Canad. J. Math. Vol. **60** (5), 2008 pp. 1050–1066

Adjacency Preserving Maps on Hermitian Matrices

Wen-ling Huang and Peter Šemrl

Abstract. Hua's fundamental theorem of the geometry of hermitian matrices characterizes bijective maps on the space of all $n \times n$ hermitian matrices preserving adjacency in both directions. The problem of possible improvements has been open for a while. There are three natural problems here. Do we need the bijectivity assumption? Can we replace the assumption of preserving adjacency in both directions by the weaker assumption of preserving adjacency in one direction only? Can we obtain such a characterization for maps acting between the spaces of hermitian matrices of different sizes? We answer all three questions for the complex hermitian matrices, thus obtaining the optimal structural result for adjacency preserving maps on hermitian matrices over the complex field.

1 Introduction and Statement of the Main Result

One can easily verify that

$$d(T, S) = \operatorname{rank}(T - S)$$

defines a metric on the real vector space H_n of all $n \times n$ hermitian matrices over the complex field. The characterization of isometries of H_n with respect to this metric was obtained by Hua [5, 6]. If such an isometry ϕ is assumed to be affine (that is, linear, after the harmless normalization $\phi(0) = 0$), then the task of describing the general form of isometries reduces to a nice exercise. It is remarkable that in Hua's result the affine character of ϕ is not an assumption but a conclusion. Moreover, instead of isometries, Hua considered maps satisfying a weaker assumption of preserving distance 1 in both directions.

Two hermitian matrices T and S are said to be *adjacent* (also *coherent*) if $\operatorname{rank}(T - S) = 1$. Hua's fundamental theorem of the geometry of hermitian matrices over the complex field reads as follows.

Theorem 1.1 *Let $n \geq 2$ be an integer and $\phi\colon H_n \to H_n$ a bijective map such that for every pair $A, B \in H_n$ the matrices A and B are adjacent if and only if $\phi(A)$ and $\phi(B)$ are adjacent. Then there exist $c \in \{-1, 1\}$, an invertible $n \times n$ complex matrix T, and $S \in H_n$ such that either*

$$\phi(A) = cTAT^* + S, \quad A \in H_n,$$

or

$$\phi(A) = cT\overline{A}T^* + S, \quad A \in H_n.$$

Received by the editors October 31, 2005; revised February 8, 2006.
The second author was partially supported by a grant from the Ministry of Science of Slovenia
AMS subject classification: Primary: 15A03; secondary: 15A04, 15A57, 15A99.
Keywords: rank, adjacency preserving map, hermitian matrix, geometry of matrices.

https://doi.org/10.4153/CJM-2008-047-1 Published online by Cambridge University Press

Here, $\overline{A}$ denotes the matrix obtained from A by applying the complex conjugation entrywise.

This result has many applications. Hua's study of the geometry of matrices is related to Siegel's work on fractional linear transformations. In the language of special relativity the above result in the case $n = 2$ can be reformulated (see [7, §5.3]) as the statement that every bijective correspondence on spacetime events preserving the coherence, that is, the property that the spatial distance between the two events is equal to the product of the time difference between the two events and the speed of light, is a Poincaré similarity, *i.e.,* the product of a Lorentz transformation and a dilation. Further, this result is of basic importance in the theory of linear preservers [10, pp. 18–19], can be applied when studying Jordan automorphisms of H_n [12, p. 355], and is closely related to the geometry of algebraic homogeneous spaces [3] (see [12, §6.8] for the detailed explanation).

Hua's theorem gives a very nice conclusion under rather weak assumptions. Still, all the applications mentioned above motivate the question whether we can prove an even stronger result. Can we replace the assumption that the adjacency is preserved in both directions by the weaker assumption that the adjacency is preserved in one direction only and still get the same conclusion? Do we need the bijectivity assumption?

The first question has been answered by Huang, Höfer, and Wan [8] who proved that every bijective map on the space of $n \times n$ hermitian matrices over more general division rings with involution that preserves adjacency in one direction preserves adjacency in both directions.

The aim of this paper is to substantionally generalize the complex case of Huang, Höfer, and Wan's result. Our generalization will also answer the second question above. Moreover, we will consider maps acting between the spaces of hermitian matrices of different sizes. Before formulating our main result let us make one more remark. Let m, n be positive integers with $n \geq 2$ and $\phi\colon H_n \to H_m$ a map preserving adjacency. Then any map defined by $A \mapsto \phi(A) + S$, $A \in H_n$, where S is any $m \times m$ hermitian matrix, preserves adjacency as well. Thus, when studying adjacency preserving maps there is no loss of generality in assuming that they map the zero $n \times n$ matrix into the zero $m \times m$ matrix. Our main result is the following.

Theorem 1.2 *Let m, n be positive integers with $n \geq 2$. Assume that $\phi\colon H_n \to H_m$ is a map such that the matrices $\phi(A)$ and $\phi(B)$ are adjacent whenever A and B are adjacent, $A, B \in H_n$. Suppose that $\phi(0) = 0$. Then one of the following holds.*

(i) *There exist a rank one matrix $R \in H_m$ and a function $\rho\colon H_n \to \mathbb{R}$ such that $\phi(A) = \rho(A)R$.*

(ii) *$m \geq n$ and there exist $c \in \{-1, 1\}$ and an invertible $m \times m$ complex matrix T such that either*

$$\phi(A) = cT \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} T^*, \quad A \in H_n, \tag{1.1}$$

or

$$\phi(A) = cT \begin{bmatrix} \overline{A} & 0 \\ 0 & 0 \end{bmatrix} T^*, \quad A \in H_n. \tag{1.2}$$

https://doi.org/10.4153/CJM-2008-047-1 Published online by Cambridge University Press

Let $\rho\colon H_n \to \mathbb{R}$ be any function and $R \in H_m$ a rank one matrix. Then the map $\phi\colon H_n \to H_m$ defined by $\phi(A) = \rho(A)R$ preserves adjacency if $\rho(A) \neq \rho(B)$ whenever A and B are adjacent. In particular, this happens when ρ is injective. If we take $\rho(A) = \operatorname{tr} A$, where $\operatorname{tr} A$ denotes the trace of A, then ϕ is a continuous (even real-linear) adjacency preserving map. Indeed, if A and B are adjacent, then $A - B$ is a rank one hermitian matrix and every rank one hermitian matrix has a nonzero trace.

We will call any adjacency preserving map whose image is contained in a linear span of a rank one matrix a degenerate adjacency preserving map. Any map of the form (1.1) or of the form (1.2) will be called a standard map.

We have several straightforward consequences of our main result. Let $\phi\colon H_n \to H_m$, $n \geq 2$, be an adjacency preserving map with $\phi(0) = 0$. If $m < n$, then ϕ is degenerate. If ϕ is surjective, then either $m = 1$, or $m = n$. If ϕ is injective and continuous, then, by the invariance of domain theorem [4, p. 344], ϕ must be a standard map. Also, if ϕ preserves adjacency in both directions, that is, for every pair $A, B \in H_n$ the matrices A and B are adjacent if and only if $\phi(A)$ and $\phi(B)$ are adjacent, then ϕ has to be standard.

We believe that the interesting corresponding problem for real symmetric matrices is still open.[1] The complex hermitian case has been studied more because of the above mentioned application in the special relativity.

We will start by introducing the notation and presenting some preliminary results. The third section will be devoted to the special case when $m = n = 2$, and the last section to the proof of the main result.

2 Notation and Preliminary Results

Let $A, B \in H_n$. Matrices A, B are said to be adjacent if $\operatorname{rank}(A - B) = 1$. The distance $d(A, B)$ between A and B is defined to be the smallest nonnegative integer k with the property that there exists a sequence of consecutively adjacent matrices $A = A_0, A_1, \dots, A_k = B$. The distance satisfies the triangle inequality

$$d(A,B) + d(B,C) \geq d(A,C) \quad \text{for all } A,B,C \in H_n.$$

For any two hermitian matrices $A, B \in H_n$, it was proved [12] that

$$d(A,B) = \operatorname{rank}(A - B).$$

For any two adjacent matrices $A, B \in H_n$ the line $l(A, B)$ joining A and B is defined to be the set consisting of A, B, and all $X \in H_n$ which are adjacent to both A and B. It was also proved in [12] that $l(A, B) = \{A + \lambda(B - A) : \lambda \in \mathbb{R}\}$.

Lemma 2.1 **([8, Lemma 2.1])** *Let $P \in H_n$ and l be a line of H_n. Then either the distance between P and any hermitian matrix of l is the same, or there is a hermitian matrix $Q \in l$ such that $d(P, X) = d(P, Q) + 1$ for all $X \in l \setminus \{Q\}$.*

Let $\phi\colon H_n \to H_m$ be any map. Note that composing such a map with $*$-congruences and the map $A \mapsto -A$ does not affect either the assumption or the conclusion

[1] *Added in proof*: The authors have been informed that this problem has now been solved.

of our main theorem. More precisely, if T and S are any $m \times m$ invertible complex matrix and $n \times n$ invertible complex matrix, respectively, and $c, d \in \{-1, 1\}$, then the map $\phi\colon H_n \to H_m$ preserves adjacency if and only if the map $\psi\colon H_n \to H_m$ defined by $\psi(A) = cT\phi(dSAS^*)T^*$, $A \in H_n$, preserves adjacency. Moreover, ϕ is degenerate if and only if ψ is degenerate, and ϕ is a standard map if and only if ψ is a standard map.

As usual, we will identify $n \times n$ matrices with linear operators acting on the n-dimensional complex vector space $\mathbb{C}^n$. Vectors will be then identified with $n \times 1$ matrices. The elements of the standard basis of $\mathbb{C}^n$ will be denoted by $e_1, \dots, e_n$ and the elements of the standard basis of the space of all $n \times n$ matrices by E_{ij}, $1 \leq i, j \leq n$. Thus, $E_{ij} = e_i e_j^*$. Every rank one projection $P \in H_n$ can be written as $P = xx^*$, where x is a vector of norm one. Further, every rank one hermitian matrix is a scalar multiple of a rank one projection. Recall that projections $P, Q \in H_n$ are called orthogonal if $PQ = 0$.

We will frequently use the following simple fact.

Lemma 2.2 *Let $A \in H_n$ be adjacent to both txx^* and sxx^*, where x is a vector of norm one and $t, s \in \mathbb{R}$, $t \neq s$. Then $A = rxx^*$ for some real number r with $r \notin \{t, s\}$.*

Proof Because A is adjacent to txx^* and sxx^*, A is contained in the line

$$l(txx^*, sxx^*) = \{\lambda(xx^*) : \lambda \in \mathbb{R}\}. \qquad \blacksquare$$

Lemma 2.3 *Let $P \in H_n$ be a projection and $A, B \in H_n$ two matrices such that $P = A + B$ and* $\operatorname{rank} P = \operatorname{rank} A + \operatorname{rank} B$. *Then A and B are orthogonal projections.*

Proof Identifying matrices with operators we first deduce from $P = A + B$ that $\operatorname{Im} P \subset \operatorname{Im} A + \operatorname{Im} B$. Applying the rank additivity condition we see that $\operatorname{Im} P = \operatorname{Im} A \oplus \operatorname{Im} B$. For an arbitrary $x \in \operatorname{Ker} P$ we have $0 = Px = Ax + Bx$, and since the image of P is the direct sum of the images of A and B, it follows that $Ax = Bx = 0$. For any $x \in \operatorname{Im} A$ we apply the inclusion $\operatorname{Im} A \subset \operatorname{Im} P$ to obtain $x = Px = Ax + Bx$. From $x, Ax \in \operatorname{Im} A$ and $Bx \in \operatorname{Im} B$ we conclude that $Ax = x$ and $Bx = 0$. Similarly, B acts as the identity on the image of B, while the restriction of A to the image of B is the zero operator. Applying the fact that $\mathbb{C}^n = \operatorname{Im} P \oplus \operatorname{Ker} P = \operatorname{Im} A \oplus \operatorname{Im} B \oplus \operatorname{Ker} P$ we see that A and B are projections and $AB = 0$. $\blacksquare$

Lemma 2.4 *Let k, n be integers, $3 \leq k \leq n$, and let $t_1, \dots, t_k$ be nonzero real numbers. Furthermore, let $P_1, \dots, P_k \in H_n$ be pairwise orthogonal rank one projections. Denote $A = \sum_{j=1}^{k} t_j P_j$. If a matrix $B \in H_n$ of rank k is adjacent to $A - t_i P_i$ and $d(B, t_i P_i) = k - 1$ for every $i = 1, \dots, k$, then $B = A$.*

Proof We have $\operatorname{rank} B = \operatorname{rank}(t_i P_i) + \operatorname{rank}(B - t_i P_i)$, and therefore, as in the previous lemma, the image of B is the direct sum of the images of $t_i P_i$ and $B - t_i P_i$. In particular, the image of P_i is contained in the image of B, and since B is of rank k, we have $\operatorname{Im} B = \operatorname{Im} P$, where $P = P_1 + \cdots + P_k$. Thus, $B = PBP$ and we may restrict our attention to the subspace PH_nP. In other words, we may assume that $k = n$ and B is invertible. Moreover, replacing B by (a not necessarily hermitian matrix) $A^{-1}B$ we may assume that $A = I$, $t_1 = \cdots = t_k = 1$, and B is adjacent to every diagonal

https://doi.org/10.4153/CJM-2008-047-1 Published online by Cambridge University Press

idempotent of rank $k-1$. Equivalently, I is adjacent to $B^{-1}Q$ for every diagonal idempotent Q of rank $k-1$. The set of all $k \times k$ complex matrices of rank $k-1$ adjacent to the $k \times k$ identity matrix is the set of all idempotent matrices of rank $k-1$. Hence, $B^{-1}Q$ is an idempotent of rank $k-1$ for every diagonal idempotent Q of rank $k-1$. It follows easily that B^{-1} is the identity matrix. ∎

Lemma 2.5 *Let m be a positive integer and $A, B \in H_m$ with* $\operatorname{rank} A = 1$. *Assume that* $\operatorname{rank}(A + tB) = 1$ *for every real number t. Then $B = 0$.*

Proof We have $\operatorname{rank} B \leq 1$, since otherwise $\operatorname{rank}(A+tB) > 1$ for large real numbers t. Thus, B is either 0 or a rank one matrix. We must show that the second possibility cannot occur. Assume on the contrary that B is a rank one hermitian matrix. Then, since $\operatorname{rank}(A + B) = 1$, we have $B = rA$ for some nonzero real number r. It follows that $A + (-\frac{1}{r})B = 0$, a contradiction. ∎

***Lemma 2.6* (See [8, Lemmas 2.5 and 2.6])** *Let $A, B \in H_n$ be invertible matrices, $A \neq B$. Then there exist a positive integer m and a sequence of invertible matrices $A = A_0, A_1, \ldots, A_m = B$ such that all the pairs A_k, A_{k+1}, $k = 0, \ldots, m-1$, are adjacent and for every $k = 0, \ldots, m-1$ there exists $C_k \in l(A_k, A_{k+1})$ of rank $n-1$.*

Let $\phi\colon H_n \to H_m$, $n \geq 2$, be an adjacency preserving map. By the definition of a line, we have the following.

Lemma 2.7 *Let $A, B \in H_n$ be an adjacent pair. Then $\phi(l(A, B)) \subset l(\phi(A), \phi(B))$, and the restriction of ϕ to the line $l(A, B)$ is injective.*

Lemma 2.8 *Let $\phi\colon H_n \to H_m$ be an adjacency preserving map such that $\phi(0) = 0$. Set* $k = \max\{\operatorname{rank}\phi(A) : \operatorname{rank} A = n\}$. *Assume that $k \geq 2$ and that for every singular $A \in H_n$ we have* $\operatorname{rank}\phi(A) < k$. *Then* $\operatorname{rank}\phi(B) = k$ *for every invertible $B \in H_n$.*

Proof Let $B \in H_n$ be any invertible matrix. Choose an invertible matrix A such that $\operatorname{rank}\phi(A) = k$ and a sequence $A = A_0, A_1, \ldots, A_t = B$ as in Lemma 2.6. Further, let C_k, $k = 0, \ldots, t-1$, be as in Lemma 2.6. By Lemma 2.7, the line $l(A_0, A_1)$ is mapped by ϕ injectively into the line $l(\phi(A_0), \phi(A_1))$. By Lemma 2.1 and $\operatorname{rank}\phi(A_0) = k$, we know that there is at most one point T on the line $l(\phi(A_0), \phi(A_1))$ such that $\operatorname{rank} T < k$. Obviously, $\phi(C_0)$ is such a point. It follows that $\operatorname{rank}\phi(A_1) = k$. Using the same argument once again, we conclude that $\operatorname{rank}\phi(A_2) = k$. After t steps we arrive at the desired equation $\operatorname{rank}\phi(B) = k$. ∎

Lemma 2.9 *Let $\phi\colon H_n \to H_m$ be an adjacency preserving map. Assume that there exist $P, Q \in H_n$ such that $d(\phi(P), \phi(Q)) = n$. Then $d(\phi(A), \phi(B)) = d(A, B)$ for every $A, B \in H_n$.*

Proof Because ϕ is an adjacency preserving map we get immediately from the definition of distance d that ϕ is a contraction with respect to d, that is, $d(\phi(A), \phi(B)) \leq d(A, B)$ for every $A, B \in H_n$.

First we prove that for $A \in H_n$, $d(A, P) = n$ implies $d(\phi(A), \phi(P)) = n$. Let σ be the map $X \mapsto \sigma(X) = X + P$ for all $X \in H_n$ and let σ' be the map $X \mapsto \sigma'(X) = X - \phi(P)$ for all $X \in H_m$. Let $\varphi = \sigma' \circ \phi \circ \sigma$. Then φ preserves adjacency,

$\operatorname{rank}\varphi(Q-P)=n$, and $\varphi(0)=0$. Because φ is a contraction with respect to d we have $\operatorname{rank}\varphi(X)<n$ for every singular $X\in H_n$. We have $d(0,A-P)=d(A,P)=n$, and by Lemma 2.8 we have $n=d(0,\varphi(A-P))=d(\phi(A),\phi(P))$.

Now we prove that for $A,B\in H_n$, $d(A,B)=n$ implies $d(\phi(A),\phi(B))=n$. We can find $C\in H_n$ such that $d(P,C)=d(C,A)=d(A,B)=n$. Applying the previous step we first see that $d(\phi(P),\phi(C))=n$, and applying it two more times we arrive at the desired conclusion $d(\phi(A),\phi(B))=n$.

Finally we prove that $d(A,B)=d(\phi(A),\phi(B))$ for all $A,B\in H_n$. If $d(A,B)=n$, then $d(\phi(A),\phi(B))=n$ from above. Suppose $d(A,B)<n$. Then there is a point C such that $d(A,B)+d(B,C)=d(A,C)=n$. This implies

$$\begin{aligned} n&=d(A,C)=d(A,B)+d(B,C)\\ &\geq d(\phi(A),\phi(B))+d(\phi(B),\phi(C))\geq d(\phi(A),\phi(C))\\ &=n. \end{aligned}$$

Hence $d(A,B)=d(\phi(A),\phi(B))$. ■

Lemma 2.10 *Let m,n be integers with $m\geq n\geq 2$. Let $A,B,C\in H_m$ with*

$$B=\begin{bmatrix}B_1 & 0\\ 0 & 0\end{bmatrix}\quad \text{and}\quad C=\begin{bmatrix}C_1 & 0\\ 0 & 0\end{bmatrix},$$

where B_1 and C_1 are $n\times n$ hermitian matrices, $B_1\neq C_1$. Assume that A and B are adjacent and A and C are adjacent. Then A is of the form

$$A=\begin{bmatrix}* & 0\\ 0 & 0\end{bmatrix},$$

where $*$ stands for an $n\times n$ hermitian matrix.

Proof After replacing A,B,C by $A-B$, $B-B$, $C-B$, respectively, we may assume that $B=0$. Thus, A is of rank one. Hence, C is either of rank one, or of rank two. In the first case we complete the proof using Lemma 2.2, while in the second case we have $C=A+(C-A)$ and $\operatorname{rank}C=\operatorname{rank}A+\operatorname{rank}(C-A)$. We already know that the rank additivity condition implies that $\operatorname{Im}A\subset\operatorname{Im}C$. This completes the proof. ■

Lemma 2.11 *Let m,n be positive integers with $n\geq 2$. Assume that $\phi\colon H_n\to H_m$ is an adjacency preserving map with $\phi(0)=0$. Suppose also that*

$$\phi(I)=\begin{bmatrix}X & 0\\ 0 & 0\end{bmatrix},$$

where $X\in H_n$ is of rank n and the zero matrices are of appropriate sizes. Then for all $A\in H_n$

$$\phi(A)=\begin{bmatrix}* & 0\\ 0 & 0\end{bmatrix}, \tag{2.1}$$

where $*$ stands for some $n\times n$ hermitian matrix.

Proof We first prove the special case when A is a projection of rank one. In this case we have $d(0, A) = 1$ and $d(A, I) = n-1$, and consequently, $\phi(A)$ is a rank one matrix with $d(\phi(A), \phi(I)) \leq n-1$. From $n = d(\phi(I), 0) \leq d(0, \phi(A)) + d(\phi(A), \phi(I)) \leq n$ we conclude that $\operatorname{rank}\phi(I) = \operatorname{rank}\phi(A) + \operatorname{rank}(\phi(I) - \phi(A))$, which yields that $\operatorname{Im}\phi(A) \subset \operatorname{Im}\phi(I)$, and thus, $\phi(A)$ is of the form (2.1). If $A = tP$ for some rank one projection P and some real number $t \neq 0, 1$, then $\phi(A)$ is adjacent to both 0 and $\phi(P)$, and by Lemma 2.10, $\phi(A)$ must be of the form (2.1). Thus, we have proved (2.1) for all hermitian matrices of rank one. Every matrix $A \in H_n$ of rank k, $2 \leq k \leq n$, is adjacent to more than one hermitian matrix of rank $k-1$. Note that by Lemma 2.9 ϕ is injective. So, we can complete the proof using Lemma 2.10 and induction. ■

3 The Special Case H_2

Let $n \geq 3$ be an integer. Minkowski space M_n can be described as $\mathbb{R}^n$ together with the indefinite inner product $(\,\cdot\,,\,\cdot\,)$ of the form

$$(x, y) := x_1 y_1 + \cdots + x_{n-1} y_{n-1} - x_n y_n,$$

where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. A theorem of A. D. Alexandrov [1] states that any bijective transformation $f \colon \mathbb{R}^n \to \mathbb{R}^n$ which preserves the Lorentz distance 0 between pairs of points in both directions, *i.e.,*

$$(3.1) \qquad (x-y, x-y) = 0 \quad \Leftrightarrow \quad \big(f(x) - f(y), f(x) - f(y)\big) = 0$$

for all $x, y \in \mathbb{R}^n$, is the product of a Lorentz transformation and a dilation.

Let $\mathcal{D}$ be a domain of $\mathbb{R}^n$, *i.e.,* an open and connected subset of $\mathbb{R}^n$. We are now interested in mappings $f \colon \mathcal{D} \to \mathbb{R}^n$, which satisfy (3.1) for all $x, y \in \mathcal{D}$. Any dilation $x \mapsto kx + t$ where $k \in \mathbb{R} \setminus \{0\}$ and $t \in \mathbb{R}^n$ satisfies (3.1). Also any Lorentz transformation $x \mapsto xL + t$ where L is a Lorentz matrix and $t \in \mathbb{R}^n$ satisfies (3.1). Finally, if $\mathcal{D}$ does not intersect the cone $\{x \in M_n : (x, x) = 0\}$, then the conformal inversion $f \colon x \mapsto x/(x, x)$ satisfies (3.1). We call a mapping a *conformal mapping* if it is the product of mappings of these three types. So any conformal mapping $f \colon \mathcal{D} \to \mathbb{R}^n$ satisfies (3.1).

The following was proved independently by A. D. Alexandrov [2] (see also [9]), and the result was generalized by I. Popovici and D. C. Rădulescu [11].

Theorem 3.1 *Let $\mathcal{D}$ be a domain of $\mathbb{R}^n$, $n \geq 3$. Let $f \colon \mathcal{D} \to \mathbb{R}^n$ be a mapping such that* (3.1) *holds for all $x, y \in \mathcal{D}$. Then f is a conformal mapping.*

Any conformal mapping $f \colon \mathbb{R}^n \to \mathbb{R}^n$ is the product of a dilation and a Lorentz transformation. So we obtain the following corollary.

Corollary 3.2 *Let $n \geq 3$. Any mapping $f \colon \mathbb{R}^n \to \mathbb{R}^n$ which satisfies* (3.1) *for all $x, y \in \mathbb{R}^n$ is the product of a dilation and a Lorentz transformation.*

From now on we consider the case $n = 4$. The mapping $\mathbb{R}^4 \to H_2$,

$$(x_1, x_2, x_3, x_4) \mapsto \begin{pmatrix} x_4 + x_1 & x_2 + i x_3 \\ x_2 - i x_3 & x_4 - x_1 \end{pmatrix},$$

takes points $x \neq y \in \mathbb{R}^4$ with $(x-y, x-y) = 0$ to adjacent matrices and vice versa (see [7]). In the language of hermitian complex 2×2 matrices, Corollary 3.2 reads as follows.

Corollary 3.3 *Let $\phi\colon H_2 \to H_2$ be a mapping such that A, B are adjacent if and only if $\phi(A), \phi(B)$ are adjacent, $A, B \in H_2$. Then there exist $c \in \{-1, 1\}$, an invertible 2×2 complex matrix T, and $S \in H_2$ such that either*

$$\phi(A) = cTAT^* + S, \quad A \in H_2,$$

or

$$\phi(A) = cT\overline{A}T^* + S, \quad A \in H_2.$$

Theorem 3.4 *Let $\phi\colon H_2 \to H_2$ be a mapping which preserves adjacency. Let there exist matrices $P, Q \in H_2$ such that $\phi(P) \neq \phi(Q)$ are not adjacent. Then there exist $c \in \{-1, 1\}$, an invertible 2×2 complex matrix T, and $S \in H_2$ such that either*

$$\phi(A) = cTAT^* + S, \quad A \in H_2,$$

or

$$\phi(A) = cT\overline{A}T^* + S, \quad A \in H_2.$$

Proof Let $P, Q \in H_2$ be matrices such that $\phi(P) \neq \phi(Q)$ are not adjacent. Then by Lemma 2.9, we have $d(\phi(A), \phi(B)) = d(A, B)$ for all $A, B \in H_2$. In particular, A, B are adjacent if and only if $\phi(A), \phi(B)$ are adjacent for all $A, B \in H_2$. Applying Corollary 3.3, we obtain the theorem. ■

4 Proof of the Main Result

We will prove our main theorem in several steps.

Claim 4.1 *Theorem* 1.2 *holds true in the special case when $n = 2$.*

Proof There is nothing to prove if $m = 1$. So, assume from now on that $m \geq 2$. By Lemma 2.9, we have that either $\phi(H_2)$ is contained in a line $\{\lambda R : \lambda \in \mathbb{R}\}$ for some rank one matrix $R \in H_m$, or $\operatorname{rank} \phi(A) = \operatorname{rank} A$ for every $A \in H_2$. In the second case we can find a standard transformation, which maps $\phi(I)$ to $\left[\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right]$, for some 2×2 invertible hermitian matrix A. By Lemma 2.11 we may consider ϕ as a map from H_2 to H_2 which preserves the adjacency in both directions. We complete the proof using Theorem 3.4. ■

Claim 4.2 *Let m, n be positive integers with $n \geq 2$. Assume that $\phi\colon H_n \to H_m$ is an adjacency preserving map such that $\phi(0) = 0$. Suppose also that*

$$\phi(I) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},$$

where I on the right-hand side stands for the $n \times n$ identity matrix and the zeroes denote the zero matrices of the appropriate sizes. Then there exists an $n \times n$ unitary matrix U such that either

$$\phi(A) = \begin{bmatrix} UAU^* & 0 \\ 0 & 0 \end{bmatrix}, \quad A \in H_n,$$

or

$$\phi(A) = \begin{bmatrix} U\overline{A}U^* & 0 \\ 0 & 0 \end{bmatrix}, \quad A \in H_n.$$

Proof We will prove this statement by induction on n. We already know that it holds true in the case when $n = 2$. So assume it holds for $n - 1$ and we want to prove it for n. If $P \in H_n$ is a projection of rank $n-1$, then by Lemma 2.9 the matrix $\phi(P)$ has rank $n - 1$ and is adjacent to $\phi(I)$. Thus, $\phi(I) = \phi(P) + R$, where R is of rank one. Applying Lemma 2.3 we see that

$$\phi(P) = \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix},$$

where Q is a projection of rank $n - 1$. The set PH_nP can be identified with H_{n-1} and $\phi(P)$ is unitarily similar to $E_{11} + \cdots + E_{n-1,n-1}$. Thus, we can apply the induction hypothesis for the restriction of ϕ to the subset PH_nP. In particular, we conclude that (because P was an arbitrary projection of rank $n - 1$) ϕ maps rank one projections into rank one projections. Moreover, if two rank one projections are orthogonal, then their ϕ-images are orthogonal as well. Thus, after replacing ϕ by the map

$$X \mapsto \begin{bmatrix} V & 0 \\ 0 & 0 \end{bmatrix} \phi(X) \begin{bmatrix} V^* & 0 \\ 0 & 0 \end{bmatrix},$$

where V is an appropriate unitary $n \times n$ matrix, we may assume that $\phi(E_{ii}) = E_{ii}$, $i = 1, \ldots, n$. Here, of course, E_{ii} appearing on the left-hand side is an $n \times n$ matrix, while E_{ii} on the right-hand side is of the size $m \times m$.

Let $i, j \in \{1, \ldots, n\}$, $i \neq j$. Applying Claim 4.1 together with $\phi(E_{ii}) = E_{ii}$ and $\phi(E_{jj}) = E_{jj}$ we see that either

$$\phi(tE_{ii} + \alpha E_{ij} + \overline{\alpha} E_{ji} + sE_{jj}) = tE_{ii} + w_{ij}\alpha E_{ij} + \overline{w_{ij}\alpha} E_{ji} + sE_{jj},$$

or

$$\phi(tE_{ii} + \alpha E_{ij} + \overline{\alpha} E_{ji} + sE_{jj}) = tE_{ii} + w_{ij}\overline{\alpha} E_{ij} + \overline{w_{ij}}\alpha E_{ji} + sE_{jj}$$

for some complex number w_{ij} with $|w_{ij}| = 1$.

Let A and B be two hermitian $n \times n$ matrices of rank one. Then we can find a rank two projection Q such that $A, B \in QH_nQ$. Applying Claim 4.1 for the restriction of ϕ to QH_nQ together with the fact that ϕ preserves projections and orthogonality of rank one projections we see that $\phi(ABA) = \phi(A)\phi(B)\phi(A)$.

Now let B be any rank one matrix and A any rank one matrix of the form $tE_{11} + \alpha E_{12} + \overline{\alpha} E_{21} + sE_{22}$. We know that either $\phi(A) = tE_{11} + w_{12}\alpha E_{12} + \overline{w_{12}\alpha} E_{21} + sE_{22}$, or $\phi(A) = tE_{11} + w_{12}\overline{\alpha} E_{12} + \overline{w_{12}}\alpha E_{21} + sE_{22}$. Let us consider only the first case. Then

https://doi.org/10.4153/CJM-2008-047-1 Published online by Cambridge University Press

since ABA is a matrix having nonzero entries only in the upper-left 2×2 corner, we have

$$\phi(ABA) = \operatorname{diag}(1, \overline{w_{12}}, 0, \dots, 0)\, ABA\, \operatorname{diag}(1, w_{12}, 0, \dots, 0) = \phi(A)\phi(B)\phi(A).$$

It is now straightforward to verify that we obtain the upper-left 2×2 corner of $\phi(B)$ by multiplying the upper-left 2×2 corner of B by $\operatorname{diag}(1, \overline{w_{12}})$ on the left and by $\operatorname{diag}(1, w_{12})$ on the right.

In the preceding arguments we can replace $E_{11}, E_{12}, E_{21}, E_{22}$ by $E_{ii}, E_{ij}, E_{ji}, E_{jj}$, $i \neq j$. In this way we show that for every $[a_{ij}] \in H_n$ of rank one we have $\phi([a_{ij}]) = [w_{ij} f_{ij}(a_{ij})]$, where $w_{ii} = 1$, $i = 1, \dots, n$, w_{ij} are complex numbers of modulus 1 with $w_{ij} = \overline{w_{ji}}$, $1 \leq i, j \leq n$, $i \neq j$, $f_{ii} = id$, $i = 1, \dots, n$, and every function $f_{ij} = f_{ji} \colon \mathbb{C} \to \mathbb{C}$ is either the identity or the complex conjugation, for $1 \leq i, j \leq n$, $i \neq j$. Now, for every $[a_{ij}] \in H_n$ of rank one, the matrix $[w_{ij} f_{ij}(a_{ij})]$ has rank one. It follows easily that there exist complex numbers $z_1, \dots, z_n$ of modulus 1 such that $w_{ij} = z_i \overline{z_j}$ and either $f_{ij} = id$ for all $i \neq j$, or f_{ij} is the complex conjugation for all $i \neq j$. Composing ϕ with a similarity transformation induced by the unitary $m \times m$ matrix $\operatorname{diag}(z_1, \dots, z_n, 1, \dots, 1)$ and with the complex conjugation, if necessary, we may assume that

$$\phi(A) = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \tag{4.1}$$

for every $A \in H_n$ of rank one.

We apply once again the induction hypothesis for the restriction of ϕ to PH_nP, where P is any projection of rank $n-1$. We conclude that (4.1) holds true for every $A \in H_n$ of rank at most $n-1$. So, it remains to prove this equation for all invertible matrices $A \in H_n$.

So, let $A \in H_n$ be invertible. Then $A = \sum_{j=1}^{n} t_j P_j$, where the P_j s are pairwise orthogonal rank one projections and the t_j s are nonzero real numbers. Denote

$$Q_i = \begin{bmatrix} P_i & 0 \\ 0 & 0 \end{bmatrix}, \quad i = 1, \dots, n.$$

Clearly, $\phi(A)$ is adjacent to $\phi(A - t_i P_i) = \sum_{j \neq i} t_j Q_j$, $i = 1, \dots, n$. By Lemma 2.9, $\operatorname{rank} \phi(A) = n$. We have $d(\phi(A), t_i Q_i) \leq d(A, t_i P_i) = n - 1$, and since $\phi(A)$ is a matrix of rank n, while $t_i Q_i$ has rank one, we actually have $d(\phi(A), t_i Q_i) = n - 1$. Applying Lemma 2.4 we see that $\phi(A) = \sum_{j=1}^{n} t_j Q_j$. Thus, (4.1) holds true for all $A \in H_n$. ■

Claim 4.3 *Let m, n be positive integers with $n \geq 3$. Assume that $\phi \colon H_n \to H_m$ is an adjacency preserving map such that $\phi(0) = 0$. Suppose also that for every projection P of rank $n-1$ there exists a nonzero vector $x \in \mathbb{C}^m$ such that $\phi(PH_nP) \subset \operatorname{span}\{xx^*\}$. Then ϕ is a degenerate adjacency preserving map.*

Proof Our assumption is that for every rank $n-1$ projection $P \in H_n$ there exists a rank one projection $Q \in H_m$ such that $\phi(PH_nP)$ is contained in the linear span of Q. It is rather easy to see that Q is independent of P. Indeed, let $P_1, P_2 \in H_n$ be projections of rank $n-1$ and $\phi(P_j H_n P_j) \subset \operatorname{span}\{Q_j\}$, $j = 1, 2$. Then we can find a rank one projection R such that $R \in P_j H_n P_j$, $j = 1, 2$. Since $\phi(R)$ is adjcent to 0, we have $\phi(R) = t_1 Q_1 = t_2 Q_2$ for some nonzero t_1, t_2. It follows that $Q_1 = Q_2$, as desired.

We have thus shown that there exists a rank one projection Q such that $\phi(A) \in \operatorname{span}\{Q\}$ for every $A \in H_n$ of rank at most $n-1$. We may assume with no loss of generality that $Q = E_{11}$.

Since every invertible $n \times n$ hermitian matrix is adjacent to some matrix of rank $n-1$, we have $\operatorname{rank} A \leq 2$ for every $A \in H_n$.

Assume first that every invertible matrix A from H_n is sent by ϕ into a matrix of rank at most one. We have to show that if $\phi(A) \neq 0$, then $\phi(A) \in \operatorname{span}\{E_{11}\}$. Assume on the contrary that $\phi(\sum_{j=1}^{n} t_j P_j) = Q \notin \operatorname{span}\{E_{11}\}$, where the P_j s are pairwise orthogonal rank one projections, the t_j s are nonzero real numbers and $Q \in H_m$ is a rank one matrix. We know that $\phi(\sum_{j=1}^{n-1} t_j P_j)$ is contained in the linear span of E_{11} and is adjacent to Q. Hence, $\phi(\sum_{j=1}^{n-1} t_j P_j) = 0$. By Lemma 2.7 the line $\{\sum_{j=1}^{n-1} t_j P_j + tP_n : t \in \mathbb{R}\}$ is mapped by ϕ injectively into the linear span of Q. Thus, there exists a real number s, $s \neq t_n$, such that $\phi(\sum_{j=1}^{n-1} t_j P_j + sP_n) = pQ$ with $p \neq 0$. We know that both $\phi(\sum_{j=2}^{n-1} t_j P_j + t_n P_n)$ and $\phi(\sum_{j=2}^{n-1} t_j P_j + sP_n)$ belong to the linear span of E_{11}. The first one is adjacent to Q and the second one to pQ. Thus, $\phi(\sum_{j=2}^{n-1} t_j P_j + t_n P_n) = \phi(\sum_{j=2}^{n-1} t_j P_j + sP_n) = 0$, contradicting the fact that $\sum_{j=2}^{n-1} t_j P_j + t_n P_n$ and $\sum_{j=2}^{n-1} t_j P_j + sP_n$ are adjacent.

The other possibility we have to treat is that there exists an invertible $A \in H_n$ such that $\operatorname{rank} \phi(A) = 2$. We will complete the proof of this step by showing that this possibility cannot occur. Indeed, if $\operatorname{rank} \phi(A) = 2$ for some invertible $A \in H_n$, then by Lemma 2.8, $\operatorname{rank} \phi(A) = 2$ for every invertible $A \in H_n$. There exists a matrix of rank $n-1$, say $\sum_{j=2}^{n} t_j P_j$ such that $\phi(\sum_{j=2}^{n} t_j P_j) = sE_{11}$ with $s \neq 0$. Here, the P_j s are pairwise orthogonal rank one projections and the t_j s are nonzero real numbers. Let P_1 be a rank one projection orthogonal to $P_2, \dots, P_n$. Now, both $P_1 + \sum_{j=2}^{n} t_j P_j$ and $2P_1 + \sum_{j=2}^{n} t_j P_j$ are adjacent to $\sum_{j=2}^{n} t_j P_j$, and they are mapped by ϕ into matrices of rank two. Thus, $\phi(P_1 + \sum_{j=2}^{n} t_j P_j) = sE_{11} + T$, where T is of rank one and $d(T, E_{11}) = 2$. Now, $\phi(P_1 + \sum_{j=2}^{n-1} t_j P_j)$ is contained in the linear span of E_{11} and is adjacent to $sE_{11} + T$. Hence, $\phi(P_1 + \sum_{j=2}^{n-1} t_j P_j) = sE_{11}$. Similarly, $\phi(2P_1 + \sum_{j=2}^{n-1} t_j P_j) = sE_{11}$. But $P_1 + \sum_{j=2}^{n-1} t_j P_j$ and $2P_1 + \sum_{j=2}^{n-1} t_j P_j$ are adjacent, and therefore, $\phi(P_1 + \sum_{j=2}^{n-1} t_j P_j) \neq \phi(2P_1 + \sum_{j=2}^{n-1} t_j P_j)$, a contradiction. ∎

Claim 4.4 *Let m, n be positive integers with $n \geq 3$. Assume that $\phi\colon H_n \to H_m$ is an adjacency preserving map such that $\phi(0) = 0$. Suppose also that for every projection P of rank $n-1$ the restriction of ϕ to PH_nP is a standard map. Then ϕ is a standard adjacency preserving map.*

Proof By our assumption the restriction of ϕ to PH_nP, where $P = E_{11} + \cdots + E_{n-1,n-1}$, is a standard map. Thus, after composing ϕ with an appropriate congruence transformation and the entrywise complex conjugation, if necessary, and after multiplying the map so obtained by -1, if necessary, we may assume that

$$\phi\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \tag{4.2}$$

for every $(n-1) \times (n-1)$ hermitian matrix A.

Our next observation is that $\phi(Q) \geq 0$ for every rank one projection Q. Indeed, for every such Q we can find a rank $n-1$ projection R such that $E_{11}, Q \in RH_nR$. The restriction of ϕ to RH_nR is of standard form. Every standard map either sends every positive definite matrix into a positive definite matrix or sends every positive definite matrix into a negative definite matrix. Since $\phi(E_{11}) = E_{11}$, we have necessarily $\phi(Q) \geq 0$.

We know that $\phi(E_{nn}) = sxx^*$ for some norm one vector x and some positive real number s. We will show that x is linearly independent of $e_1, \dots, e_{n-1} \in \mathbb{C}^m$ (in particular, this shows that $m \geq n$). Otherwise we would have $\phi(E_{nn}) = \phi(Q)$ for some rank one matrix $Q \in (E_{11} + \cdots + E_{n-1,n-1})H_n(E_{11} + \cdots + E_{n-1,n-1})$. Then we would be able to find a rank $n-1$ projection R such that $E_{nn}, Q \in RH_nR$. Since $\phi(E_{nn}) = \phi(Q)$, the restriction of ϕ to RH_nR would not be standard, a contradiction.

Thus, x has to be linearly independent of $e_1, \dots, e_{n-1}$. Therefore there exists an invertible $m \times m$ matrix T such that $Te_i = e_i$, $i = 1, \dots, n-1$, and $Tx = \frac{1}{\sqrt{s}}e_n$. Replacing ϕ with the map $X \mapsto T\phi(X)T^*$ we may assume that $\phi(E_{nn}) = E_{nn}$ and (4.2) hold true.

Set $R_i = I - E_{ii} \in H_n$, $i = 1, \dots, n$. By the induction hypothesis the restriction of ϕ to $R_iH_nR_i$ is linear. It follows that $\phi(R_i) = E - E_{ii} \in H_m$ for every $i = 1, \dots, n$. Here, $E = E_{11} + \cdots + E_{nn}$.

In the next step we will show that $\phi(I)$ is of rank n. Indeed, since I is adjacent to R_i, $i = 1, \dots, n$, the matrix $\phi(I)$ is adjacent to $E - E_{ii}$, $i = 1, \dots, n$. Thus, $\operatorname{rank}\phi(I)$ is either $n-2$, or $n-1$, or n. We have to show that the first two possibilities cannot occur.

Assume first that $\operatorname{rank}\phi(I) = n-2$. Then $E - E_{ii} = \phi(I) + T$ for some rank one matrix T and since $\operatorname{rank}(E - E_{ii}) = \operatorname{rank}\phi(I) + \operatorname{rank} T$, the image of $E - E_{ii}$ is the direct sum of the image of $\phi(I)$ and the image of T. In particular, the image of $\phi(I)$ is a subspace of the linear span of $\{e_1, \dots, e_n\} \setminus \{e_i\}$, $i = 1, \dots, n$. It follows that the image of $\phi(I)$ is the zero subspace, a contradiction.

Consider now the case when $\operatorname{rank}\phi(I) = n-1$. It is easy to see that if two hermitian matrices of the same rank are adjacent, then they have the same image. Indeed, let $T = \sum_{j=1}^r t_jx_jx_j^*$, where the $x_jx_j^*$ s are orthogonal projections and the t_j s are nonzero real numbers, and let S be a rank r hermitian matrix adjacent to T. Then $S = \sum_{j=1}^r t_jx_jx_j^* + syy^*$ for some nonzero vector y and some nonzero real number s. If y was linearly independent of the x_j s, then S would be of rank $r+1$. Thus, y is contained in the linear span of $x_1, \dots, x_r$, and consequently, the image of S is a subspace of the image of T. Because they have the same rank, these two images are actually equal. Using this observation, we see that $\operatorname{Im}\phi(I) = \operatorname{Im}(E - E_{ii}) = \operatorname{span}(\{e_1, \dots, e_n\} \setminus \{e_i\})$ for every $i = 1, \dots, n$, a contradiction.

Thus, we have proved that $\operatorname{rank}\phi(I) = n$. Applying Lemma 2.4 with $P_i = E_{ii}$,

$t_i = 1$, $i = 1, \dots, n$, we can easily get that

$$\phi(I) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

The desired conclusion follows now directly from Claim 4.2. ■

Claim 4.5 *Theorem* 1.2 *holds true in the special case when* $n = 3$.

Proof Let $P \in H_3$ be any projection of rank two. By Claim 4.1, the restriction of ϕ to PH_3P is either standard or degenerate. We already know that the statement holds true if either the restriction of ϕ to PH_3P is degenerate for every projection P of rank two, or the restriction of ϕ to PH_3P is standard for every projection P of rank two.

In order to complete the proof of this special case, we must show that the possibility that there exist rank two projections P and Q such that the restriction of ϕ to PH_3P is degenerate and the restriction of ϕ to QH_3Q is standard cannot occur. Assume, on the contrary, that this is true. Then $m \geq 2$. Without loss of generality we may assume that $Q = E_{11} + E_{22}$. Then, after composing ϕ with a congruence transformation, and with the entrywise complex conjugation, if necessary, and multiplying with -1, if necessary, we may assume that

$$\phi\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \tag{4.3}$$

for every hermitian 2×2 matrix A. Here, of course, the zeroes on the left-hand side stand for the 2×1 zero matrix, the 1×2 zero matrix, and the 1×1 zero matrix, while on the right-hand side, the zeroes denote the $2 \times (m-2)$ zero matrix, the $(m-2) \times 2$ zero matrix, and the $(m-2) \times (m-2)$ zero matrix (in the special case when $m = 2$, the zeroes on the right-hand side are absent). We will show that $\phi(tR) = t\phi(R)$ for every real number t and every hermitian matrix R of rank one. Indeed, this is true if $R \in QH_3Q$. If $R \notin QH_3Q$, then we can find a projection $R_1 \in QH_3Q$ of rank one such that rank one matrices $\phi(R)$ and $\phi(R_1)$ are linearly independent. There exists a rank two projection R_2 such that $R, R_1 \in R_2H_3R_2$. Clearly, the restriction of ϕ to $R_2H_3R_2$ is not degenerate. Hence, it is standard and therefore real-linear, and consequently, $\phi(tR) = t\phi(R)$, $t \in \mathbb{R}$, in this case as well.

Let $T \in H_3$ be any matrix. Define a map $\phi_T \colon H_3 \to H_m$ by

$$\phi_T(X) = \phi(T + X) - \phi(T), \quad X \in H_3.$$

Obviously, ϕ_T is an adjacency preserving map satisfying $\phi_T(0) = 0$. We will show that ϕ_T is of the same type as ϕ, that is, there exist rank two projections P_T and Q_T such that the restriction of ϕ_T to $P_TH_3P_T$ is degenerate and the restriction of ϕ_T to $Q_TH_3Q_T$ is standard. If this were not true, then we already know that ϕ_T would be either standard or degenerate.

If ϕ_T is standard, then

$$\begin{aligned} \phi(Y) &= \phi(T + (Y - T)) = \phi(T) + \phi_T(Y - T) = \phi(T) + \phi_T(Y) - \phi_T(T) \\ &= \phi_T(Y) + Z, \quad Y \in H_3, \end{aligned}$$

where $Z = \phi(T) - \phi_T(T)$. Applying $\phi(0) = \phi_T(0) = 0$, we conclude that $Z = 0$. Thus, ϕ is standard, a contradiction.

If ϕ_T is degenerate, then

$$\phi(Y) = \phi(T + (Y - T)) = \phi(T) + \phi_T(Y - T) = \phi(T) + f(Y)xx^*, \quad Y \in H_3,$$

for some function $f\colon H_3 \to \mathbb{R}$ and some nonzero vector x. Applying $\phi(0) = 0$, we conclude that $\phi(T) \in \operatorname{span}\{xx^*\}$. Thus, ϕ is degenerate, a contradiction.

We have thus proved that ϕ_T is of the same type as ϕ. By the previous step, $\phi_T(tR) = t\phi_T(R)$ for every $t \in \mathbb{R}$, every rank one matrix R, and every $T \in H_3$. Equivalently,

$$\phi(T + tR) = \phi(T) + t(\phi(T + R) - \phi(T)), \quad t \in \mathbb{R}, \tag{4.4}$$

for every $T \in H_3$ and every rank one matrix $R \in H_3$.

We will next prove that ϕ is real-linear. It is enough to show that

$$\phi(A_1 + \cdots + A_p) = \phi(A_1) + \cdots + \phi(A_p)$$

for every positive integer p and arbitrary rank one hermitian matrices $A_1, \dots, A_p$. We will prove this by induction on p. Assume that the statement holds true for p and we want to prove it for $p + 1$. Let $A_1, \dots, A_{p+1} \in H_3$ be any rank one matrices. Using (4.4), we see that for every real t we have

$$\begin{aligned}\phi(A_1 + \cdots + A_p + tA_{p+1}) = \phi(A_1 + \cdots + A_p)&\\ + t[\phi(A_1 + \cdots + A_{p+1}) - \phi(A_1 + \cdots + A_p)]&.\end{aligned}$$

Applying the induction hypothesis, we get

$$\begin{aligned}\phi(A_1 + \cdots + A_p + tA_{p+1}) = \phi(A_1) + \cdots + \phi(A_p)&\\ + t[\phi(A_1 + \cdots + A_{p+1}) - \phi(A_1) - \cdots - \phi(A_p)]&.\end{aligned}$$

Now, for every real t the matrix $\phi(A_1 + \cdots + A_p + tA_{p+1})$ is adjacent to

$$\phi(A_2 + \cdots + A_p + tA_{p+1}) = \phi(A_2) + \cdots + \phi(A_p) + t\phi(A_{p+1}).$$

Hence, the matrix

$$\begin{aligned}&\phi(A_1) + \cdots + \phi(A_p) + t[\phi(A_1 + \cdots + A_{p+1}) - \phi(A_1) - \cdots - \phi(A_p)]\\ &\qquad - \phi(A_2) - \cdots - \phi(A_p) - t\phi(A_{p+1})\\ &= \phi(A_1) + t[\phi(A_1 + \cdots + A_{p+1}) - \phi(A_1) - \cdots - \phi(A_p) - \phi(A_{p+1})]\end{aligned}$$

is of rank one for every real number t. It follows from Lemma 2.5 that

$$\phi(A_1 + \cdots + A_{p+1}) - \phi(A_1) - \cdots - \phi(A_p) - \phi(A_{p+1}) = 0,$$

as desired.

We will show that our assumptions together with (4.3) yield that $\phi(E_{33})$ has all nonzero entries in the upper-left 2×2 corner. If not, then $\phi(E_{33}) = sxx^*$ for some vector $x \notin \operatorname{span}\{e_1, e_2\}$ and some nonzero real number s, which implies that $\phi(I) = \phi(E_{11}+E_{22})+\phi(E_{33}) = E_{11}+E_{22}+sxx^*$ has rank 3. But then, because of the adjacency preserving property, every projection Q of rank two is mapped into a matrix of rank two, and therefore, every restriction of ϕ to QH_3Q, where Q is any projection of rank two, is standard, a contradiction.

Next, there exists a rank one projection $R \in (E_{11} + E_{22})H_3(E_{11} + E_{22})$ such that $\phi(R)$ and $\phi(E_{33})$ are linearly independent. Thus, if R_1 is a projection of rank two with $R_1RR_1 = R$ and $R_1E_{33}R_1 = E_{33}$, then the restriction of ϕ to $R_1H_3R_1$ is standard, and thus, $\phi(E_{33}) \geq 0$. It follows that

$$\phi(E_{33}) = \begin{bmatrix} cP & 0 \\ 0 & 0 \end{bmatrix},$$

where P is a 2×2 projection of rank one and c a positive real number. Let U be a 2×2 unitary matrix such that $U^*PU = E_{22}$. Define a 3×3 matrix T and $m \times m$ matrix V by

$$T = \begin{bmatrix} U & 0 \\ 0 & \frac{1}{\sqrt{c}} \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix}.$$

After replacing ϕ by $X \mapsto V^*\phi(TXT^*)V$, we may assume that both (4.3) and $\phi(E_{33}) = E_{22}$ hold true. Applying Claim 4.1 first for the restriction of ϕ to $(E_{11}+E_{33})H_3(E_{11}+E_{33})$, and then for the restriction of ϕ to $(E_{22}+E_{33})H_3(E_{22}+E_{33})$, using the real-linearity of ϕ we get the existence of real numbers φ, a, b such that either

$$\phi\left(\begin{bmatrix} t_1 & u+iv & w+iz \\ u-iv & t_2 & x+iy \\ w-iz & x-iy & t_3 \end{bmatrix}\right)$$
$$= \begin{bmatrix} t_1 & u+iv+(w+iz)\exp(i\varphi) & 0 \\ u-iv+(w-iz)\exp(-i\varphi) & t_2+t_3+ax+by & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

or

$$\phi\left(\begin{bmatrix} t_1 & u+iv & w+iz \\ u-iv & t_2 & x+iy \\ w-iz & x-iy & t_3 \end{bmatrix}\right)$$
$$= \begin{bmatrix} t_1 & u+iv+(w-iz)\exp(i\varphi) & 0 \\ u-iv+(w+iz)\exp(-i\varphi) & t_2+t_3+ax+by & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$t_1, t_2, t_3, u, v, w, z, x, y \in \mathbb{R}$. We start with the first case. Since

$$\phi\left(\begin{bmatrix} 1 \\ \alpha \\ \beta \end{bmatrix} \begin{bmatrix} 1 & \overline{\alpha} & \overline{\beta} \end{bmatrix}\right)$$

is of rank one for every pair of complex numbers α, β, the determinant of the upper-left 2×2 corner of this matrix must be zero. Hence,

$$|\alpha|^2 + |\beta|^2 + a\operatorname{Re}(\alpha\overline{\beta}) + b\operatorname{Im}(\alpha\overline{\beta}) - |\alpha|^2 - |\beta|^2 - \alpha\overline{\beta}\exp(i\varphi) - \overline{\alpha}\beta\exp(-i\varphi) = 0$$

for every pair of complex numbers α, β. Setting $\beta = 1$ we see that $a\operatorname{Re}\alpha + b\operatorname{Im}\alpha = 2\operatorname{Re}(\alpha\exp(i\varphi))$ for every $\alpha \in \mathbb{C}$. It follows that

$$\phi\left(\begin{bmatrix} 0 \\ 1 \\ -\exp(i\varphi) \end{bmatrix} \begin{bmatrix} 0 & 1 & -\exp(-i\varphi) \end{bmatrix}\right) = 0,$$

a contradiction.

In the second case, in the same way we get that

$$|\alpha|^2 + |\beta|^2 + a\operatorname{Re}(\alpha\overline{\beta}) + b\operatorname{Im}(\alpha\overline{\beta}) - |\alpha|^2 - |\beta|^2 - \overline{\alpha\beta}\exp(-i\varphi) - \alpha\beta\exp(i\varphi) = 0$$

for every pair of complex numbers α, β. Replacing first α by 1, then β by 1, and then comparing the obtained equations one can easily see that $b = 0$. Thus,

$$a\operatorname{Re}(\alpha\overline{\beta}) - 2\operatorname{Re}(\alpha\beta\exp(i\varphi)) = 0,$$

or equivalently, $\operatorname{Re}(\alpha(a\overline{\beta} - 2\beta\exp(i\varphi))) = 0$ for every pair of complex numbers α and β. It follows that $a\overline{\beta} = 2\beta\exp(i\varphi)$ for every $\beta \in \mathbb{C}$. This contradiction completes the proof. ∎

Now we are ready to prove our main result. We will prove it by induction on n. The cases $n = 2$ and $n = 3$ have already been proved. So, let $n \geq 4$ and assume that for every positive integer m every adjacency preserving map from $H_{n-2} \to H_m$ as well as every adjacency preserving map from $H_{n-1} \to H_m$ is either standard, or degenerate.

Let $P \in H_n$ be any projection of rank $n-1$. By the induction hypothesis, the restriction of ϕ to PH_nP is either standard, or degenerate. We will show that if such a restriction is degenerate for one projection P of rank $n-1$, then it is degenerate for all projections P of rank $n-1$. Indeed, assume that P is a projection of rank $n-1$ such that $\phi(PH_nP)$ is contained in the linear span of some projection of rank one and let $Q \in H_n$ be any projection of rank $n-1$. Then there exists a projection R of rank $n-2$ such that $RH_nR \subset PH_nP$ and $RH_nR \subset QH_nQ$. We know that the restriction of ϕ to RH_nR is either standard or degenerate. Since $\phi(PH_nP)$ is contained in the linear span of some projection of rank one, the restriction of ϕ to RH_nR is degenerate, and consequently, the restriction of ϕ to QH_nQ cannot be standard, and therefore it has to be degenerate.

Hence, if a restriction of ϕ to PH_nP is degenerate for some rank $n-1$ projection P, then by Claim 4.3, ϕ is degenerate.

In the remaining case when the restriction of ϕ to PH_nP is of a standard form for every projection P of rank $n-1$ we complete the proof using Claim 4.4. ∎

References

[1] A. D. Alexandrov, *Seminar report.* Uspehi Mat. Nauk **37**(1950), no. 3, 187.

[2] A. D. Alexandrov, *On the axioms of relativity theory.* Vestnik Leningrad Univ. Math. **19**(1976), 5–28.

[3] W.-L. Chow, *On the geometry of algebraic homogeneous spaces.* Ann. of Math. **50**(1949), 32–67.

[4] W. Fulton, *Algebraic Topology: A First Course.* Graduate Texts in Mathematics 153, Springer, New York, 1995.

[5] L. K. Hua, *Geometries of matrices. I. Generalizations of von Staudt's theorem.* Trans. Amer. Math. Soc. **57**(1945), 441–481.

[6] ———, *Geometries of matrices* I_1. *Arithmetical construction.* Trans. Amer. Math. Soc. **57**(1945), 482-490.

[7] ———, *Starting with the Unit Circle.* Springer-Verlag, New York, 1981.

[8] W.-l. Huang, R. Höfer, and Z.-X. Wan, *Adjacency preserving mappings of symmetric and Hermitian matrices.* Aequationes Math. **67**(2004), no. 1-2, 132–139.

[9] J. Lester, *A physical characterization of conformal transformations of Minkowski spacetime.* Ann. Discrete Math. **18**(1983), 567–574.

[10] M. H. Lim, *Rank and tensor rank preservers.* In: A survey of linear preserver problems. Linear and Multilinear Algebra **33**(1992), no. 1-2, 7–21.

[11] I. Popovici and D. C. Rădulescu, *Characterizing the conformality in a Minkowski space.* Ann. Inst. H. Poincaré. Sect. A **35**(1981), no. 2, 131–148.

[12] Z.-X. Wan, *Geometry of Matrices.* World Scientific Publishing, River Edge, NJ, 1996.

Fachbereich Mathematik, Schwerpunkt GD, Universität Hamburg, D-20146 Hamburg, Germany
e-mail: huang@math.uni-hamburg.de

Department of Mathematics, University of Ljubljana, SI-1000 Ljubljana, Slovenia
e-mail: peter.semrl@fmf.uni-lj.si