A RESULT ON HERMITIAN OPERATORS

by J. E. JAMISON and PEI-KEE LIN†

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1. Introduction. Let X be a complex Banach space. For any bounded linear operator T on X, the (*spatial*) *numerical range* of T is defined as the set

$$V(T) = \{f(Tx) : x \in X, f \in X^*, \|x\| = \|f\| = 1 = f(x)\}.$$

If $V(T) \subseteq \mathbf{R}$, then T is called *hermitian*. Vidav and Palmer (see Theorem 6 of [**3**, p. 78] proved that if the set $\{H + iK : H \text{ and } K \text{ are hermitian}\}$ contains all operators, then X is a Hilbert space. It is natural to ask the following question.

QUESTION. *Is X a Hilbert space if $\{H + iK : H$ and K are hermitian$\}$ contains all compact operators?*

In this article, we have proved the following theorem.

THEOREM. *Let P be a norm-1 projection on X. If there exist two hermitian operators H and K such that $P = H + iK$, then P is hermitian and $P = H$.*

Recall that an element $x \in X$ is *hermitian* if the span of x is the range of a rank-1 hermitian projection $P \in \mathscr{L}(X)$. Berkson [**1**] (also see [**5**, p. 499] proved that if every nonzero element is hermitian, then X is isometrically isomorphic to a Hilbert space. Hence, the theorem shows the answer of the above question is affirmative.

2. Proof of theorem. Let $Y = \text{range } P$ and $Z = \ker P$. Then the matrices of $H, K : X = Y \oplus Z \to X = Y \oplus Z$ have the following forms:

$$H = \begin{pmatrix} E + I_Y & F \\ C & D \end{pmatrix}$$

and

$$K = \begin{pmatrix} iE & iF \\ iC & iD \end{pmatrix}$$

where $E : Y \to Y$, I_Y is the identity on Y, $F : Z \to Y$, $C : Y \to Z$ and $D : Z \to Z$. Since H and K are hermitian it follows that

$$i[H, K] = iHK - iKH = H(P - H) - (P - H)H = HP - PH = \begin{pmatrix} 0 & -F \\ C & 0 \end{pmatrix}$$

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is hermitian (see Lemma 5.4 of [3]). If $[H, K]=0$ (i.e. P is normal), then Y and Z are invariant subspaces of H and K. So the restrictions of H and K to Y and Z are hermitian. This implies that

$$I_Y = H|_Y + iK|_Y \quad \text{and} \quad 0 = H|_Z + iK|_Z.$$

So by Lemma 1.1 of [6], $I_Y = H|_Y$, $K|_Y = 0$ and $H|_Z = 0 = K|_Z$. Now, we claim that $[H, K]=0$. It is known [6] that if T is a non-zero hermitian operator on X, the ultraproduct $\tilde{T}$ of T has at least one non-zero eigenvalue (for definition and detail, see [6]). Moreover, if T is hermitian and $T(Y) \subseteq Z$, then $\tilde{T}$ is hermitian and $\tilde{T}(\tilde{Y}) \subseteq \tilde{Z}$, where $\tilde{Z} = \{(z_i) \in \tilde{X} : z_i \in X\}$ and $\tilde{Y} = \{(y_i) \in \tilde{X} : y_i \in Y\}$. So we may assume $i[H, K]$ has a non-zero real eigenvalue λ. Let $x = y \oplus z$ be a corresponding eigenvector. Then $i[H, K]y = \lambda z$ and $i[H, K]z = \lambda y$ (since $i[H, K]Y \subseteq Z$ and $i[H, K]Z \subseteq Y$). So $y \neq 0 \neq z$ (we can assume that $\|y\| = 1$), and there exist $y' \in Y$ and $z' \in Z$ such that

$$Hy = y' + \lambda z \quad \text{and} \quad Hz = -\lambda y + z'.$$

Therefore,

$$\begin{aligned} [i[H, K], H]y &= i[H, K]Hy - iH[H, K]y \\ &= i[H, K](y' + \lambda z) - \lambda Hz \\ &= i[H, K]y' + \lambda^2 y + \lambda^2 y - \lambda z'. \end{aligned}$$

Since P is a norm-1 projection onto Y, there is $f \in X^*$ such that $\|f\| = 1 = \|y\| = f(y)$ and $f|_Z = 0$. So

$$f([i[H, K], H]y) = 2\lambda^2 \qquad (\text{since } i[H, K]y' \in Z).$$

This contradicts the fact that $i[i[H, K], H]$ is hermitian.

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Department of Mathematics
Memphis State University
Memphis
Tennessee 38152, U.S.A.