3

Theories invariant under affine current algebras

In addition to being conformal invariant, it was shown in Chapter 1 that the theory of a free massless scalar field admits also affine algebra currents $J(z)$ and $\bar{J}(\bar{z})$ which are holomorphically and anti-holomorphically conserved, namely $\bar{\partial}J(z,\bar{z}) = \partial\bar{J}(z,\bar{z}) = 0$. The existence of these currents, as was the case with the energy-momentum tensor, implies that the theory is invariant under an infinite-dimensional group of transformations. Inspired by this invariance of the free scalar theory we would like to identify and investigate field theories equipped with an affine current algebra (ACA), which is often refered to as Kac–Moody algebra or affine Lie algebra (ALA). Conformal field theories (CFT) are characterized by the Virasoro anomaly, the set of primary fields and the corresponding structure constants. It will be shown in this chapter that theories with ACA admit a similar algebraic structure and moreover that they are necessarily also CFTs. Thus every ACA theory will be characterized by the Virasoro anomaly as well as its ACA analog, the ACA level. Primary fields that have been defined so far via their operator product expansion (OPE) with the $T(z)$ and $\bar{T}(\bar{z})$ will have to obey certain OPE also with currents $J(z)$ and $\bar{J}(\bar{z})$. The zero modes of the free scalar affine currents J_0 and $\bar{J}_0$ (1.57) commute, namely, they generate an abelian group. For theories with "ordinary" non-affine currents the generalization of the abelian group to non-abelian ones led (in four dimensions) to the standard model of the basic interactions and in fact to an enormously rich spectrum of interesting theories. It is thus very natural to explore the generalization of the abelian ACA to non-abelian affine current algebras. The investigation of two-dimensional theories which are invariant under transformations generated by such affine currents is the subject of this chapter. We start with a brief reminder of the properties of finite dimensional Lie algebras.

The topics included in this chapter are covered in several books and review papers. In particular we have made use of the famous review by Goddard and Olive [111], and the book by Di Francesco, Mathieu and Senechal [77].

3.1 Simple finite-dimensional Lie algebras

Consider the Lie algebra $\mathcal{G}$,

$$[T^a, T^b] = if^{ab}_c T^c, \tag{3.1}$$

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associated with a group G, namely, the set T^a are the generators of the group G.[1] We will consider simple groups, namely those that do not contain invariant subgroups. Denote the maximal set of commuting Hermitian generators by $H^i, i = 1, \ldots, r$ so that

$$[H^i, H^j] = 0 \quad i, j = 1, \ldots, r. \tag{3.2}$$

This abelian subalgebra of $\mathcal{G}$ is referred to as the *Cartan subalgebra*. It can be shown that any two such abelian subalgebras with generators H^i and $\tilde{H}^i$ are conjugate under the action of the group, namely, $\tilde{H}^i = gH^ig^{-1}$ for some $g \in G$. The dimension of the Cartan subalgebra, which is the maximal number of commuting generators is defined as the *rank* of the algebra $\mathcal{G}$, rank $(\mathcal{G}) = r$.

A basis of the full algebra $\mathcal{G}$ constitutes H^i and the *step operators* or *ladder operators* E^α defined by,

$$[H^i, E^\alpha] = \alpha^i E^\alpha, \quad i = 1, \ldots, r. \tag{3.3}$$

The r-dimensional vector α is called a *root* associated with the step operator E^α. The roots are real and up to multiplication with a scalar there is a single E^α associated with α via (3.3). No multiple of a given root α is a root apart from $-\alpha$ which is the root paired with $E^{-\alpha} = E^{\alpha\,\dagger}$. The number of roots is obviously $(\dim\mathcal{G} - \mathrm{rank}\mathcal{G})$.

The rest of the algebra are the commutation relations $[E^\alpha, E^\beta]$ which follow from the Jacobi identity,

$$[H^i, [E^\alpha, E^\beta]] = (\alpha^i + \beta^i)[E^\alpha, E^\beta], \tag{3.4}$$

so that if $(\alpha^i + \beta^i) \neq 0$ and is not a root, then $[E^\alpha, E^\beta] = 0$. If on the other hand $(\alpha^i + \beta^i)$ is a root then $[E^\alpha, E^\beta]$ must be a multiple of $E^{\alpha+\beta}$. If $(\alpha^i + \beta^i) = 0$ it follows that $[E^\alpha, E^{-\alpha}] \sim \alpha \cdot H$.

To summarize the full algebra reads,

$$\begin{aligned}
[H^i, H^j] &= 0 \quad i, j = 1, \ldots, \mathrm{rank}(\mathcal{G}) \\
[H^i, E^\alpha] &= \alpha^i E^\alpha \quad \alpha = 1, \ldots, (\dim G - \mathrm{rank}(\mathcal{G})) \\
[E^\alpha, E^\beta] &= \epsilon(\alpha\beta) E^{\alpha+\beta} \quad \text{if } \alpha + \beta \text{ is a root} \\
&= \frac{2\alpha \cdot H}{\alpha^2} \quad \text{if } \alpha + \beta = 0 \\
&= 0 \quad \text{otherwise.}
\end{aligned} \tag{3.5}$$

This basis of the algebra is a modified version of the *Cartan–Weyl basis*. The constants $\epsilon(\alpha\beta)$ can be chosen to be ± 1 if all the root vectors have the same length.

[1] Finite-dimensional Lie algebra is covered in many books, for instance Cahn [54].

It is straightforward to realize that the triplet of generators $E^{\alpha}, E^{-\alpha}$, and $\frac{\alpha \cdot H}{\alpha^2}$ is isomorphic to J_+, J_-, J_3 of an $SU(2)$ algebra, namely,

$$\left[\frac{\alpha \cdot H}{\alpha^2}, E^{\pm\alpha}\right] = \pm E^{\pm\alpha}, \quad [E^{+\alpha}, E^{-\alpha}] = 2\frac{\alpha \cdot H}{\alpha^2}. \tag{3.6}$$

Consequently the eigenvalues of $2\frac{\alpha \cdot H}{\alpha^2}$, just like those of $2J_3$, are integers in any unitary representation. The eigenvalues associated with each root β is given by $\frac{2\alpha \cdot \beta}{\alpha^2} \in \mathcal{Z}$. It is natural to define the notion of coroot $\alpha^\wedge = \frac{2\alpha}{\alpha^2}$.

3.1.1 The Weyl group

Consider a root β such that $2\alpha \cdot \beta/\alpha^2 \neq 0$ is its eigenvalue under the operation of $2\frac{\alpha \cdot H}{\alpha^2}$. There must be another step operator $E^{\beta+m\alpha}$ which is a member of the $SU(2)$ multiplet with the opposite eigenvalue, namely,

$$2\alpha \cdot \beta/\alpha^2 + 2m = -2\alpha \cdot \beta/\alpha^2. \tag{3.7}$$

Then $m = -2\alpha \cdot \beta/\alpha^2$, and

$$\beta + m\alpha = \beta - 2\frac{(\alpha \cdot \beta)\alpha}{\alpha^2} = \beta - (\alpha^\wedge, \beta)\alpha \equiv \sigma_\alpha(\beta) \tag{3.8}$$

is a root for each pair of roots α and β; $\sigma_\alpha(\beta)$ is a reflection in the hyperplane perpendicular to α. The set of these reflections that permute the roots, generate a finite group $W(\mathcal{G})$, the *Weyl group* of $\mathcal{G}$.

3.1.2 Cartan matrix and Dynkin diagrams

It is convenient to define the notion of *simple roots* as follows. Select a rank($\mathcal{G}$) dimensional basis of the roots that consists of $\alpha_{(i)}$, $i = 1, \ldots, \text{rank}(\mathcal{G})$ in such a way that any root α can be written as,

$$\alpha = \sum_{i}^{\text{rank } \mathcal{G}} n_i \alpha_{(i)}, \tag{3.9}$$

where the n_i are integers which are either all $n_i \geq 0$ or all $n_i \leq 0$. In the former case α is *positive*, while in the latter it is *negative*. This base is called the basis of *simple roots*. Associated with the simple roots one defines the *simple Weyl reflections* $\sigma_{\alpha_{(i)}}$ that generate the entire Weyl group.

The scalar products of simple roots define the *Cartan matrix* as follows:

$$A_{ij} = \frac{2\alpha_{(i)} \cdot \alpha_{(j)}}{\alpha_{(j)}^2}. \tag{3.10}$$

The Cartan matrix is a rank($\mathcal{G}$) $\times$ rank($\mathcal{G}$) matrix with integer components and with diagonal elements which take the value of 2. The off diagonal elements are either negative or vanishing.

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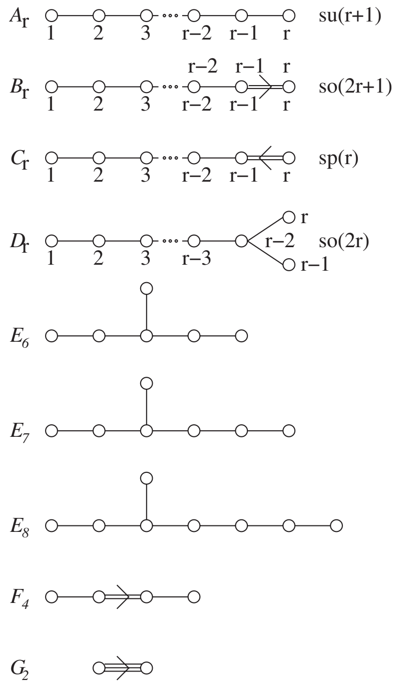

Fig. 3.1. Dynkin diagrams for all the simple Lie algebras.

It can be shown that the roots of a simple Lie algebra can have at most two different lengths, a long root and a short one. The ratio between their lengths are either 2 or 3. When all the roots have the same length the algebra is a *simply laced algebra*. From the Cartan matrix A_{ij} one can reconstruct a basis of simple roots up to a scale and overall orientation. In fact constructing all the roots from the simple roots, one finds that full information on $\mathcal{G}$ is encoded in A_{ij}.

The information contained in the Cartan matrix A_{ij} may be encoded in a planar diagram, the *Dynkin diagram*. The construction of such a diagram is as follows:

- To each simple root $\alpha_{(i)}$ assign a node in the diagram.
- If a node represents a short root mark it by a black dot, and if a long one by a white dot.
- Join the points $\alpha_{(i)}$ and $\alpha_{(j)}$ by $A_{ij}A_{ji}$ lines. For $i \neq j$, $A_{ij}A_{ji}$ can take the values of $0, 1, 2, 3$. In fact since $A_{ij}A_{ji} = 4\cos^2\theta_{ij}$, where θ_{ij} is the angle between the two roots, then orthogonal simple roots are not connected, and those with an angle of $120, 135, 150$ degrees are connected with one, two, or three lines.
- In some conventions one does not separate between black and white dots, but rather one draws an arrow pointing from $\alpha_{(i)}$ to $\alpha_{(j)}$ when $\alpha^2_{(i)} > \alpha^2_{(j)}$.

The Dynkin diagrams for all simple Lie algebras are given in Fig. [3.1].

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3.1.3 Highest weight states

We now consider finite-dimensional representations of $\mathcal{G}$ other than the adjoint representation that has been analyzed so far, the latter being the same as the generators. We can always choose a basis $\{|\mu\rangle\}$ for which,

$$H^i|\mu\rangle = \mu^i|\mu\rangle. \tag{3.11}$$

The rank($\mathcal{G}$) dimensional vector μ of eigenvalues of the Cartan subalgebra generators is called the *weight vector*. A root is a weight of the adjoint representation. In a similar manner to their action as roots, the triplet $E^\alpha, E^{-\alpha}$ and $2\alpha \cdot H/\alpha^2$ form an $SU(2)$ algebra and hence $\{|\mu\rangle\}$ must be an $SU(2)$ multiplet, and in particular,

$$2\alpha \cdot \mu/\alpha^2 \in \mathcal{Z}, \tag{3.12}$$

for any root α.

This property of any given weight defines a lattice $\Lambda_W(\mathcal{G})$, the *weight lattice* of the algebra $\mathcal{G}$. The weights associated with a representation must be mapped into one another under the operation of σ_α and in fact the whole Weyl group. One can choose a basis for the weight lattice $\Lambda_W(\mathcal{G})$ consisting of *fundamental weights* $\lambda_{(i)}$ such that,

$$2\lambda_{(i)} \cdot \alpha_{(j)}/\alpha_{(j)}^2 = \delta_{ij}. \tag{3.13}$$

Any weight λ can then be expanded as $\lambda = \sum n_i \lambda_{(i)}$ with integer coefficients n_i. If all $n_i \geq 0$, the weight λ is called a *dominant weight*. Every weight can be mapped into a unique dominant weight by action of the Weyl group. The dominant weight $\rho = \sum_i \lambda_{(i)}$, where $i = 1, \ldots, \text{rank}\mathcal{G}$, is characterized by $\rho \cdot \alpha > 0$ for any positive α and $\rho \cdot \alpha < 0$ for any negative α. In fact $\rho = 1/2 \sum \alpha$ where the sum is over all the positive roots.

For any given finite-dimensional representation of $\mathcal{G}$ one defines the *highest weight state* $|\mu_0\rangle$ for which $\rho \cdot \mu_0$ is the largest. Such a state is annihilated by all the raising operators,

$$E^\alpha|\mu_0\rangle = 0, \tag{3.14}$$

for every $\alpha > 0$. All the states of a given irreducible representation can be built by acting on the highest weight state with lowering operators, namely, each state takes the form,

$$E^{-\beta_1} \ldots . E^{-\beta_n}|\mu_0\rangle. \tag{3.15}$$

In fact every irreducible representation has a unique highest weight state $|\mu_0\rangle$ and the other weights μ have the property that $\mu_0 - \mu$ is a sum of positive roots. The highest weight state is a dominant weight. In the opposite direction for each dominant weight there is a unique irreducible representation for which it is the highest weight state. As was mentioned for the adjoint representation the weights are

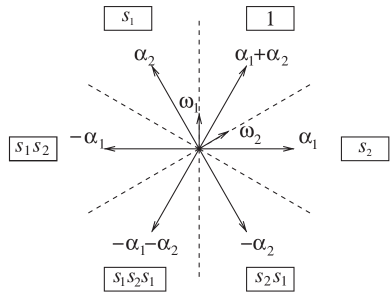

Fig. 3.2. Root system and Weyl chambers of $SU(3)$.

the roots so that the corresponding highest weight state is the *highest root*. The difference between the highest root and any other root is a sum of positive roots.

We end this subsection with an example. Consider the $SU(3)$ algebra. The Cartan matrix for this algebra takes the form,

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \tag{3.16}$$

The simple roots are related to the fundamental weights as $\alpha_{(1)} = 2\lambda_{(1)} - \lambda_{(2)}$ and $\alpha_{(2)} = -\lambda_{(1)} + 2\lambda_{(2)}$. The scalar products between the fundamental weights are $(\lambda_{(1)}, \lambda_{(1)}) = (\lambda_{(2)}, \lambda_{(2)}) = 2/3$ and $(\lambda_{(1)}, \lambda_{(2)}) = 1/3$, using the standard normalization of $(\alpha_{(i)}, \alpha_{(i)}) = 2$. The full Weyl group is given by $W = \{1, \sigma_1, \sigma_2, \sigma_1\sigma_2, \sigma_2\sigma_1, \sigma_1\sigma_2\sigma_1\}$. The action of the different elements of the Weyl group on the two simple roots gives all roots.

The root system and the Weyl chambers are given in Fig. 3.2. The Weyl chambers are separated by the dashed lines, and are specified here by the elements of the Weyl group, with the latter denoted in the figure by s_i. The Weyl chambers are defined by,

$$C_\omega = \{\lambda | (\omega\lambda, \alpha_i) \geq 0, i = 1 \ldots r\}.$$

3.2 Affine current algebra

In the previous subsection we acquired a certain familiarity with the notions of roots, highest weights, Cartan matrices, Dynkin diagrams etc., in the context of a simple Lie algebra. As was explained in the introduction to this chapter, two-dimensional CFTs are characterized by an extended algebraic structure, that of affine Lie algebra.[2] We now describe the basic properties of the affine Lie algebra using the notions of the previous subsection.

[2] Affine Lie algebras were introduced into the physics literature by Bardacki and Halpern in [27]. Independently V. Kac [136] and R.V. Moody [164] introduced them in the mathematical literature.

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The basic ALA is given by,

$$[J^a_m, J^b_n] = i f^{ab}_c J^c_{m+n} + \hat{k} m \delta_{ab} \delta_{m+n,0}, \tag{3.17}$$

where the central element $\hat{k}$ commutes with all the generators J^a_m, namely, $[J^a_m, \hat{k}] = 0$.

We now use a generalization of the Cartan–Weyl basis to the affine algebra as follows:

$$\begin{aligned}
[H^i_m, H^j_n] &= \hat{k} m \delta^{ij} \delta_{m+n,0} \\
[H^i_m, E^\alpha_n] &= \alpha^i E^\alpha_{m+n}, \\
[E^\alpha_m, E^\beta_n] &= \epsilon(\alpha\beta) E^{\alpha+\beta}_{m+n} \quad \text{if } \alpha+\beta \text{ is a root} \\
&= \frac{2}{\alpha^2}(\alpha \cdot H_{m+n} + \hat{k} m \delta_{m+n}) \quad \text{if } \alpha+\beta = 0 \\
&= 0 \quad \text{otherwise.} \\
[H^i_m, \hat{k}] &= [E^\alpha_n, \hat{k}] = 0.
\end{aligned} \tag{3.18}$$

We have used the normalization $(H^i, H^j) = \delta^{ij}$, $(E^\alpha, E^\beta) = \frac{2}{\alpha^2}\delta_{\alpha+\beta}$, where (X, Y) denotes the Killing form, defined as the trace of the product in the adjoint representation, $Tr(a\mathrm{d}X, a\mathrm{d}Y)$. The hermiticity properties are,

$$H^i_m{}^\dagger = H^i_{-m}, \quad E^\alpha_m{}^\dagger = E^{-\alpha}_{-m}, \quad \hat{k}^\dagger = \hat{k} \tag{3.19}$$

Unlike the simple Lie algebra, here we have an $r+1$ dimensional abelian subalgebra consisting of $[H^1_0, \ldots, H^r_0, \hat{k}]$. With respect to these generators, E^α_m are step operators,

$$[H^i_0, E^\alpha_m] = \alpha^i E^\alpha_m \quad [\hat{k}, E^\alpha_n] = 0. \tag{3.20}$$

Each of the eigenvectors $(\alpha^1, \ldots, \alpha^r, 0)$ is infinitely degenerate since it is independent of m. Moreover this abelian subalgebra is not maximal since $[H^i_0, H^j_n] = 0$. Thus one has to extend the algebra by adding a grading operator which can be taken to be L_0 such that,

$$[L_0, J^a_n] = -n J^a_n \quad [L_0, \hat{k}] = 0. \tag{3.21}$$

Using the generators $(H^i_0, \hat{k}, -L_0)$ as in the Cartan subalgebra we have as the step operators E^α_n corresponding to a root $(\alpha, 0, n)$ and H^i_n corresponding to $(0, 0, n)$.

The root system of ALA is thus infinite but spans an $(r+1)$ dimensional space. We can divide the roots into positive and negative according to the following rule:

$$(\alpha, 0, n) > 0 \quad \text{if } n > 0, \text{ or if } n = 0 \text{ and } \alpha > 0 \tag{3.22}$$

The basis of the simple roots can therefore be taken as,

$$\begin{aligned}
\alpha_{(i)} &= (\alpha_i, 0, 0), \quad 1 \le i \le r \\
\alpha_{(0)} &= (-\theta, 0, 1),
\end{aligned} \tag{3.23}$$

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where α_i is the basis of simple roots of the Lie algebra, and θ is the highest root. Thus an arbitrary root of the ALA can be expressed as,

$$\alpha = \sum_{i=0}^{r} n_i \alpha_{(i)}. \tag{3.24}$$

It is positive if $n_i \geq 0$ and negative when $n_i \leq 0$, and these are the only two posibilities.

3.2.1 Cartan matrix and Dynkin diagrams

The first step is to define the scalar product $\langle X, Y \rangle$ which has to be symmetric and obey the relation,

$$\langle X, [Y, Z] \rangle + \langle Y, [X, Z] \rangle = 0$$

for $X, Y, Z \in \hat{g}$, the ALA. Upon using a convenient normalization one can bring the basic scalar products to the following form,

$$\begin{aligned} \langle T^a_m, T^b_n \rangle &= \delta^{ab} \delta_{m+n} \\ \langle T^a_m, -L_0 \rangle &= 0 \\ \langle T^a_m, \hat{k} \rangle &= 1 \\ \langle \hat{k}, \hat{k} \rangle &= 1 \\ \langle \hat{k}, -L_0 \rangle &= 1 \\ \langle -L_0, -L_0 \rangle &= 0. \end{aligned} \tag{3.25}$$

The last relation is actually a choice, following on from the invariance of the algebra under a shift of L_0 by a multiple of $\hat{k}$. Here we use T instead of δ used previously.

In the following Hermitian basis,

$$T^i_0, \quad \frac{(T^i_m + T^i_{-m})}{\sqrt{2}}, \quad \frac{(\hat{k} - L_0)}{\sqrt{2}}, \quad \frac{(\hat{k} + L_0)}{\sqrt{2}}, \tag{3.26}$$

the scalar product is Lorentzian since the norm of all the first three basis vectors is $+1$ while that of $\frac{(\hat{k}+L_0)}{\sqrt{2}}$ is -1.

The Lorentzian signature holds also for the Cartan sub-algebra (CSA) generators. One can define the scalar product of two vectors of simultaneous eigenvalues of the CSA,

$$m^i = (\mu^i, \mu^i_k, \mu^i_{-L_0}), \quad m^j = (\mu^j, \mu^j_k, \mu^j_{-L_0})$$

to be,

$$m^i \cdot m^j = \mu^i \cdot \mu^j + \mu^i_k \cdot \mu^j_{-L_0} + \mu^i_{-L_0} \cdot \mu^j_k. \tag{3.27}$$

In particular the scalar product of two roots,

$$a^i = (\alpha^i, 0, n^i), \quad a^j = (\alpha^j, 0, n^j)$$

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is

$$a^i \cdot a^j = \alpha^i \cdot \alpha^j. \tag{3.28}$$

The root that corresponds to E_n^α, $a = (\alpha, 0, n)$ has a norm $a^2 = \alpha^2 > 0$ and hence is referred to as a space-like root, whereas the root that is associated with H_n^i, $n\delta = (0, 0, n)$ has zero norm (light-like) and is orthogonal to all other roots. We have used the "unit" of $\delta = (0, 0, 1)$.

The *Cartan matrix* of $\hat{g}$, which is an $(r+1) \times (r+1)$ matrix, is defined in a similar way to one of the Lie algebra, namely,

$$A_{ij} = \frac{2a_{(i)} \cdot a_{(j)}}{a_{(j)}^2} \quad 0 \le i, j \le r. \tag{3.29}$$

We add to the Cartan matrix of the Lie algebra, the extra row and column A_{i0} and A_{0i} which can be found using (3.10) with $\alpha_0 = -\theta$. Now from the definition of the fundamental weight it follows that $\theta = -\sum_{i=0}^r A_{0i}\lambda_i$. Since θ is a long root of g, namely, $\theta^2 \ge \alpha_{(i)}^2$ one gets that,

$$\begin{aligned} -A_{i0} &= 1 \quad \text{if } A_{0i} \neq 0 \\ -A_{i0} &= 0 \quad \text{if } A_{0i} = 0, \end{aligned} \tag{3.30}$$

provided that θ is not itself a simple root, as happens for $SU(2)$. The Dynkin diagram of $\hat{g}$ is obtained using that of g appended with an extra point that corresponds to α_0 connected by $-A_{0i}$ lines to the points $a_{(i)}$. If $-A_{0i} > 1$ an arrow is drawn which points toward $a_{(0)}$. We demonstrate the construction in the following example:

$\hat{SU}(2)$ - There are only two simple roots $a_{(0)} = (-\alpha, 0, 1)$ and $a_{(1)} = (\alpha, 0, 0)$ so that $A_{0i} = A_{i0} = -2$ and the Cartan matrix is

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}. \tag{3.31}$$

Thus, there are two roots of equal length connected by four lines with arrows pointing in both directions to indicate that $a_{(0)}^2$ is equal to $a_{(1)}^2$.

The Dynkin diagrams of the affine simple algebra are shown in Fig. 3.3. The point that corresponds to $\alpha_{(0)}$ is marked by a zero. The black dots relate to the notion of twisted affine Lie algebra which we do not discuss here (see for example [111]).

3.2.2 The Weyl group

The Weyl group of $\hat{g}$ is defined in a similar way to that of g, namely it is generated by reflections in the hyperplanes normal to a,

$$\sigma_a(b) = b - 2\left(\frac{b \cdot a}{a^2}\right) a = (\sigma_\alpha(\beta), 0, n_a - 2\left(\frac{\alpha \cdot \beta}{\alpha^2}\right) n_\beta), \tag{3.32}$$

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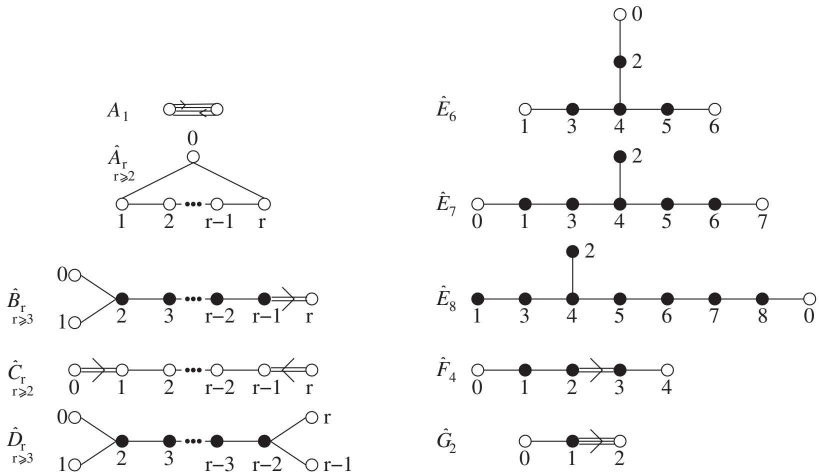

Fig. 3.3. Dynkin diagrams for all the affine simple Lie algebras.

for $a = (\alpha, 0, n_\alpha)$ and $b = (\beta, 0, n_\beta)$ a space-like root. Light-like roots are invariant under such reflections $\sigma_a(n\delta) = n\delta$. In fact the Weyl group of $\hat{g}$ is a semi-direct product of the Weyl group of g and the coroot lattice of g which is the lattice generated by the coroots $\alpha^\nu = \frac{\alpha}{\alpha^2}$. The coroots form the root system of the Lie algebra dual to g obtained by interchanging the root lengths. The simply laced algebras A_n, D_n, E_n are obviously self-dual, as are F_4 and G_2, whereas $B_n^\nu = C_n$ and $C_n^\nu = B_n$.

3.2.3 Highest weight representations

A highest weight state $|\hat{\mu}_0\rangle$ is a state that is annihilated by all the raising operators for positive roots, namely,

$$E_0^\alpha|\hat{\mu}_0\rangle = E_n^{\pm\alpha}|\hat{\mu}_0\rangle = H_n^i|\hat{\mu}_0\rangle = 0, \tag{3.33}$$

for $n > 0, \alpha > 0$. The eigenvalue of this state is the highest weight vector $\hat{\mu}_0 = (\mu_0^i, k, h)$ given by,

$$H_0^i|\hat{\mu}_0\rangle = \mu_0^i|\hat{\mu}_0\rangle, \quad \hat{k}|\hat{\mu}_0\rangle = k|\hat{\mu}_0\rangle, \quad L_0|\hat{\mu}_0\rangle = h|\hat{\mu}_0\rangle. \tag{3.34}$$

We can set h to zero as a matter of convention. A highest weight representation is characterized by a unique highest weight state. To have a unitary highest weight representation the following necessary and sufficient conditions have to be obeyed:

$$\frac{2k}{\theta^2} \in \mathcal{Z} \quad k \geq \theta \cdot \mu_0 \geq 0. \tag{3.35}$$

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The non-negative integer $\frac{2k}{\theta^2}$ is the *level* of the representation. Any state in the representation is characterized by a weight vector $\hat{\mu} = (\mu^i, k, h)$ such that $\mu_0 - \mu$ is a sum of positive roots. Introduce now a set of fundamental weights $l_{(i)}$ for $\hat{g}$, $i = 0, \ldots r$, such that $2l_{(i)} \cdot a_{(j)}/a^2_{(j)} = \delta_{ij}$. The general solution of the condition $2a \cdot \hat{\mu}/\theta^2 \in \mathcal{Z}$, which is equivalent to the condition Eq. (3.35) for $a = a_{(0)} = (-\theta, 0, 1)$, now takes the form $\hat{\mu}_0 = \sum_{i=0}^{r} n_i l_{(i)}$, where n_i are non-negative integers, apart from the indeterminate component in the L_0 direction. For $l_{(i)}$ one finds,

$$l_{(i)} = \left(\lambda_{(i)}, \frac{1}{2} m_i \theta^2, 0\right) \quad l_{(0)} = \left(0, \frac{1}{2} m_0 \theta^2, 0\right), \tag{3.36}$$

where $m_0 = 1$ and where the integers m_i are defined via $\theta/\theta^2 = \sum_{i=0}^{r} m_i \alpha_{(i)}/\alpha^2_{(i)}$. The corresponding level is given by,

$$\text{level} = \sum_{i=0}^{r} n_i m_i. \tag{3.37}$$

Level 1 representations are thus associated with highest weights $l_{(i)}$ with all $m_i = 1$. Those are indicated by open points in Fig. 3.3.

From the definition of m_i it follows that,

$$\sum_{i=0}^{r} A_{ij} m_j = 0. \tag{3.38}$$

Since the Cartan matrix has the basic symmetry of the extended Dynkin diagram, also the positions of the open dots have to preserve this symmetry. For the classical groups A_r, B_r, C_r, D_r the values of m_i for the closed dots is 2. For the exceptional groups the vector $(m_0, \ldots, m_r)$ is as follows

$$\begin{array}{cc} \hat{E}^6 & (1,1,2,2,3,2,1) \\ \hat{E}^7 & (1,2,2,3,4,3,2,1) \\ \hat{E}^8 & (1,2,4,6,5,4,3,2,1) \\ \hat{F}^4 & (1,2,3,2,1) \\ \hat{G}^2 & (1,2,1) \end{array} \tag{3.39}$$

3.3 Current OPEs and the Sugawara construction

In Section 1.8 for the free scalar theory it was shown that the OPE of two currents $J(z)J(w)$ takes the form of $J(z)J(w) = \frac{1}{(z-w)^2} +$ finite terms. This type of OPE, which corresponds to the abelian ALA, is generalized following the discussion in Section 3.2 to,

$$J^a(z)J^b(w) = \frac{k\delta^{ab}}{(z-w)^2} + i\frac{f^{ab}_c J^c(w)}{(z-w)} + \text{finite terms}. \tag{3.40}$$

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We can now use the OPE to evaluate the infinitesimal transformation of the current under ALA transformations,

$$\begin{aligned}\delta_\epsilon J^a(w) &= \frac{1}{2\pi i}\oint_w \mathrm{d}z\epsilon^b(z)J^b(z)J^a(w)\\ &= \frac{1}{2\pi i}\oint_w \mathrm{d}z\epsilon^b(z)\left[\frac{k\delta^{ab}}{(z-w)^2}+i\frac{f^{ab}_c J^c(w)}{(z-w)}\right] = if^a_{bc}\epsilon^b J^c - k\partial\epsilon^a. \end{aligned}\quad (3.41)$$

The same structure also holds for $\bar{J}^a(\bar{z})$.

The OPE form of the ALA can be transformed into a commutator form of the algebra. We introduce a Laurent expansion of the currents,

$$J^a(z) = \sum_n J^a_n z^{-(n+1)} \qquad J^a_n = \frac{1}{2\pi i}\oint \mathrm{d}z z^n J^a(z). \quad (3.42)$$

Substituting the OPE into the expression of the commuation relation we indeed find the ALA of (3.17), namely

$$[J^a_m, J^b_n] = if^{ab}_c J^c_{m+n} + \hat{k}m\delta_{ab}\delta_{m+n,0}. \quad (3.43)$$

In free scalar theory there are two "currents" which are holomorphically conserved, J and T, and moreover the energy-momentum tensor is bilinear in J, as was shown in Section 1.5. We now elevate this special case into a general construction of T for theories which admit ALA structure. The construction is known as the *Sugawara construction*. One writes T as a normal ordered product of the currents,

$$T(z) = \frac{1}{2\kappa} : J^a(z)J_a(z) : \quad (3.44)$$

with a coefficient κ that has to be determined quantum mechanically. In fact, one way to determine κ is by requiring that J^a is a primary field of weight 1, namely,

$$T(z)J^a(w) = \frac{J^a}{(z-w)^2} + \frac{\partial J^a(w)}{(z-w)}. \quad (3.45)$$

Using the OPE (3.40) and the relation $-f^{ab}_c f^c_{bd} = 2C\delta^a_d$, where C is the dual Coxeter number, we find that $\frac{(k+C)}{\kappa} = 1$ so that the form of the Sugawara constructed T is,

$$T(z) = \frac{1}{2(k+C)} : J^a(z)J_a(z) : \quad (3.46)$$

Note that the Casimir of the adjoint is $2C$. Note also that in Section 1.5, for the free scalar case, we had a relative minus sign, due to a difference of factor i in defining the currents there.

In the WZW models discussed in the next section, classically one has T with a coefficient of $\frac{1}{2k}$. It is thus clear that for those models quantum mechanically, due to the double contraction, we get a finite renormalization of the level $k \to k + C$.

Here we have used currents which are in an orthonormal basis. If instead we express the current in the Cartan–Weyl basis used in the previous sections, the form of T in terms of the Cartan sub-algebra generators H^i and the step operators E^α is,

$$T(z) = \frac{1}{2(k+C)}\left[: H^i(z)H^i(z) : + \frac{|\alpha|^2}{2}(E^\alpha E^{-\alpha} + E^{-\alpha}E^\alpha)\right]. \tag{3.47}$$

The OPEs (3.45) and (3.40) also enable us to determine c, the Virasoro anomaly of the model, via the computation,

$$\begin{aligned}
T(z)T(w) &= T(z)\frac{1}{2(k+C)} : J^a J^a : (w) \\
&= \frac{1}{(k+C)}\left\{\frac{J^a(z)J^a(w)}{(z-w)^2} + \frac{\partial J^a(z)J^a(w)}{(z-w)}\right\} \\
&= \frac{1}{(k+C)}\left\{\frac{k(dim\ G)}{(z-w)^4} + \frac{: J^a J^a : (w)}{(z-w)^2} + \frac{1}{2}\frac{\partial : J^a J^a : (w)}{(z-w)}\right\} \\
&= \frac{1}{(k+C)}\frac{k(dim\ G)}{(z-w)^4} + 2\frac{T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}.
\end{aligned} \tag{3.48}$$

We thus read off the Virasoro anomaly

$$c = \frac{k\text{dim } G}{k+C}. \tag{3.49}$$

The construction of T in terms of a normal ordered product of two currents calls for combining together the ALA and the Virasoro algebra. Substituting into (3.46) the mode expansions, of $T(z)$ in terms of L_n and of $J(z)$ in terms of J_n, one finds that,

$$L_n = \frac{1}{2(k+C)}\sum_m : J^a_{n-m}J^a_m : \tag{3.50}$$

where here the normal ordering implies putting the currents with positive m to the right. In fact normal ordering is required only for L_0.

Using this relation, we write down the full Virasoro algebra and ALA,

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}(n-1)n(n+1)\delta(n+m) \\
[L_n, J^a_m] &= -mJ^a_{m+n} \\
[J^a_m, J^b_n] &= if^{ab}_c J^c_{m+n} + \hat{k}m\delta_{ab}\delta_{m+n,0}.
\end{aligned} \tag{3.51}$$

In mathematical terminology the Virasoro algebra belongs to the enveloping algebra of the ALA.

3.4 Primary fields

Recall that the operators of any CFT were shown to be either Virasoro primaries or descendants. The former were defined by their OPE with T. In a similar manner ALA primaries $\Phi_{l,\bar{l}}(z,\bar{z})$ are defined via their OPE with J,

$$\begin{aligned} J^a(z)\Phi_{l,\bar{l}}(w,\bar{w}) &= \frac{T^a_l \Phi_{l,\bar{l}}(w,\bar{w})}{(z-w)} \\ \bar{J}^a(z)\Phi_{l,\bar{l}}(w,\bar{w}) &= \frac{T^a_{\bar{l}} \Phi_{l,\bar{l}}(w,\bar{w})}{(\bar{z}-\bar{w})}, \end{aligned} \tag{3.52}$$

where $T^a_l, T^a_{\bar{l}}$ are the matrix T^a in the $l, \bar{l}$ representations, for the holomorphic and antiholomorphic sectors, respectively. From here on we discuss only holomorphic properties. In terms of the Laurent components J^a_n the condition for a primary field reads,

$$J^a_n \Phi_{l,\bar{l}}(0,\bar{z}) = 0 \quad \text{for } n > 0; \quad J^a_0 \Phi_{l,\bar{l}}(z,\bar{z}) = T^a_l \Phi_{l,\bar{l}}(z,\bar{z}). \tag{3.53}$$

In theories where the energy-momentum tensor can be constructed in a Sugawara construction it is easy to see that the ALA primaries are also Virasoro primaries. Indeed, using (3.50) we see that L_n for $n > 0$ annihilates the ALA primary. For L_0 acting on the primaries we get,

$$L_0 \Phi_l = \frac{1}{2(k+C)} J^a_0 J^a_0 \Phi_l = \frac{C_2(l)}{2(k+C)} \Phi_l. \tag{3.54}$$

Thus the primaries in theories equipped with the Sugawara construction, for instance the WZW models that will be discussed in the next section, obey (3.53) and also,

$$L_n \Phi_{l,\bar{l}}(0,\bar{z}) = 0 \quad \text{for } n > 0; \quad L^a_0 \Phi_{l,\bar{l}}(z,\bar{z}) = \frac{C_2(l)}{2(k+C)} \Phi_{l,\bar{l}}(z,\bar{z}). \tag{3.55}$$

Recall that T is not a Virasoro primary but rather is a descendant of the identity $T(0) = L_{-2}I$. The same applies to $J(z)$. From the mode expansion (3.42) it is clear that,

$$J^a(0) = J^a_{-1} I. \tag{3.56}$$

Note however that $J^a(z)$ is a Virasoro primary. Apart from the distinguished descendant J there are descendant operators of all the primaries. In fact all the local operators can be written as,

$$J^{a_1}_{-n_1} \dots J^{a_N}_{-n_N} \bar{J}^{\bar{a}_1}_{-\bar{n}_1} \dots \bar{J}^{\bar{a}_{\bar{N}}}_{-\bar{n}_{\bar{N}}} \Phi_{l,\bar{l}}(z,\bar{z}), \tag{3.57}$$

and in the case of a Sugawara construction all the operators are of the form,

$$L_{-m_1} \dots L_{-m_M} \bar{L}_{-\bar{m}_1} \dots \bar{L}_{-\bar{m}_{\bar{M}}} J^{a_1}_{-n_1} \dots J^{a_N}_{-n_N} \bar{J}^{\bar{a}_1}_{-\bar{n}_1} \dots \bar{J}^{\bar{a}_{\bar{N}}}_{-\bar{n}_{\bar{N}}} \Phi_{l,\bar{l}}(z,\bar{z}). \tag{3.58}$$

3.5 ALA characters

In Section 2.8 we introduced the notion of the Virasoro character (2.45) which characterizes the structure of the Virasoro Verma module. In a complete analogy

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let us now define a character of the CFT and ALA module $\hat{\lambda}$ as follows:

$$\chi_{\hat{\lambda}}(z^j,\tau) = \mathrm{e}^{-im_{\hat{\lambda}}\delta} Tr_{\hat{\lambda}}[\mathrm{e}^{2\pi i\tau L_0} e^{-2\pi i\sum_j z_j h^j}], \tag{3.59}$$

where $m_{\hat{\lambda}}$, δ and h^j are the generators of the Cartan subalgebra associated with the group, and z_j are complex numbers. The character can also be expressed in terms of the generalized theta function $\Theta_{\hat{\lambda}}$ in the following form:

$$\chi_{\hat{\lambda}}(z^j,\tau) = \frac{\sum_{w\in W}\epsilon(w)\Theta_{w(\hat{\lambda}+\hat{\rho})}}{\sum_{w\in W}\epsilon(w)\Theta_{w\hat{\rho}}}, \tag{3.60}$$

where the sums are over the elements of the finite Weyl group, $\epsilon(w) = (-1)^{l(w)}$ with $l(w)$ the length of w.

Rather than defining the generalized theta function for any ALA at any level, we define here only the function for $\hat{SU}(2)$ level k. For this case we have,

$$\Theta^{(k)}_{\lambda_1}(z;\tau;t) = \mathrm{e}^{-2\pi kt}\sum_{n\in\mathcal{Z}} \mathrm{e}^{-2\pi i[knz+\frac{1}{2}\lambda_1 z - kn^2\tau - \frac{\lambda_1^2\tau}{4k}]}. \tag{3.61}$$

with $\Theta^{(k)}_{\lambda_1}(z;\tau;o) \equiv \Theta(k-\lambda_1,\lambda_1)$ (see [77] for details).

In terms of this function the character of $\hat{SU}(2)_k$ takes the form,

$$\chi_{\hat{\lambda}} = \frac{\Theta^{(k+2)}_{\lambda_1+1} - \Theta^{(k+2)}_{-\lambda_1-1}}{\Theta^{(2)}_1 - \Theta^{(2)}_{-1}}, \tag{3.62}$$

where $\hat{\lambda} = [k-\lambda_1,\lambda_1]$. For the special point $(z=0, t=0)$ the character expressed in terms of $q = \mathrm{e}^{2\pi i\tau}$ reads,

$$\chi_{\hat{\lambda}}(q) = q^{\frac{(\lambda_1+1)^2}{4(k+2)}-\frac{1}{8}} \frac{\sum_{n\in\mathcal{Z}}[\lambda_1+1+2n(k+2)]q^{n[\lambda_1+1+(k+2)n]}}{\sum_{n\in\mathcal{Z}}[1+4n]q^{n[1+2n]}}. \tag{3.63}$$

For level one and for $k=\lambda_1=1$ we get,

$$\begin{aligned}\chi_{\hat{\lambda}}(q) &= q^{\frac{5}{24}}\frac{(2-4q+8q^5-10q^8+\ldots)}{(1-3q+5q^3-7q^6+9q^{10}+\ldots)} \\ &= q^{\frac{5}{24}}(2+2q+6q^2+8q^3+\ldots).\end{aligned} \tag{3.64}$$

The content of the four first grades of the module $[k-\lambda_1=0,\lambda_1=1]$ is $(1),(1),(3)\oplus(1),(3)\oplus 2(1)$, so that the number of the states in these different grades is indeed 2,2,6 and 8 as in the expression of the character.

3.6 Correlators, null vectors and the Knizhnik–Zamolodchikov equation

Correlators of Virasoro primaries were subjected to local and global Ward identities, (2.61) and (2.56), respectively. We now derive their ALA duals. Performing a group transformation of a given correlator,

$$\begin{aligned}&\left\langle \oint \frac{\mathrm{d}z}{2\pi i}\epsilon^a(z)J^a(z)\phi_{l_1}(w_1,\bar{w}_1)\ldots\phi_{l_n}(w_n,\bar{w}_n)\right\rangle\\&=\sum_{i=1}^{n}\left\langle \phi_{l_1}(w_1,\bar{w}_1)\ldots\oint \frac{\mathrm{d}z}{2\pi i}\epsilon^a(z)J^a(z)\phi_{l_i}(w_i,\bar{w}_i)\ldots\phi_{l_n}(w_n,\bar{w}_n)\right\rangle\\&=\sum_{i=1}^{n}\langle \phi_{l_1}(w_1,\bar{w}_1)\ldots\delta_\epsilon\phi_{l_i}(w_i,\bar{w}_i)\ldots\phi_{l_n}(w_n,\bar{w}_n)\rangle\,.\end{aligned} \tag{3.65}$$

Now from the OPE (3.52) we know that $\delta_\epsilon\phi_{l_i}(w_i,\bar{w}_i)=\epsilon^a(w_i)T^a_{l_i}\phi_{l_i}(w_i,\bar{w}_i)$. Since this holds for arbitrary ϵ we can get a local form of the Ward identity in the form,

$$\langle J^a(z)\phi_{l_1}(w_1,\bar{w}_1)\ldots\phi_{l_n}(w_n,\bar{w}_n)\rangle=\sum_{i=1}^{n}\frac{T^a_{l_i}}{(z-w_i)}\langle \phi_{l_1}(w_1,\bar{w}_1)\ldots\phi_{l_n}(w_n,\bar{w}_n)\rangle\,. \tag{3.66}$$

As for the global Ward identity, we use the fact that the correlator has to be invariant under global g transformations (constant ϵ^a), namely,

$$\delta^a_\epsilon\,\langle \phi_{l_1}(w_1,\bar{w}_1)\ldots\phi_{l_n}(w_n,\bar{w}_n)\rangle=0,$$

leading to

$$\sum_{i=1}^{n}T^a_{l_i}\,\langle \phi_{l_1}(w_1,\bar{w}_1)\ldots\phi_{l_n}(w_n,\bar{w}_n)\rangle=0. \tag{3.67}$$

Null vectors of CFTs were found to be useful in Section 2.9, since they lead to differential equations for certain correlators. In a similar manner one can write down null vectors of ALA. In the context of the Sugawara construction, due to the link between the Virasoro algebra generator T and the ALA generators J^a, there are null vectors that combine generators from both infinite algebras. We discuss now an important example of this class that leads to the *Knizhnik–Zamolodchikov* equations. Consider, at Virasoro level one, the following null vector,

$$|\mathrm{null}\rangle=\left\{L_{-1}-\frac{1}{k+C}J^a_{-1}T^a_{l_i}\right\}|\Phi_{l_i}\rangle\,. \tag{3.68}$$

It is easy to see that this is indeed a null state, following (3.52). If we insert the corresponding null operator into a correlation function of primary fields, like $<\Phi_1(z_1)\ldots\mathrm{null}(z_i)\ldots\Phi_n(z_n)>$, the latter must vanish and hence we get,

$$\left\{\partial_i-\frac{1}{k+C}\sum_{j\neq i}\frac{T^a_iT^a_j}{(z_i-z_j)}\right\}<\Phi_1(z_1)\ldots\Phi_n(z_n)>=0. \tag{3.69}$$

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In the derivation, we use,

$$
\begin{aligned}
&<\Phi_1(z_1)\dots J^a_{-1}\Phi_i(z_i)\dots\Phi_n(z_n)> \\
&= \frac{1}{2\pi i}\oint_{z_i}\frac{\mathrm{d}z}{z-z_i}<\Phi_1(z_1)\dots J^a(z)\Phi_i(z_i)\dots\Phi_n(z_n)> \\
&= \frac{1}{2\pi i}\oint_{z_i,j\neq i}\frac{\mathrm{d}z}{z-z_i}\sum_{j\neq i}\frac{T^a_j}{(z-z_j)}<\Phi_1(z_1)\dots\Phi_j(z_j)\dots\Phi_n(z_n)> \\
&= \sum_{j\neq i}\frac{T^a_j}{(z_i-z_j)}<\Phi_1(z_1)\dots\Phi_n(z_n)> \, .
\end{aligned}
\tag{3.70}
$$

For the case of four-point functions, as the correlator depends only on the cross-ratio coordinate $\mathcal{Z}=\frac{z_{12}z_{34}}{z_{13}z_{24}}$, the partial differential equations reduce to an ordinary differential equation. In Section 4.4 we will demonstrate a solution of the Knizhnik–Zamolochikov equation for a four-point function.

3.7 Free fermion realization

In the previous chapter the theories of massless free Dirac and Majorana fermions were analyzed as examples of CFTs. In particular it was shown that the Dirac fermion admits an abelian ALA structure. It is thus natural to expect that the theory of N fermions should be invariant under the transformations associated with non-abelian ALAs.[3] Indeed, it will be shown in this section that an $\hat{SO}(N)$ ALA, and a $\hat{U}(N)$ are the underlying algebraic structures of N free massless Majorana fermions and N Dirac fermions, respectively. We start with the former case.

3.7.1 Free Majorana fermions and $\widehat{SO}(N)$

Consider a generalization of the action given in Section 2.11 for N Majorana fermions,

$$
S=\frac{1}{8\pi}\int \mathrm{d}^2z\sum_{i=1}^{N}\{\psi_i\bar{\partial}\psi_i+\tilde{\psi}_i\partial\tilde{\psi}_i\},
\tag{3.71}
$$

where ψ and $\tilde{\psi}$ are left and right Weyl–Majorana fermions, respectively. Note that this is possible in 2d, and in any other dimension that is 2 modulu 8. In 4d, for example, we do not have a Weyl–Majorana fermion, as in the case of a single Majorana fermion, due to the equations of motion,

$$
\psi\equiv\psi_i(z)\quad \tilde{\psi}\equiv\tilde{\psi}_i(\bar{z}).
\tag{3.72}
$$

[3] The free fermion realization of ALA was presented for the first time in [27].

However, unlike the case of a single fermion, here the action is invariant under transformations associated with $\widehat{SO}(N)$ affine algebra generated by the following holomorphically (anti-holomorphically) conserved currents,

$$J^a(z) = \frac{1}{2}\psi^i T^a_{ij}\psi^j \quad \bar{J}^a(\bar{z}) = \frac{1}{2}\tilde{\psi}^i T^a_{ij}\tilde{\psi}^j, \tag{3.73}$$

where T^a are $SO(N)$ matrices which can be expressed as,

$$T^a_{ij} \equiv t^{(kl)}_{ij} = i(\delta^k_i\delta^l_j - \delta^k_j\delta^l_i). \tag{3.74}$$

The coefficients (halfs) are not determined by the Noether procedure, but are chosen in a manner that will be explained below.

The T^a_{ij} matrices obey the relations,

$$\begin{aligned} Tr[T^aT^b] &= 2\delta^{ab} \\ \sum_a T^a_{ij}T^a_{kl} &= -\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} \\ \sum_{ab} f_{abc}f_{abd} &= 2(N-2)\delta_{cd}. \end{aligned} \tag{3.75}$$

The anticommutation relation and the OPE generalize in an obvious way those of the single Majorana fermion, namely,

$$\{\psi^i(x_0,x_1)\psi^j(y_0,y_1)\}|_{x_0=y_0} = \frac{1}{2}\delta^{ij}\delta(x_1-y_1) \tag{3.76}$$

and

$$\psi^i(z)\psi^j(w) = \frac{\delta^{ij}}{z-w} \quad \tilde{\psi}^i(z)\tilde{\psi}^j(w) = \frac{\delta^{ij}}{\bar{z}-\bar{w}}. \tag{3.77}$$

Now using this OPE one can derive the OPE of two currents and verify that they take the form of (3.40),

$$\begin{aligned} J^a(z)J^b(w) &= \frac{1}{4} :\psi^i(z)T^a_{ij}\psi^j(z)::\psi^k(w)T^b_{kl}\psi^l(w): \\ &= \frac{1}{4}T^a_{ij}T^b_{kl}\left[-\left(:\psi^i(z)\psi^k(w): + \frac{\delta^{ik}}{z-w}\right)\psi^j(z)\psi^l(w)\right. \\ &+ \left.\left(:\psi^i(z)\psi^l(w): \ \frac{\delta^{il}}{z-w}\right)\psi^j(z)\psi^k(w)\right] = \frac{1}{4}T^a_{ij}T^b_{kl}\frac{1}{z-w} \\ &[-\delta^{ik}:\psi^j(z)\psi^l(w): + \delta^{il}:\psi^j(z)\psi^k(w): + \delta^{jk}:\psi^i(z)\psi^l(w): \\ &- \delta^{jl}:\psi^i(z)\psi^k(w):] + \frac{1}{4}T^a_{ij}T^b_{kl}\frac{1}{(z-w)^2}\left[-\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}\right]. \end{aligned} \tag{3.78}$$

By expanding the fields that are functions of z around w and using the relations above one finds that indeed the OPE of the two currents take the form of (3.40), namely,

$$J^a(z)J^b(w) = \frac{1\delta^{ab}}{(z-w)^2} + \frac{f^{ab}_c J^c(w)}{(z-w)} + \text{finite terms} \tag{3.79}$$

It is thus clear that this is a realization of an $\widehat{S}O(N)$ ALA of level $k=1$.

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The Noether currents associated with the conformal transformations, the energy-momentum tensor T ($\bar{T}$) is just the sum of T ($\bar{T}$) associated with each one of the N Majorana fermions, hence,

$$T(z) = -\frac{1}{2}\sum_i : \psi^i \partial \psi^i : \quad \bar{T}(\bar{z}) = -\frac{1}{2}\sum_i : \tilde{\psi}^i \bar{\partial} \tilde{\psi}^i : . \tag{3.80}$$

Since the Virasoro anomaly of a single Majorana fermion is $c = \frac{1}{2}$ it is clear that the theory of N fermions has $c = \frac{N}{2}$.

In Section 2.12 it was shown that T of a Dirac fermion could be transformed into a Sugawara form, $T(z) = -\frac{1}{2} : J(z)J(z) :$, where $J(z)$ was the $U(1)$ current. Since we will show below that the Sugawara form is the underlying structure of the important class of WZW models, it is a natural question to ask whether also in the present case for the N fermions T can be put into a Sugawara construction.

Now, using the expression for the Virasoro anomaly for a theory with $\widehat{SO}(N)_1$, we find, as we saw before, that,

$$c = \frac{\dim G}{k+C} = \frac{\frac{1}{2}N(N-1)}{1+(N-2)} = \frac{N}{2}. \tag{3.81}$$

3.7.2 Primary fields

Similarly to the case of a single Majorana field, the OPE of $T(z)\psi^i(w)$ is,

$$T(z)\psi^i(w) = \frac{1}{2}\frac{\psi^i}{(z-w)^2} + \frac{\partial\psi^i}{z-w}, \tag{3.82}$$

which implies that ψ^i are N *primary* fields of conformal dimensions $(\frac{1}{2}, 0)$, and similarly $\tilde{\psi}^i$ has dimension $(0, \frac{1}{2})$.

Is the primary field $\Phi^{(1/2,1/2)}(z\bar{z}) = \psi(z)\tilde{\psi}(\bar{z})$ the only primary operator (in addition to the identity operator that corresponds to the vacuum state)? For the primaries of the ALA $\widehat{SO}(N)_1$ we find (see (2.13)) that there is also one primary operator with dimension $\frac{N}{16}$ for odd N, and two primary operators for even N. These additional primaries transform in the spinor representation of $\widehat{SO}(N)_1$.

Can one construct these primaries in terms of the fermionic fields ψ and $\tilde{\psi}$? The situation here is similar to the one in the Ising model. In fact, using the spin operator $\sigma(z, \bar{z})$ or its dual, one indeed gets from the N independent Majorana fermion theories, the dimension $\frac{N}{16}$ and the number of degrees of freedom 2^N, which are identical to the dimension of the spinor representation.

So far we have shown the free fermion construction of $\widehat{SO}(N)_1$, namely, of the ALA at level 1. We would now like to investigate the possibility of having free fermion realization also to the affine Lie algebra at higher levels. Going through our previous derivation it is clear that the ALA structure of the OPE of two currents (3.40), applies to fermions at any representation. For a given representaion ρ the corresponding level k is determined from the first term on the right-hand side, namely $Tr(T^aT^b) = 2k\delta^{ab}$. Now since for a representation ρ,

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$Tr(T^a_\rho T^b_\rho) = 2D_2(\rho)\delta^{ab}$ where $D_2(\rho)$ is the Dynkin index of the representation, it is clear that free fermions constitute a realization of $\widehat{SO}(N)$ at level $D_2(\rho)$.

3.8 Free Dirac fermions and the $\widehat{U}(N)$

Consider the theory of N Dirac fermions described by the following action,

$$S = \frac{1}{4\pi}\int d^2z\{\psi^{i\dagger}\bar{\partial}\psi_i + \tilde{\psi}^{i\dagger}\partial\tilde{\psi}_i\}. \tag{3.83}$$

In terms of symmetries, the difference between this theory and the one of a single Dirac fermion, is that now there is an invariance under $U(N)$ left holomorphic and right anti-holomorphic transformations, namely,

$$\psi \to \psi' = g(z)\psi \quad \tilde{\psi} \to \tilde{\psi}' = \bar{g}(\bar{z})\tilde{\psi}, \tag{3.84}$$

where $g(z), \bar{g}(\bar{z}) \in U(N)$. The associated holomorphic currents are given by,

$$J^a = {\psi^i}^\dagger T^{a\,j}_{\ i}\psi_j \quad J = {\psi^i}^\dagger\psi_i, \tag{3.85}$$

where J is the $U(1)$ current, $J^a(z)$ are the $SU(N)$ currents and $T^{a\,j}_{\ i}$ are matrices in the adjoint of $SU(N)$, that obey

$$\begin{aligned}
Tr[T^aT^b] &= \delta^{ab} \\
\sum_a T^a_{ij}T^a_{kl} &= \delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl} \\
\sum_{ab} f_{abc}f_{abd} &= N\delta_{cd}.
\end{aligned} \tag{3.86}$$

Using the OPEs of the fermions, it is straightforward to realize that the currents indeed constitute the OPEs that correspond to a $\widehat{U}(N)$ of level $k = 1$,

$$\begin{aligned}
J^a(z)J^b(w) &= \frac{1\delta^{ab}}{(z-w)^2} + \frac{f^{ab}_c J^c(w)}{(z-w)} + \text{finite terms} \\
J(z)J(w) &= \frac{1}{(z-w)^2} + \text{finite terms} \\
J^a(z)J(w) &= \text{finite terms}.
\end{aligned} \tag{3.87}$$

Similar to the case of Majorana fermions, the Noether current T is given by,

$$T(z) = T(z)_{U(1)} + T(z)_{SU(N)} = -\frac{1}{2}[{\psi^i}^\dagger\partial\psi_i - \partial{\psi^i}^\dagger\psi_i], \tag{3.88}$$

and can be reexpressed in terms of a Sugawara form,

$$\begin{aligned}
T(z)_{U(1)} &= \frac{1}{4N} : {\psi^\dagger}^i\psi_i{\psi^\dagger}^j\psi_j : \\
T(z)_{SU(N)} &= \frac{1}{2(N+1)}\sum_a : {\psi^\dagger}^iT^a\psi_i{\psi^\dagger}^jT^a\psi_j : .
\end{aligned} \tag{3.89}$$

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Since a Dirac fermion has a $c = 1$ Virasoro anomaly, it is clear that the theory of N Dirac fermions has $c = N$. This is also the outcome of the Virasoro anomaly associated with the Sugawara form as follows,

$$c_{U(1)} + c_{SU(N)} = 1 + \frac{N^2 - 1}{N + 1} = N. \tag{3.90}$$

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