FINITE SIMPLE GROUPS WITH NILPOTENT THIRD MAXIMAL SUBGROUPS

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(Received 29 September 1965)

We say that a subgroup H is an n-th maximal subgroup of G if there exists a chain of subgroups $G = G_0 > G_1 > \cdots > G_n = H$ such that each G_i is a maximal subgroup of G_{i-1}, $i = 1, 2, \cdots, n$. The purpose of this note is to classify all finite simple groups with the property that every third maximal subgroup is nilpotent.

THEOREM. *If G is a finite simple group such that every third maximal subgroup of G is nilpotent, then G is isomorphic either to a linear fractional group $PSL(2, q)$, for certain $q > 3$, or to $Sz(2^3)$, the Suzuki simple group over the field of 2^3 elements.*

REMARK. If the group $PSL(2, q)$ satisfies the condition that all third maximal subgroups are nilpotent then it follows that

(a) $q = 2^r$, 3^s or t, where r, s, t are primes, $r > 2$,

and

(b) if $q = 3^r$ or 2^s, then $(q+1)/\varepsilon$, $(q-1)/\varepsilon$, where $\varepsilon = 2$ if q is odd, and $\varepsilon = 1$ if q is even, are products of at most two (not necessarily distinct) primes; if $q = t$, then $(t-1)/2$ is a product of at most two primes and $(t+1)/2$ is either a product of at most two primes or a power of 2.

Conversely, the groups $Sz(2^3)$ and $PSL(2, q)$, where q satisfies the conditions (a) and (b) above, have the property that every third maximal subgroup is nilpotent.

NOTATIONS AND KNOWN RESULTS. We let $H \leqq G$, $H < G$, $H \trianglelefteq G$ mean that H is a subroup, a proper subgroup, and a normal subgroup of G, respectively. We let $N(S)$, $C(S)$, for any subset S of G, denote the normalizer and the centralizer of S in G, respectively. We let $Z(G)$ denote the centre of G. If x is any element of G we let $\langle x \rangle$ be the group generated by x.

The following result of Janko [3] and Berkovič [1] is essential.

LEMMA 1. *If G is a finite group all of whose second maximal subgroups are nilpotent, then G is either soluble or isomorphic to $PSL(2,5)$ or $SL(2,5)$.*

PROOF OF THEOREM. According to a definition of J. G. Thompson [6], we say that a finite group G is an N-group if the normalizer of any non-trivial soluble subgroup of G is itself soluble.

If G is a non-abelian simple group all of whose third maximal subgroups are nilpotent, then G is an N-group. For suppose that there exists a non-trivial soluble subgroup $S \leqq G$ such that $N(S)$ is non-soluble. The group $H = N(S)$ is clearly a maximal subgroup of G. For if not, there exists a subgroup M, with $H \leqq M$, which is second maximal in G. But then M has the property that every proper subgroup of M is nilpotent, and hence, by a result of Iwasawa [2], we see that M is soluble, a contradiction. Now H, being maximal in G, has the property that every second maximal subgroup of H is nilpotent. By Lemma 1 we see that $H \cong PSL(2,5)$ or $H \cong SL(2,5)$. Since $S \neq 1$, we have that $H \cong SL(2,5)$ and $S = Z(H)$. The Theorem D of Suzuki [5] p. 682, gives that G is non-simple, a contradiction.

Therefore G is a N-group. Now Thompson [6] has classified all finite N-groups and since G is simple we see that G is isomorphic to one of the following groups:

$PSL(2, q)$, q a prime power, $q > 3$,

$Sz(2^{2n+1})$,

$PSL(3, 3)$,

M_{11},

A_7,

or

$PSU_3(3^2)$.

Case 1. The groups $Sz(q)$, $q = 2^{2n+1}$, $n \geqq 1$.

Let $2q = r^2$, $r = 2^{n+1}$. Then if $G = Sz(q)$, the following relations must be satisfied:

$$q-1 = \text{prime},$$
$$q+r+1 = \text{prime},$$
$$q-r+1 = \text{prime}.$$

These are satisfied only if $n = 1$, since at least one is divisible by 5.

Case 2. The group $PSL(3, 3)$ is inadmissible since this contains the Hessian group H as a subgroup. The group H has subgroups $H > F > E$, in the notation of Miller-Blichfeldt-Dickson [4] p. 239 and E is non-nilpotent of order 36.

Case 3. The groups M_{11} and A_7.

Both these groups are inadmissible since they contain a subgroup isomorphic to A_6, while non-soluble subgroups of any group with our property are either $PSL(2, 5)$ or $SL(2, 5)$.

Case 4. The group $PSU_3(3^2)$.

This group is inadmissible since $PSU_3(3^2) > U_2(3^2)$. The group $SU_2(3^2)$ is contained in $U_2(3^2)$ with index 4 and is non-nilpotent.

Thus we have ruled out all possibilities except the linear fractional groups and the group $Sz(2^3)$, as stated in the theorem.

Now suppose that the group G is isomorphic to $PSL(2, q)$, $q > 3$. Then G has the property that every third maximal subgroup is nilpotent if and only if every maximal subgroup H has the property $(*)$ every second maximal subgroup of H is nilpotent.

Let $q = p^n > 3$, p a prime. A p-Sylow normalizer N is isomorphic to the groups of transformations of $GF(q)$

$$x \to \alpha^2 x + \beta, \qquad \alpha, \beta \in GF(q), \qquad \alpha = 0.$$

It follows that N satisfies $(*)$ if and only if

(a) $(q-1)/\varepsilon$ is a product of less than or equal to two primes (not necessarily distinct);

(b) $n = 1$ when $p > 3$;

(c) n is a prime greater than 2 when $p = 3$;

(d) n is a prime when $p = 2$.

The dihedral subgroup $D_{2(q+1)/\varepsilon}$ satisfies $(*)$ if and only if

(e) $(q+1)/\varepsilon$ is a product of at most two primes or a power of 2.

The condition (a) implies that $(*)$ holds for the dihedral groups $D_{2(q-1)/\varepsilon}$. The only other possible maximal subgroups are $PSL(2, 5)$, $PSL(2, 3)$, S_4 or $PSL(2, 2)$, and these groups automatically satisfy $(*)$.

References

[1] Ja. G. Berkovič, 'The existence of subgroups of a finite non-soluble group', *Dokl. Akad. Nauk* 156 (1964), 1255—1257.

[2] K. Iwasawa, 'Über die Struktur der endlichen Gruppen, deren echte Untergruppen sämtlich nilpotent sind', *Proc. Phys. Math. Soc. Japan* 23 (1941), 1—4.

[3] Z. Janko, 'Endliche Gruppen mit lauter nilpotenten zweitmaximalen Untergruppen', *Math. Zeitschr.* 79 (1962), 422—424.

[4] G. A. Miller, H. F. Blichfeldt and L. E. Dickson, *Theory and applications of finite groups* (New York 1938).

[5] M. Suzuki, 'On finite groups with cyclic Sylow subgroups for all odd primes', *Amer. J. Math.* 77 (1955), 657—691.
[6] J. G. Thompson, 'Some simple groups', *Symposium on group theory* (Harvard, 1963).

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https://doi.org/10.1017/S1446788700004936 Published online by Cambridge University Press