fiber type in case (iii).

5. Fano manifolds with a quasi-elementary contraction onto a surface

In this section we show Theorem 1.1(i). So let us consider a smooth Fano variety X of dimension $n \geq 3$, and a quasi-elementary contraction $f: X \rightarrow Y$ onto a surface. We know by Lemma 3.10(iii) that f is equidimensional and Y is smooth. Moreover, Y is rational, for instance because X is rationally connected.

Remark 5.1. Let $\psi: Y \rightarrow Z$ be a birational elementary contraction. Then Z is smooth, ψ is the blow-up of a point $z_0 \in Z$, and there exists a diagram

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & W \\ f \downarrow & & \downarrow g \\ Y & \xrightarrow{\psi} & Z \end{array}$$

where W is a smooth Fano variety, g is a quasi-elementary contraction with smooth fiber F_0 over z_0 , φ is the blow-up of F_0 , $\text{Exc}(\varphi) \cong \mathbb{P}^1 \times F_0$, and $f|_{\text{Exc}(\varphi)}, \varphi|_{\text{Exc}(\varphi)}$ are the two projections.

In fact ψ has a lifting as explained in § 2.5, which gives the diagram above. Since X is Fano, φ is a Mori contraction. Then Theorem 3.14(v) gives the statement.

Notice moreover that the lifting φ is uniquely determined by ψ , because $\text{Exc}(\varphi) = f^{-1}(\text{Exc}(\psi)) \cong \mathbb{P}^1 \times F_0$ and $\text{NE}(\varphi)$ is determined by the curves $\mathbb{P}^1 \times \{pt\}$ in X . As described in § 2.5, every choice of an extremal ray $\tilde{\alpha}$ in $\text{NE}(\psi \circ f)$ such that $\tilde{\alpha} \not\subseteq \text{NE}(f)$, gives rise to a lifting of ψ . We deduce that such an extremal ray $\tilde{\alpha} = \text{NE}(\varphi)$ is unique, and that $\text{NE}(\psi \circ f) = \tilde{\alpha} + \text{NE}(f)$.

Proof of Theorem 1.1(i). If $\rho_Y = 1$ then $Y \cong \mathbb{P}^2$, so we can assume that $\rho_Y > 1$. In order to show that Y is Del Pezzo, it is enough to show that any elementary contraction of Y is a Mori contraction, see Remark 3.11.

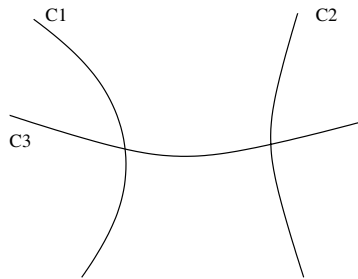
Let us fix such a contraction $\psi: Y \rightarrow Z$. If ψ is birational, then Remark 5.1 says that Z is smooth and ψ is a blow-up, hence a Mori contraction.

Let us consider the case where ψ is not birational, so that $\dim Z \leq 1$. Since we have assumed that $\rho_Y > 1$, we have $\dim Z = 1$, and Y is a smooth rational surface with $\rho_Y = 2$. Thus Y is a Hirzebruch surface and $Z \cong \mathbb{P}^1$. If we consider the other elementary contraction of Y , we find two possibilities: either it is again a contraction over \mathbb{P}^1 and $Y \cong \mathbb{P}^1 \times \mathbb{P}^1$, or it is birational. In this case, by Remark 5.1 it must be a smooth blow-up, so $Y \cong \mathbb{F}_1$.

Hence Y is a Del Pezzo surface, and we have the first part of the statement.

Let us assume that $\rho_Y \geq 3$ and prove the second part of the statement. Since Y is Del Pezzo, there is a morphism $Y \rightarrow \mathbb{P}^2$ that is a blow-up of $\rho_Y - 1$ distinct points p_i . Thus we can apply Remark 5.1 to each blow-up and get a smooth Fano variety X_0 with a quasi-elementary contraction $f_0: X_0 \rightarrow \mathbb{P}^2$, such that X is obtained from X_0 by blowing up the fibers of f_0 over p_i .

Therefore it is enough to prove the statement in the case $\rho_Y = 3$, where Y is the blow-up of \mathbb{P}^2 in two points p_1 and p_2 . Call C_1 and C_2 the two corresponding exceptional curves, and C_3 the proper transform in Y of the line through p_1 and p_2 in \mathbb{P}^2 (see diagram below).



These are all the (-1) -curves in Y . Call α_i the extremal ray of $\text{NE}(Y)$ containing the numerical class of C_i ; then $\text{NE}(Y) = \alpha_1 + \alpha_2 + \alpha_3$ is simplicial.

Let us make a preliminary remark on the cone $\text{NE}(X)$. The surface Y has three elementary contractions, which are blow-ups with exceptional curves C_1, C_2 , and C_3 . Recall from Remark 5.1 that the lifting of each of these blow-ups is unique. This means that for every $i = 1, 2, 3$ there is a unique extremal ray $\tilde{\alpha}_i$ of $\text{NE}(X)$ such that $\tilde{\alpha}_i + \text{NE}(f)$ is a face of $\text{NE}(X)$ and $f_*(\tilde{\alpha}_i) = \alpha_i$.

Let us show that X has no small elementary contraction $\xi: X \rightarrow V$ such that $\text{NE}(\xi) \not\subseteq \text{NE}(f)$. By contradiction, let $\xi: X \rightarrow V$ be such a contraction. Since f is finite on fibers of ξ , these fibers have dimension at most 2. If F_0 is an irreducible component of a fiber of ξ with $\dim F_0 = 2$, then $f(F_0) = Y$, so

$$(f_*)|_{\mathcal{N}_1(F_0, X)}: \mathcal{N}_1(F_0, X) \rightarrow \mathcal{N}_1(Y)$$

is surjective. This is impossible because $\dim \mathcal{N}_1(F_0, X) = 1$ while $\rho_Y = 3$.

On the other hand, a small contraction cannot have only one-dimensional fibers; see [Wis91a, Corollary on p. 145]. Thus we have a contradiction.

Consider now an extremal ray β of $\text{NE}(X)$ that is not contained in $\text{NE}(f)$. Since the contraction of β is not small, Theorem 4.4(ii) implies that there is a face $\text{NE}(h)$ (h as in (4.2)) of $\text{NE}(X)$, of dimension $1 + \dim \text{NE}(f)$, containing both β and $\text{NE}(f)$. Then $f_*(\beta)$ is an extremal ray of $\text{NE}(Y)$, say α_1 . This implies that $\beta = \tilde{\alpha}_1$.

Summing up, we have shown that $\tilde{\alpha}_1, \tilde{\alpha}_2,$ and $\tilde{\alpha}_3$ are the only extremal rays of $\text{NE}(X)$ outside $\text{NE}(f)$.

Let us show that X is a product. Applying Remark 5.1 we get the sequence

$$\begin{array}{ccccc}
 & & \zeta & & \\
 & & \curvearrowright & & \\
 X & \xrightarrow{\varphi_1} & W_1 & \longrightarrow & X_0 \\
 \downarrow f & & \downarrow & & \downarrow f_0 \\
 Y & \xrightarrow{\psi_1} & \mathbb{F}_1 & \longrightarrow & \mathbb{P}^2
 \end{array}$$

where $\psi_1: Y \rightarrow \mathbb{F}_1$ is the contraction of C_1 and $\mathbb{F}_1 \rightarrow \mathbb{P}^2$ is the contraction of (the image of) C_2 ; hence $\zeta: X \rightarrow X_0$ is the contraction of $\tilde{\alpha}_1 + \tilde{\alpha}_2$.

Since $\dim X_0 \geq 3$, $\text{NE}(f_0)$ is a proper face of $\text{NE}(X_0)$, and there is at least one extremal ray β of $\text{NE}(X_0)$ outside $\text{NE}(f_0)$. As explained in §2.5, β must be the image via ζ_* of some extremal ray $\tilde{\beta}$ of $\text{NE}(X)$ not lying in $\text{NE}(\zeta) = \tilde{\alpha}_1 + \tilde{\alpha}_2$. Since $\zeta_*(\text{NE}(f)) = \text{NE}(f_0)$ and $\beta \not\subseteq \text{NE}(f_0)$, it must be $\beta \not\subseteq \text{NE}(f)$. Thus the only possibility is $\tilde{\beta} = \tilde{\alpha}_3$ and $\beta = \zeta_*(\tilde{\alpha}_3)$.

Let

$$h: X_0 \longrightarrow W_0$$

be the contraction of β . Observe that $\dim W_0 \geq n - 2$, because h is finite on fibers of f_0 . Our goal is to show that $X_0 \cong \mathbb{P}^2 \times W_0$ and f_0 is the projection, which implies the statement. The first step is to show that $\dim W_0 = n - 2$.

For $i = 1, 2, 3$ let $\varphi_i: X \rightarrow W_i$ be the smooth blow-up obtained by contracting $\tilde{\alpha}_i$. Let $E_i \subset X$ be the exceptional divisor of φ_i , F the fiber of f over $C_1 \cap C_3$, and F' the one over $C_2 \cap C_3$. By Remark 5.1 we have $E_3 \cong \mathbb{P}^1 \times F \cong \mathbb{P}^1 \times F'$, so $F \cong F'$; let us call it F . Thus $E_i \cong \mathbb{P}^1 \times F$, $E_1 \cap E_2 = \emptyset$, while $E_1 \cap E_3$ and $E_2 \cap E_3$ are fibers of f .

Let $\tilde{C}_i \subset E_i$ be a curve corresponding to $\mathbb{P}^1 \times \{*\}$. Then $[\tilde{C}_i] \in \tilde{\alpha}_i$, $\tilde{C}_i \cdot E_i = -1$, $f_*(\tilde{C}_i) = C_i$, and $f^*(C_i) = E_i$. This gives

$$\tilde{C}_1 \cdot E_3 = \tilde{C}_3 \cdot E_1 = \tilde{C}_2 \cdot E_3 = \tilde{C}_3 \cdot E_2 = 1 \quad \text{while} \quad \tilde{C}_1 \cdot E_2 = \tilde{C}_2 \cdot E_1 = 0.$$

Finally set $E'_3 := \zeta(E_3)$ and $C'_3 := \zeta_*(\tilde{C}_3)$, so that $[C'_3] \in \beta$.

Observe that E'_3 is a divisor in X_0 and there are $m_1, m_2 \in \mathbb{Z}_{\geq 0}$ such that

$$\zeta^*(E'_3) = E_3 + m_1 E_1 + m_2 E_2.$$

Then

$$0 = E'_3 \cdot \zeta_*(\tilde{C}_1) = E_3 \cdot \tilde{C}_1 + m_1 E_1 \cdot \tilde{C}_1 + m_2 E_2 \cdot \tilde{C}_1 = 1 - m_1$$

so $m_1 = 1$, and similarly $m_2 = 1$. Hence

$$E'_3 \cdot C'_3 = (E_3 + E_1 + E_2) \cdot \tilde{C}_3 = 1.$$

This says that C'_3 is a minimal element in the extremal ray β , namely that, for every curve $C \subset X_0$ such that $[C] \in \beta$, we have $C \equiv mC'_3$ with $m \in \mathbb{Z}_{>0}$.

Moreover

$$K_X = \varphi_1^*(K_{W_1}) + E_1 = \zeta^*(K_{X_0}) + E_1 + E_2,$$

which gives

$$-K_{X_0} \cdot C'_3 = \zeta^*(-K_{X_0}) \cdot \tilde{C}_3 = (-K_X + E_1 + E_2) \cdot \tilde{C}_3 = 3.$$

This says that the length $l(\beta)$ of the extremal ray β , that is the minimal anticanonical degree of rational curves whose class is in β , is 3. Now [Wi91a, Theorem 1.1] yields that for every non-trivial

fiber F of h we have

$$\dim F + \dim \text{Locus}(\beta) \geq n + 2.$$

On the other hand f_0 is finite on fibers of h , so $\dim F \leq 2$. Therefore $\dim \text{Locus}(\beta) = n$, every fiber of h has dimension 2, and $\dim W_0 = n - 2$.

Now f_0 and h induce a finite morphism $X_0 \rightarrow \mathbb{P}^2 \times W_0$; let d be its degree.

Let $w_0 \in W_0$ be a general point and $S_0 = h^{-1}(w_0)$. Then S_0 is a smooth surface and $g := (f_0)|_{S_0} : S_0 \rightarrow \mathbb{P}^2$ is finite of degree d .

Since in X the divisor E_3 intersects transversally E_1 and E_2 , $\zeta|_{E_3}$ is an isomorphism, so that $E'_3 \cong \mathbb{P}^1 \times F$ and $(f_0)|_{E'_3}$ is the first projection.

Observe that $f_0(E'_3)$ is a line l (the line through p_1 and p_2), and $h|_{E'_3} : E'_3 \rightarrow W_0$ is surjective, so it factors as

$$E'_3 \xrightarrow{\pi_2} F \xrightarrow{\xi} W_0$$

where π_2 is the second projection and ξ is a finite morphism. We have

$$g^{-1}(l) = (f_0)^{-1}(l) \cap S_0 = E'_3 \cap S_0 = E'_3 \cap h^{-1}(w_0) = (h|_{E'_3})^{-1}(w_0) \cong \mathbb{P}^1 \times \xi^{-1}(w_0).$$

On the other hand $g^{-1}(l)$ is the support of an ample divisor in S_0 and hence it is connected. This implies that ξ is an isomorphism and that $\tilde{l} := g^{-1}(l) \simeq \mathbb{P}^1 \times w_0$.

Then $g^*(l) = d\tilde{l}$; on the other hand $g|_{\tilde{l}} = (f_0)|_{\tilde{l}}$ is an isomorphism, so $g_*(\tilde{l}) = l$. Now we have

$$1 = l^2 = l \cdot g_*(\tilde{l}) = g^*(l) \cdot \tilde{l} = d(\tilde{l})^2,$$

so $d = 1$ and we are done; Theorem 1.1(i) is proved. □

6. Fano manifolds with a quasi-elementary contraction onto a 3-fold

In this section we prove Theorem 1.1(ii). Let us consider a smooth Fano variety X of dimension $n \geq 4$, and a quasi-elementary contraction $f : X \rightarrow Y$ with $\dim Y = 3$.

Recall that Y is factorial and has at most isolated canonical singularities by Lemma 3.10(i) and (iv). Let $\psi : Y \rightarrow Z$ be an elementary contraction of Y . We consider all possibilities for ψ .

If ψ is of type (3,0), then Z is a point and Y is Fano with $\rho_Y = 1$.

If ψ is of type (3,1), then $Z \simeq \mathbb{P}^1$ and $\rho_Y = 2$. In this case the other elementary contraction of Y cannot be again of type (3,1), because non-trivial fibers of distinct elementary contractions can intersect at most in finitely many points, and Y is factorial.

CLAIM 6.1. If $\psi : Y \rightarrow Z$ is of type (3,2), then ψ is equidimensional, and we have the following possibilities:

- (i) $\rho_Y = 2$ and $Z \cong \mathbb{P}^2$;
- (ii) $\rho_Y = 3$ and Z is either $\mathbb{P}^1 \times \mathbb{P}^1$ or \mathbb{F}_1 ; or
- (iii) Z is a smooth Del Pezzo surface with $\rho_Z \geq 3$, $Y \cong Z \times \mathbb{P}^1$, ψ is the projection, and $X \cong Z \times W$, where W is a smooth Fano variety of dimension $n - 2$ with a quasi-elementary contraction onto \mathbb{P}^1 .

Proof. We have $\dim Z = 2$ and $\psi \circ f : X \rightarrow Z$ is quasi-elementary (see Example 3.2). By Theorem 1.1(i), $\psi \circ f$ is equidimensional and Z is a smooth Del Pezzo surface. Then ψ must be equidimensional too.

If $\rho_Z = 1, 2$, we get the first two cases. If $\rho_Z \geq 3$, again by Theorem 1.1(i) we see that $X \cong Z \times W$, where W is a smooth Fano variety of dimension $n - 2$, and $\psi \circ f$ is the projection onto Z . Then

any intermediate contraction $Z \times W \rightarrow V \rightarrow Z$ must be onto a product $V \cong Z \times F$, hence we get $Y \cong Z \times \mathbb{P}^1$. Moreover, f induces a contraction $g: W \rightarrow \mathbb{P}^1$. Since f is quasi-elementary, it is easy to see that g is quasi-elementary too. □

CLAIM 6.2. If ψ is of type (2,0), then we have the following possibilities:

- (i) $\rho_Y = 2$ and Y has an elementary contraction of type (3,2); or
- (ii) $\rho_Y = 3$ and Y has a birational elementary contraction with fibers of dimension at most 1.

Proof. The divisor $\text{Exc}(\psi)$ satisfies $\dim \mathcal{N}_1(\text{Exc}(\psi), Y) = 1$, hence Proposition 3.16 yields $\rho_Y \leq 3$. However, Y has a non-trivial elementary contraction, hence $\rho_Y > 1$. Then the statement follows from Remark 3.17. □

CLAIM 6.3. Let $\psi: Y \rightarrow Z$ be an elementary contraction that is birational with fibers of dimension at most 1. Then ψ is the blow-up of a smooth curve $C \subset Z_{\text{reg}}$, and we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & W \\ f \downarrow & & \downarrow g \\ Y & \xrightarrow{\psi} & Z \end{array}$$

where W is smooth, φ is the blow-up of a smooth subvariety of codimension 2, $\text{Exc}(\varphi) = f^*(\text{Exc}(\psi))$, and g is a quasi-elementary Mori contraction.

Proof. Recall that ψ has a lifting as explained in § 2.5, so we have a diagram as above. Since X is Fano, φ is a Mori contraction. Then Theorem 3.14 yields the statement. □

In particular, Claim 6.3 implies that Y has no small elementary contractions.

CLAIM 6.4. In the setting of Claim 6.3, suppose that Y is Fano. Set $E := \text{Exc}(\varphi)$ and $S := \text{Exc}(\psi)$. Then the following statements are equivalent:

- (i) W is not Fano;
- (ii) Z is not Fano; and
- (iii) $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ with normal bundle $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1)$, and there is an extremal ray β of $\text{NE}(Y)$, different from $\text{NE}(\psi)$, such that $S \cdot \beta < 0$.

Proof. (i) \Rightarrow (ii) If W is not Fano, by [Wis91a, Proposition 3.4] there exists an extremal ray α of $\text{NE}(X)$ such that $\alpha \neq \text{NE}(\varphi)$ and $\alpha \cdot E < 0$. Observe that α is not contracted by f , because $E \cdot C = 0$ for every curve $C \subset X$ contracted by f .

Let $\varphi_2: X \rightarrow W_2$ be the contraction of α and let F_0 be an irreducible component of a non-trivial fiber of φ_2 . Then $F_0 \subset E$ and f is finite on F_0 , hence $f(F_0) \subseteq S$ and $\dim F_0 \leq 2$. Moreover, if $\dim F_0 = 2$, then $f(F_0) = S$. This would imply that $\dim \mathcal{N}_1(S, Y) = 1$, while $\dim \mathcal{N}_1(S, Y) = 2$. Thus $\dim F_0 = 1$.

Now [Wis91a, Theorem 1.2] yields that φ_2 is a smooth blow-up with exceptional divisor E . Observe that S is a smooth \mathbb{P}^1 -bundle over $\psi(S)$. It is not difficult to see that the \mathbb{P}^1 -bundle $(\varphi_2)|_E$ induces a second rational fibration on S , so that $S \cong \mathbb{P}^1 \times \mathbb{P}^1$. Moreover, if $C \subset E$ is a non-trivial fiber of φ_2 , then

$$-1 = C \cdot E = C \cdot f^*(S) = f_*(C) \cdot S,$$

and this gives $\mathcal{N}_{S/Y} \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1)$. In particular, $\psi(S)$ is a curve of anticanonical degree 0 in Z , so Z is not Fano.

(ii) \Rightarrow (iii) Suppose that Z is not Fano. Observe that Y and Z may be singular; however they are factorial and $K_Y = \psi^*(K_Z) + S$. Then reasoning as in [Wis91a, Proposition 3.4] one gets statement (iii).

(iii) \Rightarrow (i) The contraction of β is birational with fibers of dimension at most 1; let $\varphi_2: X \rightarrow W_2$ be the smooth blow-up given by Claim 6.3. Then $E = \text{Exc}(\varphi_2)$, and $\varphi_2 \neq \varphi$. If $C \subset E$ is a non-trivial fiber of φ_2 , then $K_X \cdot C = E \cdot C = -1$, which yields $-K_W \cdot \varphi_*(C) = 0$. Hence W is not Fano. \square

CLAIM 6.5. Suppose that Y is Fano, and let β_1 and β_2 be two extremal rays of $\text{NE}(Y)$ with distinct divisorial loci $S_i = \text{Locus}(\beta_i)$. If $S_1 \cap S_2 \neq \emptyset$, then the contraction $\psi: Y \rightarrow Z$ of one of the β_i is the blow-up of a smooth curve $C \subset Z_{\text{reg}}$, and Z is Fano.

Proof. Let $\psi_i: Y \rightarrow Z_i$ be the contraction of β_i . Since $S_1 \cap S_2 \neq \emptyset$ and Y is factorial, we have $\dim S_1 \cap S_2 = 1$. Thus ψ_1 and ψ_2 cannot both be of type (2,0).

So let us assume that ψ_1 is of type (2,1). If Z_1 is Fano, we are done. If Z_1 is not Fano, Claim 6.4 yields the existence of a second extremal ray $\tilde{\beta}_1$ of $\text{NE}(Y)$, distinct from β_1 , with $\tilde{\beta}_1 \cdot S_1 < 0$. Then $\mathcal{N}_1(S_1, Y)$ is generated by β_1 and $\tilde{\beta}_1$, and no other extremal ray of $\text{NE}(Y)$ can be contained in $\mathcal{N}_1(S_1, Y)$.

Thus ψ_2 cannot be of type (2,0). Let us show that Z_2 must be Fano. If not, by Claim 6.4 there is another extremal ray $\tilde{\beta}_2$ of $\text{NE}(Y)$, distinct from β_2 , with $\tilde{\beta}_2 \cdot S_2 < 0$. Since β_2 and $\tilde{\beta}_2$ are not contained in $\mathcal{N}_1(S_1, Y)$, we have $\beta_2 \cdot S_1 \geq 0$ and $\tilde{\beta}_2 \cdot S_1 \geq 0$.

Now if $C \subset S_1 \cap S_2$ is an irreducible curve, we get $C \cdot S_1 < 0$ because $\beta_1 \cdot S_1 < 0$ and $\tilde{\beta}_1 \cdot S_1 < 0$. On the other hand, $C \cdot S_1 \geq 0$ because $\beta_2 \cdot S_1 \geq 0$ and $\tilde{\beta}_2 \cdot S_1 \geq 0$. Thus we have a contradiction. \square

CLAIM 6.6. In the setting of Claim 6.3, suppose that Y is Fano. Then we are in one of the following situations:

- (i) W and Z are Fano;
- (ii) $\rho_Y \geq 3$ and Y has another elementary contraction $\tilde{\psi}$ as in Claim 6.3, such that the corresponding \tilde{W}, \tilde{Z} are Fano; or
- (iii) $\rho_Y = 3$ and Y has an elementary contraction of type (3,2).

Proof. Let us assume that we are not in situation (i), so W and Z are not Fano. By Claim 6.4 we have $S \cong \mathbb{P}^1 \times \mathbb{P}^1$, and there is a second extremal ray β of $\text{NE}(Y)$ with $\beta \cdot S < 0$.

There exists an extremal ray $\tilde{\beta}$ of $\text{NE}(Y)$ such that $S \cdot \tilde{\beta} > 0$; let $\tilde{\psi}: Y \rightarrow \tilde{Z}$ be its contraction.

Clearly $\tilde{\beta}$ is distinct from $\text{NE}(\psi)$ and β , and the elements of three distinct extremal rays must be linearly independent in $\mathcal{N}_1(Y)$. Hence $\rho_Y \geq 3$ and $\tilde{\beta} \cap \mathcal{N}_1(S, Y) = \{0\}$, which implies that $\tilde{\psi}$ is finite on S . On the other hand, S must intersect every non-trivial fiber of $\tilde{\psi}$, hence $\tilde{\psi}$ has fibers of dimension at most 1.

If $\tilde{\psi}$ is of fiber type, then it is of type (3,2). Let us consider

$$(\tilde{\psi}_*)|_{\mathcal{N}_1(S, Y)}: \mathcal{N}_1(S, Y) \rightarrow \mathcal{N}_1(\tilde{Z}).$$

Since $\ker \tilde{\psi}_*$ is the line spanned by $\tilde{\beta}$ in $\mathcal{N}_1(Y)$, we have $\ker \tilde{\psi}_* \cap \mathcal{N}_1(S, Y) = \{0\}$ and $(\tilde{\psi}_*)|_{\mathcal{N}_1(S, Y)}$ is injective. On the other hand, $\tilde{\psi}(S) = \tilde{Z}$, hence $(\tilde{\psi}_*)|_{\mathcal{N}_1(S, Y)}$ is surjective too. This gives $\rho_{\tilde{Z}} = 2$ and $\rho_Y = 3$, and thus situation (iii).

Suppose that $\tilde{\psi}$ is birational. Then $\text{Exc}(\tilde{\psi}) \cap \text{Exc}(\psi) \neq \emptyset$, and Claim 6.5 implies that \tilde{Z} is Fano. Finally also \tilde{W} is Fano by Claim 6.4, and we get situation (ii). \square

CLAIM 6.7. Let $\pi: Y \rightarrow T$ be a contraction onto a surface. Let $\alpha_1, \dots, \alpha_m$ be the extremal rays of $\text{NE}(\pi)$ and $\psi_i: Y \rightarrow Z_i$ be the contraction of α_i . Then $\text{Exc}(\psi_i) \cap \text{Exc}(\psi_j) \neq \emptyset$ for some $i \neq j$.

Proof. Set $S_i := \text{Exc}(\psi_i)$, and assume by contradiction that $S_i \cap S_j = \emptyset$ for every $i \neq j$. Then each ψ_i is birational and $S_i \cdot \alpha_i < 0$ for every i by Remark 2.4; moreover $S_i \cdot \alpha_j = 0$ for every $i \neq j$.

Let $C \subset Y$ be an irreducible curve that is contracted by π but not contained in $S_1 \cup \dots \cup S_m$. Then

$$C \equiv \sum_{i=1}^m \lambda_i l_i \quad \text{where } [l_i] \in \alpha_i \text{ and } \lambda_i \in \mathbb{Q}_{\geq 0} \text{ for every } i = 1, \dots, m.$$

So $0 \leq C \cdot S_i = \lambda_i(l_i \cdot S_i)$ implies that $\lambda_i = 0$ for every $i = 1, \dots, m$, a contradiction. □

Proof of Theorem 1.1(ii). Let us suppose that $\rho_Y \geq 4$. By Claims 6.1, 6.2, and 6.3, any elementary contraction of Y is either a smooth blow-up or a \mathbb{P}^1 -bundle. This implies that Y is Fano (but possibly singular) by Remark 3.11.

Suppose now that Y is smooth and that $\rho_Y \geq 6$. Then $Y \cong S \times \mathbb{P}^1$ with S a Del Pezzo surface by [MM81, Theorem 2]. Thus $X \rightarrow S$ is a quasi-elementary contraction (see Example 3.2), and applying Theorem 1.1(i) as in the proof of Claim 6.1 we easily get the statement.

Thus we are left to prove that if $\rho_Y \geq 4$, then Y must be smooth.

By contradiction, assume that Y is singular and $\rho_Y \geq 4$. By Claims 6.1, 6.2, and 6.6, we can construct a sequence

$$\begin{array}{ccccccc} X = X_{\rho_Y} & \longrightarrow & X_{\rho_Y-1} & \longrightarrow & \cdots & \longrightarrow & X_4 \\ f \downarrow & & \downarrow & & & & \downarrow f_4 \\ Y = Y_{\rho_Y} & \xrightarrow{\psi_{\rho_Y}} & Y_{\rho_Y-1} & \longrightarrow & \cdots & \xrightarrow{\psi_5} & Y_4 \end{array}$$

where, for every $i = 4, \dots, \rho_Y$, X_i is smooth and Fano, f_i is a quasi-elementary contraction, Y_i is Fano with $\rho_{Y_i} = i$, and ψ_i is the blow-up of a smooth curve contained in the smooth locus of Y_{i-1} . In particular, each Y_i is singular, because Y is.

Since Y_4 is singular and $\rho_{Y_4} = 4$, Claim 6.1 implies that Y_4 has no elementary contraction of fiber type.

CLAIM 6.8. The 3-fold Y_4 has no contraction onto a surface.

Proof. By contradiction let $\pi: Y_4 \rightarrow T$ be such a contraction. Then π cannot be elementary, so $\rho_T \leq 2$.

The cone $NE(\pi)$ contains $m \geq 2$ extremal rays, whose contractions are birational. Call S_1, \dots, S_m their exceptional loci. By Claim 6.7 they cannot all be disjoint, so we can assume that $S_1 \cap S_2 \neq \emptyset$.

Using Claim 6.5, we can assume that the elementary contraction $\psi_4: Y_4 \rightarrow Y_3$ with exceptional locus S_1 is the blow-up of a smooth curve $C_3 \subset (Y_3)_{reg}$, and that Y_3 is Fano. We observe that Y_3 is singular because Y_4 is. As before we get

$$\begin{array}{ccc} X_4 & \longrightarrow & X_3 \\ f_4 \downarrow & & \downarrow f_3 \\ Y_4 & \xrightarrow{\psi_4} & Y_3 \end{array}$$

where X_3 is smooth and Fano and f_3 is a quasi-elementary contraction. Moreover, π induces a contraction $\pi': Y_3 \rightarrow T$.

If π' is elementary, then $\rho_T = 2$, π' is equidimensional, and T is either $\mathbb{P}^1 \times \mathbb{P}^1$ or \mathbb{F}_1 by Claim 6.1. Hence π' is a conic bundle (see [Sar82, § 1] for the definition and properties of conic bundles). Let $\Delta_{\pi'} \subset T$ be the discriminant locus of π' , which is non-empty because Y_3 is singular.

We recall that $C_3 \subset Y_3$ cannot intersect any curve of anticanonical degree 1 by Remark 2.3. Thus C_3 cannot be a component of a reducible fiber of π' .

If C_3 is an irreducible fiber, let $p = \pi'(C_3) \in T$ and call T' the blow-up of T in p : then Y_4 has an elementary contraction onto T' , which is impossible.

Hence $\pi'(C_3)$ is a curve in T , and it is disjoint from $\Delta_{\pi'}$ because C_3 cannot intersect singular fibers of π' .

If $T \cong \mathbb{P}^1 \times \mathbb{P}^1$, the only possibility is that $\Delta_{\pi'}$ is a union of fibers of a projection $\mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$, hence a disjoint union of smooth rational curves. But this is impossible, see for instance [Pro05, Lemma 5.3].

Consider now the case where $T \cong \mathbb{F}_1$. Then we have a contraction $Y_3 \rightarrow \mathbb{F}_1 \rightarrow \mathbb{P}^2$, which has a second factorization $Y_3 \rightarrow Y_2 \rightarrow \mathbb{P}^2$. It is not difficult to see that $\psi_3: Y_3 \rightarrow Y_2$ is birational with fibers of dimension at most 1, so by Claim 6.3 it is again a blow-up of a smooth curve $C_2 \subset (Y_2)_{\text{reg}}$, and we have the diagram

$$\begin{array}{ccc} X_3 & \longrightarrow & X_2 \\ f_3 \downarrow & & \downarrow f_2 \\ Y_3 & \xrightarrow{\psi_3} & Y_2 \xrightarrow{\xi} \mathbb{P}^2 \end{array}$$

where X_2 is smooth, Y_2 is singular with $\rho_{Y_2} = 2$, and f_2 is a quasi-elementary Mori contraction.

The only curve in Y_2 which could have non-positive anticanonical degree is C_2 . Thus $\xi: Y_2 \rightarrow \mathbb{P}^2$ is a conic bundle with non-empty discriminant locus $\Delta_\xi \subset \mathbb{P}^2$. If $\xi(C_2)$ is a curve in \mathbb{P}^2 , then C_2 must intersect some singular fiber, which is again impossible by Remark 2.3. Thus C_2 is a smooth fiber of ξ , Y_2 is Fano, and X_2 is Fano too by Claim 6.4.

Let us consider the other elementary contraction $\eta: Y_2 \rightarrow Z$ of Y_2 .

If η is again of type (3,2), reasoning as above we get that η must contract C_2 , a contradiction because $\text{NE}(\eta) \cap \text{NE}(\xi) = \{0\}$.

If η is of type (3,1), it is a fibration in Del Pezzo surfaces over \mathbb{P}^1 . Then Y_2 is a finite cover of $\mathbb{P}^1 \times \mathbb{P}^2$, and C_2 is the inverse image of $\mathbb{P}^1 \times \{pt\}$. This implies that Y_3 is a finite cover of $\mathbb{P}^1 \times \mathbb{F}_1$, which gives a surjective morphism $Y_3 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$. Taking the Stein factorization we get an elementary contraction $Y_3 \rightarrow T'$ of type (3,2), where T' is a finite cover of $\mathbb{P}^1 \times \mathbb{P}^1$. By Claim 6.1, T' can be $\mathbb{P}^1 \times \mathbb{P}^1$ or \mathbb{F}_1 , but T' has two distinct fibrations, thus $T' \cong \mathbb{P}^1 \times \mathbb{P}^1$. We have already excluded this possibility.

Therefore, η is birational; let E be its exceptional divisor. Since $E \cdot \text{NE}(\eta) < 0$ by Remark 2.4, it must be $E \cdot \text{NE}(\xi) > 0$, hence $\xi(E) = \mathbb{P}^2$ and E intersects C_2 . In particular, E cannot be covered by curves of anticanonical degree 1, thus η is of type (2,0). Moreover, $E \cdot C_2 \geq 2$, because ξ has singular fibers.

The composite contraction $Y_3 \rightarrow Z$ has a second factorization

$$Y_3 \xrightarrow{\sigma} \tilde{Y}_2 \xrightarrow{\chi} Z.$$

It is not difficult to see that σ has exceptional locus $\psi_3^*(E)$ and is the blow-up of a smooth curve $\tilde{C}_2 \subset (\tilde{Y}_2)_{\text{reg}}$. Since the image of $\psi_3^*(E)$ in Z is $z_0 := \eta(E)$, \tilde{C}_2 is contracted by χ . Then we see that χ is again a Mori contraction of type (2,1) with exceptional divisor $\sigma(\text{Exc}(\psi_3))$, and \tilde{Y}_2 is Fano. In particular, χ is again a blow-up of a smooth curve contained in Z_{reg} , so that z_0 is a smooth point. Moreover, $\psi_3^*(E)$ is contained in the smooth locus of Y_3 , so that $E \subset (Y_2)_{\text{reg}}$.

Therefore η is just the blow-up of z_0 and $E \cong \mathbb{P}^2$. If E intersects C_2 in at least two distinct points, take l to be a line in E through these two points. Let \tilde{l} be the proper transform of l in Y_3 . Then $-K_{Y_2} \cdot l = 2$ and $\tilde{l} \cdot \text{Exc}(\psi_3) \geq 2$, so by Remark 2.3 we get $-K_{Y_3} \cdot \tilde{l} \leq 0$, a contradiction.

If E intersects C_2 in a single non-reduced point y_0 , similarly as before take l to be a line in $E \cong \mathbb{P}^2$ through y_0 . Then the schematic intersection $\psi_3^*(E) \cap \text{Exc}(\psi_3)$ is non-reduced along the fiber of ψ_3 over y_0 , thus again $\tilde{l} \cdot \text{Exc}(\psi_3) \geq 2$ gives a contradiction. This concludes the case where $\rho_T = 2$.

We still have to exclude the case where $\rho_T = 1$ and Y_3 has no elementary contractions of type (3,2). Reasoning as for Y_4 , we see that one of the two extremal rays of $\text{NE}(\pi')$ yields a blow-up

$\psi_3: Y_3 \rightarrow Z_2$ of a smooth curve $C' \subset (Z_2)_{\text{reg}}$, and Z_2 is Fano with $\rho_{Z_2} = 2$. Now π' yields an elementary contraction $\pi'': Z_2 \rightarrow T$. Claim 6.1 gives $T \cong \mathbb{P}^2$, and $\Delta_{\pi''}$ is non-empty. As before we easily get a contradiction. This completes the proof of Claim 6.8. \square

Using Claims 6.8, 6.2, and 6.6, we get a sequence

$$Y_4 \xrightarrow{\psi_4} Y_3 \xrightarrow{\psi_3} Y_2 \xrightarrow{\psi_2} Y_1$$

where each Y_i is Fano and each ψ_i is the blow-up of a smooth curve $C_{i-1} \subset (Y_{i-1})_{\text{reg}}$. In particular Y_1 is factorial with isolated canonical singularities, singular, and $\rho_{Y_1} = 1$.

We first show that Y_1 must have terminal singularities, using the following lemma.

LEMMA 6.9. *Let Z be a three-dimensional \mathbb{Q} -factorial projective variety, with isolated canonical singularities and K_Z Cartier. Suppose that Z is Fano with $\rho_Z = 1$, and that the singularities of Z are not terminal. Then one of the following occurs:*

- (i) Z contains a one-dimensional family of curves of anticanonical degree 1 passing through a singular point; or
- (ii) Z is covered by a family of curves of anticanonical degree at most 2 passing through a singular point.

We postpone the proof of Lemma 6.9 and carry on with the proof of Theorem 1.1(ii).

Suppose that Y_1 has at least one non-terminal singular point: then Lemma 6.9 applies to Y_1 . If part (i) holds, let $S \subset Y_1$ be a surface covered by curves of anticanonical degree 1. Since $\rho_{Y_1} = 1$, S is ample, and $C_1 \cap S \neq \emptyset$. Observe that, even if $C_1 \subset S$, C_1 does not contain any singular point, hence it cannot be a member of the family given by part (i). Thus C_1 intersects some curve of anticanonical degree 1, which is impossible by Remark 2.3.

Suppose now that part (ii) holds for Y_1 . If C_1 is a component of some reducible curve l_1 of the family, it must be $l_1 = C_1 \cup C'_1$ with $-K_{Y_1} \cdot C_1 = -K_{Y_1} \cdot C'_1 = 1$, which again gives a contradiction. Again C_1 cannot be a member of the family, because it does not contain singular points. Hence C_1 is not contained in any member of the family; let T be an irreducible surface containing C_1 such that through every point of C_1 there is a curve of anticanonical degree at most 2 contained in T . Let \tilde{T} be the proper transform of T in Y_2 . Then through every point of $\tilde{T} \cap \text{Exc}(\psi_2)$ there is a curve of anticanonical degree 1 contained in \tilde{T} (see Remark 2.3).

Consider $C_2 \subset (Y_2)_{\text{reg}}$. If $\psi_2(C_2)$ is a point, then C_2 must intersect some curve of anticanonical degree 1 contained in \tilde{T} . On the other hand, if $\psi_2(C_2)$ is a curve, then it must intersect T , thus C_2 must intersect $\psi_2^{-1}(T) = \tilde{T} \cup \text{Exc}(\psi_2)$. In any case C_2 will intersect some curve of anticanonical degree 1, which gives a contradiction.

Hence Y_1 has terminal singularities, and the same holds for each Y_i . Consider in particular Y_4 . By [Nam97], Y_4 has a smoothing, that is an integral complex space \mathcal{Y} , with a projective flat morphism $\mathcal{Y} \rightarrow \Delta$ onto the complex unit disk, such that Y_4 is the fiber over 0 while the fiber Y_t over $t \neq 0$ is a smooth Fano 3-fold. It is proven in [JR06] that $\text{Pic } Y_4 \cong \text{Pic } Y_t$, in particular $\rho_{Y_t} = 4$. Then we know by Mori and Mukai’s classification that Y_t has a conic bundle structure (not necessarily elementary), see [IP99, Theorem on p. 141]. Our goal is to deduce from this that Y_4 must have a contraction onto a surface, contradicting Claim 6.8.

For this we need the following lemma, based on [JR06].

LEMMA 6.10. *Let Z be a three-dimensional factorial Fano variety with terminal singularities and $Z \rightarrow \Delta$ a smoothing. Consider the inclusions $i_t: Z_t \hookrightarrow Z$ and $i_0: Z_0 \hookrightarrow Z$.*

Then the push-forwards $(i_t)_*$ and $(i_0)_*$ induce bijections among the cones

$$\text{NE}(Z_t), \quad \text{NE}(\mathcal{Z}/\Delta), \quad \text{and} \quad \text{NE}(Z_0),$$

and every contraction of Z_t or of Z_0 is the restriction of the contraction of the corresponding face of $\text{NE}(\mathcal{Z}/\Delta)$.

By Lemma 6.10, the conic bundle on Y_t induces a contraction $\mathcal{Y} \rightarrow \mathcal{T} \rightarrow \Delta$. This restricts to a contraction $\pi: Y_4 \rightarrow T$ onto a surface, which contradicts Claim 6.8. Theorem 1.1(ii) is proved, and this concludes the proof of Theorem 1.1. □

Proof of Lemma 6.9. Let $\psi: Z' \rightarrow Z$ be a partial crepant resolution such that Z' has terminal and factorial singularities (see [KM98, § 6.3]). Hence $K_{Z'} = \psi^*(K_Z)$, $-K_{Z'}$ is nef and big, and $\rho_{Z'} > 1$ because ψ is not an isomorphism.

Since Z is \mathbb{Q} -factorial, for every non-terminal point $p \in Z$ the inverse image $\psi^{-1}(p)$ has pure dimension 2, therefore $\text{Exc}(\psi)$ is a divisor. Moreover, any irreducible curve contained in $\text{Exc}(\psi)$ has anticanonical degree 0.

There is at least one elementary Mori contraction $f: Z' \rightarrow W$. Observe that f must be finite on $\text{Exc}(\psi)$, because any curve contracted by f has positive anticanonical degree. Hence any fiber F of f such that $F \cap \text{Exc}(\psi) \neq \emptyset$ has dimension at most 1.

Suppose that f is of fiber type. Then it must be of type (3,2) and [Cut88, Theorem 7] says that W is a smooth surface and f is a conic bundle. Moreover, if $E \subseteq \text{Exc}(\psi)$ is an irreducible component, then $f(E) = W$, so that every fiber of f intersects E . Then the fibers of f give a covering family of curves of anticanonical degree at most 2 in Z , all passing through the singular point $\psi(E)$, and we get part (ii).

Suppose that f is birational. Since Z' is Gorenstein, f is divisorial; set $D := \text{Exc}(f)$. We claim that D cannot be disjoint from $\text{Exc}(\psi)$. In fact, if so, $\psi(D)$ would be a non-nef Cartier divisor in Z , which is impossible because $\rho_Z = 1$.

Hence f must be of type (2,1) and by [Cut88, Theorem 4] we have $-K_{Z'} \cdot l = 1$ for the general fiber l of f . Again, if E is an irreducible component of $\text{Exc}(\psi)$ intersecting D , every fiber l of f must intersect E . So we get a one-dimensional family of curves of anticanonical degree 1 in Z , passing through the singular point $\psi(E)$. □

Proof of Lemma 6.10. By [JR06, Proposition 1.1], \mathcal{Z} has at most isolated terminal factorial singularities at the singular points of Z_0 .

We refer to [KMM87, § 0-1] for the notation in the relative situation. Observe that for a projective morphism in the analytic category the standard results of the Minimal Model Program hold, see [KM98, Example 2.17] and references therein. In particular, since each fiber of $\mathcal{Z} \rightarrow \Delta$ is Fano, $\text{NE}(\mathcal{Z}/\Delta)$ is closed and polyhedral by the relative version of the cone theorem.

We first observe that the linear maps

$$(i_t)_*: \mathcal{N}_1(Z_t) \longrightarrow \mathcal{N}_1(\mathcal{Z}/\Delta) \quad \text{and} \quad (i_0)_*: \mathcal{N}_1(Z_0) \longrightarrow \mathcal{N}_1(\mathcal{Z}/\Delta)$$

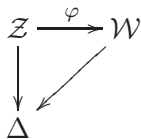
are isomorphisms. In fact they are dual to the restrictions

$$\text{Pic}(\mathcal{Z}) \otimes \mathbb{R} \longrightarrow \text{Pic}(Z_t) \otimes \mathbb{R} \quad \text{and} \quad \text{Pic}(\mathcal{Z}) \otimes \mathbb{R} \longrightarrow \text{Pic}(Z_0) \otimes \mathbb{R},$$

which are isomorphism by [JR06, Theorem 1.4]. Moreover, we have $(i_t)_* \text{NE}(Z_t) \subseteq \text{NE}(\mathcal{Z}/\Delta)$ and $(i_0)_* \text{NE}(Z_0) \subseteq \text{NE}(\mathcal{Z}/\Delta)$.

Up to shrinking Δ , we can assume that for every extremal ray α of $\text{NE}(\mathcal{Z}/\Delta)$, either $\text{Locus}(\alpha)$ dominates Δ , or $\text{Locus}(\alpha)$ is contained in Z_0 . Let us show that the second case cannot happen.

Assume by contradiction that $\text{Locus}(\alpha)$ is contained in Z_0 and consider the contraction φ of α , as below.



Then φ is an isomorphism outside the central fiber, and restricts to a contraction $\varphi_0: Z_0 \rightarrow W_0$ on the central fiber. The morphism $W \rightarrow \Delta$ is projective and the general fiber has dimension 3, hence $\dim W_0 = 3$ and φ_0 is birational.

This means that $E = \text{Exc}(\varphi) = \text{Exc}(\varphi_0)$ has dimension at most 2, namely φ is a small contraction of Z and it must be of type (2,0). Hence W is not factorial at $\varphi(E)$.

Since $(i_0)_*$ is injective, φ_0 is an elementary contraction of type (2,0) of Z_0 , and E is irreducible. Moreover, W_0 has terminal singularities and is Fano.

If W_0 were Gorenstein, W should be factorial again by [JR06, Proposition 1.1], which is not the case: so K_{W_0} is not Cartier. The possibilities for φ_0 are given in [Cut88, Theorem 5], and the only case where K_{W_0} is not Cartier is $E \cong \mathbb{P}^2$ with normal bundle $\mathcal{N}_{E/Z_0} \cong \mathcal{O}_{\mathbb{P}^2}(-2)$.

Since E is a smooth prime divisor in the factorial variety Z_0 , it is contained in $(Z_0)_{\text{reg}}$ and hence in Z_{reg} . Now [Kaw89, Theorem 2.1] yields $\mathcal{N}_{E/Z} \cong \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$, a contradiction.

Hence $\text{Locus}(\alpha)$ dominates Δ , which means that α is contained in both cones $(i_t)_* \text{NE}(Z_t)$ and $(i_0)_* \text{NE}(Z_0)$. Repeating this for every extremal ray of $\text{NE}(Z/\Delta)$, we get the statement. \square

7. Applications and examples

In this section we prove the corollaries stated in the introduction, and some other applications. We also give some related examples.

Suppose that X is a smooth Fano variety with a quasi-elementary contraction $f: X \rightarrow Y$. If X has other suitable contractions, one can use Theorems 4.1 and 4.4 to get a quasi-elementary contraction $h: X \rightarrow Z$ with $\dim Z \leq 3$, and then apply Theorem 1.1. Corollaries 1.4, 7.2, and 7.3 are obtained in this way.

Proof of Corollary 1.4. Recall that $\rho_X - \rho_{Y_i} \leq \rho_{F_i}$ because f_i is quasi-elementary. Suppose that $\dim F_1 + \dim F_2 \geq n - 2$. By Theorem 4.4(i) and Remark 4.5, there exists a contraction $h: X \rightarrow Z$ where $\dim Z \leq n - (\dim F_1 + \dim F_2)$ and

$$\rho_X - \rho_Z = (\rho_X - \rho_{Y_1}) + (\rho_X - \rho_{Y_2}) \leq \rho_{F_1} + \rho_{F_2}.$$

This immediately gives $\dim F_1 + \dim F_2 \leq n$, and, if equality holds, Z is a point so $\rho_Z = 0$. Moreover, $\dim F_1 + \dim F_2 = n - 1$ implies that $Z \cong \mathbb{P}^1$ and $\rho_Z = 1$.

Let us notice that h is quasi-elementary too. In fact let G be a general fiber of h , then G contains general fibers of f_1 and f_2 . Since both f_i are quasi-elementary, $\mathcal{N}_1(G, X)$ contains both $\ker(f_1)_*$ and $\ker(f_2)_*$. On the other hand we have

$$\mathcal{N}_1(G, X) \subseteq \ker h_* = \ker(f_1)_* + \ker(f_2)_*$$

as shown in the proof of Theorem 4.4, so $\mathcal{N}_1(G, X) = \ker h_*$ and h is quasi-elementary.

Hence if $\dim F_1 + \dim F_2 = n - 2$ then Z is a Del Pezzo surface by Theorem 1.1(i), so that $\rho_Z \leq 9$.

Finally, suppose that $\dim F_1 + \dim F_2 = n - 3$ and that f_2 is elementary. Similarly to the previous case, Theorem 4.1(i) gives a quasi-elementary contraction $h: X \rightarrow Z$ with $Z \leq 3$ and

$\rho_X \leq \rho_{F_1} + \rho_Z + 1$. If $\dim Z \leq 2$ we proceed as before. If $\dim Z = 3$ then $\rho_Z \leq 10$ by Theorem 1.1(ii), so we are done. \square

Remark 7.1. In the statement of Corollary 1.4 one can replace ρ_{F_i} by $\dim \ker(f_i)_*$, which gives a better bound for instance when f_i is elementary. Similarly in the following corollaries.

COROLLARY 7.2. *Let X be a smooth Fano variety and $f: X \rightarrow Y$ a quasi-elementary contraction of fiber type with $\dim Y \geq 3$ and general fiber F .*

Let $\varphi: X \rightarrow W$ be an elementary contraction such that $\text{NE}(\varphi) \cap \text{NE}(f) = \{0\}$. Then every fiber of φ has dimension at most $\dim Y$. Moreover:

- (i) *if φ has a fiber of dimension $\dim Y$, then $\rho_X \leq 1 + \rho_F$ and $\rho_Y = 1$; and*
- (ii) *if φ has a fiber of dimension $\dim Y - 1$, then $\rho_X \leq 3 + \rho_F$ and $\rho_Y \leq 3$.*

We state the corollary in this form for completeness; however, let us notice that only the last statement is really new.

Proof. Recall that $\rho_X \leq \rho_Y + \rho_F$ because f is quasi-elementary. Since f is finite on fibers of φ , they have dimension at most $\dim Y$. If there is a fiber with the same dimension as Y , let F_0 be an irreducible component with $\dim F_0 = \dim Y$. Then $f(F_0) = Y$, so

$$(f_*)|_{\mathcal{N}_1(F_0, X)}: \mathcal{N}_1(F_0, X) \longrightarrow \mathcal{N}_1(Y)$$

is surjective and $\rho_Y \leq \dim \mathcal{N}_1(F_0, X) = 1$.

If φ has a fiber of dimension $\dim Y - 1$, let F_0 be an irreducible component with $\dim F_0 = \dim Y - 1$. Then $f(F_0)$ is a prime divisor in Y with $\dim \mathcal{N}_1(f(F_0), Y) = 1$, so Proposition 3.16 implies that $\rho_Y \leq 3$. \square

COROLLARY 7.3. *Let X be a smooth Fano variety and $f: X \rightarrow Y$ a quasi-elementary contraction of fiber type with $\dim Y \geq 3$ and general fiber F .*

Let $\varphi: X \rightarrow W$ be a divisorial elementary contraction such that $\text{NE}(\varphi) \cap \text{NE}(f) = \{0\}$ and $f(\text{Exc}(\varphi)) = Y$.

- (i) *If the general fiber of $\varphi|_{\text{Exc}(\varphi)}$ has dimension $\dim Y - 2$, then $\rho_X \leq \rho_F + 10$.*
- (ii) *If f is elementary and the general fiber of $\varphi|_{\text{Exc}(\varphi)}$ has dimension $\dim Y - 3$, then $\rho_X \leq 12$.*

Proof. The proof is very similar to that of Corollary 1.4. We apply Theorem 4.4(iii) and Theorem 1.1(i) in the first case, Theorem 4.1(iii) and Theorem 1.1(ii) in the second case. \square

Proof of Corollary 1.2. The first two statements are a direct consequence of Theorem 1.1(i). For the last statement, suppose that $\rho_X \geq 7$ and that $f: X \rightarrow Y$ is an elementary contraction with $\dim Y = 3$. Then $\rho_Y \geq 6$, so Theorem 1.1(ii) says that $X \cong S \times S'$ where S, S' are Del Pezzo surfaces and S' has an elementary contraction onto \mathbb{P}^1 . Then $S' \cong \mathbb{P}^1 \times \mathbb{P}^1$ or $S' \cong \mathbb{F}_1$. \square

Proof of Corollary 1.3. The first two statements follow from Corollary 1.4 and Theorem 1.1. Let $f: X \rightarrow Y$ be an elementary contraction with $\dim Y = 4$, and $\varphi: X \rightarrow W$ another elementary contraction. If φ has a fiber of dimension at least 3, then $\rho_X \leq 4$ by Corollary 7.2. In particular, this holds if φ is of type (3,0), (4,0), or (4,1). Finally suppose that $f(\text{Exc}(\varphi)) = Y$. If φ is of fiber type, we have $\rho_X \leq 12$ by the previous part. If φ is birational, then it must be divisorial, so $\rho_X \leq 12$ by Corollary 7.3. \square

Finally we give an application in the spirit of § 3.15.

COROLLARY 7.4. *Let X be a smooth Fano variety and $f: X \rightarrow Y$ a non-trivial quasi-elementary contraction of fiber type with general fiber F .*

Let $D \subset X$ be a prime divisor such that $\dim \mathcal{N}_1(D, X) = 2$. Then one of the following occurs:

- (i) $\rho_X = 3$, $Y \simeq \mathbb{P}^1$, and D is a fiber of f ;
- (ii) $\rho_X \leq 4$, $\rho_Y \leq 3$, and f is elementary;
- (iii) $\rho_X \leq 10$, $X \cong F \times Y$, Y is a Del Pezzo surface, $\rho_F = 1$, and $\dim f(D) = 1$; or
- (iv) $\rho_X \leq 1 + \rho_F$, $\rho_Y = 1$, and $f(D) = Y$.

Proof. Suppose that $f(D) \subsetneq Y$. Then Lemma 3.9 says that $D = f^{-1}(f(D))$; in particular D contains a fiber F , so

$$\ker f_* = \mathcal{N}_1(F, X) \subseteq \mathcal{N}_1(D, X)$$

and $\rho_X - \rho_Y \leq 2$. If $\rho_X - \rho_Y = 2$, then $\mathcal{N}_1(D, X) = \ker f_*$, so D is a fiber and $Y \cong \mathbb{P}^1$. Thus we are in case (i).

If $\rho_X - \rho_Y = 1$, then f is elementary and D is not a fiber, hence $\dim Y \geq 2$. If $\rho_Y \leq 3$, we are in case (ii).

Suppose that $\rho_Y \geq 4$. We have $\dim \mathcal{N}_1(f(D), Y) = 1$, so Proposition 3.16 yields that $\dim Y = 2$. Thus Theorem 1.1(i) gives case (iii).

Assume now that $f(D) = Y$. Then the restriction

$$(f_*)|_{\mathcal{N}_1(D, X)}: \mathcal{N}_1(D, X) \longrightarrow \mathcal{N}_1(Y)$$

is surjective, so $\rho_Y \leq 2$. If equality holds, then $\mathcal{N}_1(D, X) \cap \ker f_* = \{0\}$ and hence $\mathcal{N}_1(D, X) \cap \text{NE}(f) = \{0\}$. This implies that $f|_D$ is finite, so f has one-dimensional fibers. Then f is elementary and $\rho_X = 3$, so we are again in case (ii). Finally if $\rho_Y = 1$ we get case (iv). \square

Example 7.5 (Elementary contractions over surfaces). It is not difficult to write down examples of smooth Fano varieties of dimension $n \geq 3$ which are not products, but have an elementary contraction of fiber type over \mathbb{P}^2 , $\mathbb{P}^1 \times \mathbb{P}^1$, or \mathbb{F}_1 . Thus the condition $\rho_Y \geq 3$ in the second part of Theorem 1.1(i) is necessary.

For instance, one can consider the \mathbb{P}^{n-2} -bundles

$$\mathbb{P}_{\mathbb{P}^2}(\mathcal{O}^{\oplus(n-2)} \oplus \mathcal{O}(1)), \quad \mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O}^{\oplus(n-2)} \oplus \mathcal{O}(1, 1)), \quad \mathbb{P}_{\mathbb{F}_1}(\mathcal{O}^{\oplus(n-2)} \oplus \mathcal{O}(l)),$$

where $l \subset \mathbb{F}_1$ is the proper transform of a general line in \mathbb{P}^2 .

A different example is given by $\mathbb{F}_1 \times \mathbb{P}^1 \times \mathbb{P}^{n-3}$, which has a quasi-elementary contraction onto $\mathbb{P}^1 \times \mathbb{P}^1$ with fiber $\mathbb{P}^1 \times \mathbb{P}^{n-3}$. In this case the variety is a product, but it is not the product of the fiber and the target of the contraction.

Example 7.6 (Elementary contractions over 3-folds). Let $Y = \mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(1, 1))$, so that Y is Fano with $\rho_Y = 3$. Observe that Y is the divisorial resolution of a quadric $Q \subset \mathbb{P}^4$ with an isolated singularity. Let $L \in \text{Pic } Y$ be the pull-back of $\mathcal{O}_Q(1)$, and consider

$$X = \mathbb{P}_Y(\mathcal{O}^{\oplus(n-3)} \oplus L).$$

Then X is Fano with dimension n and $\rho_X = 4$, and it is not a product. One can write down analogous examples with $\rho_Y = 1, 2$.

We do not know whether there are similar examples with $\rho_Y = 4, 5$. Let us point out that smooth Fano 3-folds Y with $\rho_Y = 5$ (respectively $\rho_Y = 4$) that are not products are given by just two families (respectively 12), after [MM81].

Example 7.7 (Conic bundles). Let X be a smooth Fano variety and $f: X \rightarrow Y$ an equidimensional contraction with $\dim Y = n - 1$. Then f is a conic bundle, and it is quasi-elementary if and only if it is elementary.

We know that Y is smooth (see [And85, Theorem 3.1] and [AW98, Proposition 4.1]) and if $\dim X \leq 4$ then Y is also Fano, by [Wiś91a, Corollary on p. 156]. However, this is not true in higher dimensions, see [Wiś91a, Example on p. 156].

Example 7.8 (Elementary contractions in dimension 4). Let X be a smooth Fano 4-fold and consider an elementary contraction $f: X \rightarrow Y$ with $\dim Y = 3$.

If f is equidimensional then Y is smooth and Fano, see Example 7.7. However, it is well known that f can have isolated two-dimensional fibers, which are classified, see [AW98, Kac97]. In [Kac97, § 11] we find several examples where f is not equidimensional and $Y \cong \mathbb{P}^3$; in particular, in this case X is not a product. However, we are not aware of similar examples with Y singular.

Example 7.9 (A Fano 4-fold with $\rho = 6$ and only small elementary contractions). In the toric case, smooth Fano varieties are classified up to dimension 7, the cases of dimensions 5, 6 and 7 being quite recent [KN07, Øbr07]. They are a good source of explicit examples.

After the classification in [Bat99] (see also [Sat00]), toric Fano 4-folds have Picard number at most 8. The ones with $\rho = 7, 8$ are just $S_3 \times S_3$ and $S_3 \times S_2$, S_2 and S_3 being the blow-up of \mathbb{P}^2 in two points and in three non-collinear points respectively. Among the ones with $\rho = 6$ there is a case with no (non-trivial) quasi-elementary contractions, the toric Del Pezzo 4-fold V (n. 118 in [Bat99]).

The Mori cone $\text{NE}(V)$ has dimension 6 but has 20 extremal rays. Every elementary contraction is a small contraction with exceptional locus a \mathbb{P}^2 with normal bundle $\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$. Every such exceptional \mathbb{P}^2 intersects three others in a point.

One can see that V has a contraction of fiber type $f: V \rightarrow Y$ with $\dim Y = 3$ and $\rho_Y = 1$, so f is not quasi-elementary. There are six two-dimensional fibers, which are unions of two exceptional \mathbb{P}^2 intersecting in one point. Moreover, Y has six isolated non- \mathbb{Q} -factorial points in the images of these fibers.

To our knowledge, among the known examples of Fano 4-folds with no (non-trivial) quasi-elementary contractions of fiber type, V is the one with largest Picard number.

This example has an analog V_n in each even dimension $n = 2m \geq 4$. This is a smooth toric Fano variety with $\rho_{V_n} = n + 2$ and $2 \binom{n+1}{m}$ extremal rays. Every elementary contraction is a small contraction with exceptional locus a \mathbb{P}^m with normal bundle $\mathcal{O}_{\mathbb{P}^m}(-1)^{\oplus m}$. The varieties V_n are called toric Del Pezzo varieties and were introduced in [VK84].

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