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ON THE DUALS OF FLAT BANACH SPACES

BY

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ABSTRACT. We give a simpler proof to a theorem of L. A. Karlovitz that the dual of a flat Banach space is flat, and also study some geometric properties of the dual space.

1. **Introduction.** The notion of "flat Banach spaces" was introduced by Harrel and Karlovitz ([2], [3]). Since we shall make repeated use of some of their definitions and basic facts we reproduce them here. A real Banach space X is said to be *flat* if the *girth* of its unit ball (defined by Schaffer [7] to be the infimum of the lengths of all centrally-symmetric closed curves which lie in the surface of the unit ball) is four and if the girth is achieved by some curve. This is equivalent to the existence of a function $g: R \to X$ such that for each $s, t \in R$

$$\|g(t)\| = 1, \qquad g(t) = -g(t+2), \qquad \|g(t)-g(s)\| \leq |t-s| \tag{1}$$

These conditions easily imply (see [3]) that

$$\|g(t)-g(s)\| = |t-s| \quad \text{for} \quad |t-s| \leq 2 \tag{2}$$

The restriction of g to every closed interval of length 4 is a centrally-symmetric closed curve of length 4 on the surface of the unit ball which is arc-length parametrized. g (or its restriction) is called a *girth curve.* A flat Banach space X is called *completely flat* if $X = \overline{\text{span}}\, g(R)$ (we shall denote the right hand side by $\overline{\text{span}}\, g$) where g is a girth curve.

In the sequel we shall denote the unit ball and its boundary in a Banach space X by $B(X)$ and $S(X)$, respectively.

Let X be a flat Banach space with a girth curve g. For each $t \in [0, 2)$ choose $f_t \in X^*$ such that

$$\|f_t\| = 1, \qquad f_t(g(t)) = 1 \tag{3}$$

We define $f_t = -f_{t-2}$ for $t \in [2, 4)$, and extend the definition periodically for all $t \in R$, and now (3) is satisfied for each $t \in R$. $\{f_t; t \in R\}$ are uniquely determined on $\overline{\text{span}}\, g$, since (see [3])

$$f_t(g(s)) = 1 - |s-t| \quad \text{for} \quad |s-t| \leq 2 \tag{4}$$

It also follows that

$$f_t(g(s)) = f_s(g(t)) \quad \text{for } all \quad t, s \in R \tag{5}$$

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We shall use the notation

$$\Delta_g(s, h)=\frac{1}{h}[g(s-h)-g(s)] \quad \text{for} \quad s\in R, h\in(0,2)$$

We have $\|\Delta_g(s,h)\|=1$, and, for $h\in(0,2)$, $|s-t|\leq 2$, $|t-s+h|\leq 2$

$$(6) \qquad f_t(\Delta_g(s,h))=\frac{1}{h}[-|s-h-t|+|s-t|]=\begin{cases} 1 & t\leq s-h \\ 2s/h-2t/h-1 & s-h\leq t\leq s \\ -1 & s\leq t \end{cases}$$

We shall need the following lemma.

LEMMA 1.

$$\lim_{t\to t_0} f_t(x)=f_{t_0}(x) \quad \text{for each} \quad x\in\overline{\text{span}}\, g$$

In particular, if $X=\overline{\text{span}}\, g$, *then the function* $t\to f_t$ *from* R *into* $S(X^*)$ *is* w^**-continuous and the set* $\{f_t; t\in R\}$ *is* w^**-homeomorphic to the circle.*

Proof. Since $\{f_t\}\subset S(X^*)$, it is sufficient to consider $x\in g(R)$, but then it is immediate from (5).

If $X=\overline{\text{span}}\, g$, we can regard the function $t\to f_t$ as a function from the circle into $S(X^*)$ which is w^*-continuous and one-to-one, that is, homeomorphism onto its image.

2. **The flatness of the dual space.** We now present a simpler and "more continuous" proof for results of Karlovitz ([5], Theorems 1,3. The oversight that $B(X^*)$ need not be w^*-sequentially compact is also corrected.)

THEOREM 2. *If X is a flat Banach space, then X^* is flat.*

If X is completely flat, there exists in $S(X^*)$ a girth curve such that no functional from the curve attains its norm.

Proof. (a) Suppose first that X is completely flat, $X=\overline{\text{span}}\, g$, where g is a girth curve and suppose $\{f_t\}$ are the corresponding functionals.

For each $r\in R$ consider the functional γ_r defined by

$$\gamma_r(x)=\frac{1}{2}\int_r^{r+2} f_t(x)\,dt \qquad x\in X$$

(i.e. $\gamma_r=\frac{1}{2}\int_r^{r+2} f_t\,dt$ where the integral is a w^*-Riemann-integral). By Lemma 1 the integrand is a continuous function of t and so the integral exists. Clearly $\|\gamma_r\|\leq 1$, and since $\gamma_r(\Delta_g(r+2,h))=(2-h)/2$ it follows that $\|\gamma_r\|=1$. It is easily verified that $\gamma_{r+2}=-\gamma_r$. For $r\le s\le r+2$ we have

$$(7)\qquad (\gamma_r-\gamma_s)(x)=\frac{1}{2}\left|\int_r^s f_t(x)\,dt-\int_{r+2}^{s+2} f_t(x)\,dt\right|=\frac{1}{2}\left|\int_r^s (f_t-f_{t+2})(x)\,dt\right|$$
$$=\left|\int_r^s f_t(x)\,dt\right|\le\|x\|(s-r);$$

hence $\|\gamma_r-\gamma_s\|\le|r-s|$ for $|r-s|\le 2$ and it follows that the mapping $r\to\gamma_r(r\in[0,4])$ defines a girth curve of length four.

Now if $\gamma_r(x)=1$ for some $x\in S(X)$ and $r\in R$, it follows that $f_t(x)=1$ for each $t\in[r,r+2]$, but $f_r=-f_{r+2}$—a contradiction.

(b) Suppose now that X is a flat Banach space with a girth curve g and corresponding functionals $\{f_t\}$. For each $r\in R$ consider the functionals

$$G_r^n=\frac{1}{2n}\sum_{i/n\in[r,r+2)} f_{i/n}\qquad n=1,2,\ldots$$

Clearly $G_r^n\in B(X^*)$, and hence there exists a net $\{n_\alpha\}$ of positive integers such that $\{G_r^{n_\alpha}\}_\alpha$ converges $-w^*$, for each $r\in R$, to some $G_r\in B(X^*)$ (For $B(X^*)^R$ is compact in the product topology when $B(X^*)$ is taken with the w^*-topology). Clearly G_r is an extension of $\gamma_r\in(\overline{\text{span}}\ g)^*$ from part (a), hence $\|G_r\|=1$. It is easily seen that $G_r^n=-G_{r+2}^n$ and (analogous to (7)) $|G_r^n-G_s^n|\le|r-s|$ for $|r-s|\le 2$, hence also $G_r=-G_{r+2}$ and $\|G_r-G_s\|\le|r-s|$ for $|r-s|\le 2$. Thus the mapping $r\to G_r$ defines a girth curve in $S(X^*)$.

3. **The geometry of the dual space.** Let X be a flat Banach space with a girth curve g and corresponding functionals $\{f_t\}$. We turn to study the role played by $\{f_t\}$ in the geometry of the dual space.

PROPOSITION 3. (a) $\overline{\text{conv}}\,\{f_t: t\in[\alpha,\alpha+2)\}\subset S(X^*)$
(b) $[f_t,f_s]\subset S(X^*)$ *for* $t\ne s+2 (\text{mod } 4)$
(c) $\overline{\text{conv}}^{w^*}\{f_t;\ t\in[\alpha,\beta]\}\subset S(X^*)$ *for* $\alpha\le\beta<\alpha+2$
(d) $\|f_t-f_s\|=2$ *for* $t\ne s (\text{mod } 4)$
(e) $\{f_t;\ t\in[0,2)\}$ *are linearly independent*

Proof. (a) Let $f=\sum_{i=1}^n \lambda_i f_{t_i}$ where $\sum_{i=1}^n \lambda_i=\sum_{i=1}^n|\lambda_i|=1$ and, $\alpha\le t_1<t_2<\cdots<t_n<\alpha+2$. Then $\|f\|\le 1$ and, by (6), $f(\Delta_g(\alpha+2,h))=1$ for $h\in(0,\alpha+2-t_n)$, hence $\|f\|=1$.
This implies the desired result.

(b) Because of periodicity in the index, it can be assumed that $s\le t<s+2$, and then (b) is a result of (a)

(c) Let $\alpha\le\beta<\alpha+2$, and take $h\in(0,\alpha+2-\beta)$. Then, as in (a), we get $f(\Delta_g(\alpha+2,h))=1$ for each $f\in\text{conv}\,\{f_t,\ t\in[\alpha,\beta]\}$ and therefore also for each $f\in\overline{\text{conv}}^{w^*}\{f_t,\ t\in[\alpha,\beta]\}$

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(d) (see [5]. Th. 2) As in (b), we can assume $s<t\le s+2$. Then $\|f_s-f_t\|\le 2$ and $(f_s-f_t)(\Delta_g(t,h))=2$ for $h\in(0,t-s)$, hence $\|f_s-f_t\|=2$

(e) Analogous to the proof that $\{g(t); t\in[0,2)\}$ are linearly independent ([4]. Cor. 2). Suppose $\sum_{i=1}^{n}\alpha_i f_{t_i}=0$ where $\alpha_i\in R$ and $0\le t_1<t_2<\cdots<t_r<2$. Then, for each $s\in[t_j,t_{j+1}]$

$$\begin{aligned}0&=\sum_{i=1}^{n}\alpha_i f_{t_i}(g(s))\\&=\sum_{i=1}^{j}\alpha_i(1-s+t_i)+\sum_{i=j+1}^{n}\alpha_i(1-t_i+s)\\&=[\text{term independent of } s]+s\left(-\sum_{i=1}^{j}\alpha_i+\sum_{i=j+1}^{n}\alpha_i\right)\end{aligned}$$

Therefore $-\sum_{i=1}^{j}\alpha_i+\sum_{i=j+1}^{n}\alpha_i=0$ $j=1,\ldots,n$, which imply $\alpha_j=0$ $j=1,\ldots,n$.

PROPOSITION 4. (a) *If X is completely flat, then $\{f_t\}$ are exposed points of $B(X^*)$ (where X^* is with the normed topology or w^*-topology)*

(b) *If X is flat, $\{f_t\}$ can be chosen so that they are extreme points of $B(X^*)$.*

Proof. (a) If X is completely flat, then $\{f_t\}$ are uniquely determined. This means that for a fixed $t\in R$, the functional $X^*\to R$ defined by $f\to f(g(t))$ attains its norm only at f_t.

(b) Suppose X is flat, and let $Y=\overline{\text{span}}\, g$, where g is a girth curve. Let $\{f_t\}\subset S(Y^*)$ be the corresponding functionals. Then by (a), they are extreme points of $B(Y^*)$, and they can be extended to extreme points of $B(X^*)$ by taking $F_t\in S(X^*)$ to be an extreme point of the convex and w^*-compact set of the Hahn-Banach extensions of f_t.

COROLLARY 5. *The Banach space $C(K)$, where K is compact Hausdorff, is not completely flat.*

Proof. It is known that $\operatorname{ext} B(C^*(K))=\hat K\cup(-\hat K)$, where $\hat K$ is the set of evaluation functionals. $\hat K\cap(-\hat K)=\phi$ and $\hat K, -\hat K$ are w^*-closed, that is, $\operatorname{ext} B(C^*(K))$ is not w^*-connected. Now if $C(K)$ is completely flat, and $\{f_t\}$ are the corresponding functionals, then this set has members in both $\hat K$ and $-\hat K$ (since $f_t=-f_{t+2}$), therefore it is not w^*-connected, a contradiction to Lemma 1.

The question now arises when are $\{f_t\}$ *all* the extreme points of $B(X^*)$? The answer will follow from the next theorem.

Let $C_\sigma(T)$ be the Banach space of all the real continuous functions f defined on the circle T such that $f(t)=-f(-t)$ for each $t\in T$, with the sup-norm.

THEOREM 6. *$C_\sigma(T)$ is a completely flat Banach space. There exists a girth curve $g:R\to C_\sigma(T)$ such that $C_\sigma(T)=\overline{\text{span}}\, g$ and such that if $X=\overline{\text{span}}\, g_1$ is an arbitrary completely flat Banach space spanned by a girth curve g_1, then there* *exists a linear operator* $P:X\to C_\sigma(T)$ *such that* $\|P\|=1$, $g=P\circ g_1$ *and* $\overline{PX}=C_\sigma(T)$.

Proof. We shall identify $C_\sigma(T)$ with the space of the real continuous functions f defined on the real line R which satisfy $f(s)=-f(s+2)$ for each $s\in R$.

Let $X=\overline{\text{span}}\, g_1$ be a completely flat Banach space spanned by a girth curve g_1, and let $\{f_t\}$ be the corresponding functionals.

For each $t\in R$, consider the function $F_t\equiv f_t\circ g_1: R\to R$. Obviously $F_t\in C_\sigma(T)$, $\|F_t\|=1$ and $F_t=-F_{t+2}$. For $s,t\in R$ we have

$$\|F_t-F_s\|=\max_r |f_t(g_1(r))-f_s(g_1(r)|$$
$$=\max_r |f_r(g_1(t))-f_r(g_1(s))|\le \|g_1(t)-g_1(s)\|\le |t-s|$$

Thus the mapping $g:R\to C_\sigma(T)$ defined by $t\to F_t$ is a girth curve in $S(C_\sigma(T))$. We note that $F_t(s)=1-|s-t|$ for $|t-s|\le 2$, thus each F_t is uniquely determined on R, independently of the space X and the girth curve g_1. Consider now the linear operator $P:X\to C_\sigma(T)$ defined by $(Px)(t)=f_t(x)$. Clearly $\|P\|\le 1$. $P(g_1(s))(t)=f_t(g_1(s))=f_s(g_1(t))=F_s(t)=g(s)(t)$, so $P\circ g_1=g$ which implies that $\|P\|=1$ and $\overline{PX}=\overline{\text{span}}\, g$.

It is left to prove that $C_\sigma(T)=\overline{\text{span}}\, g$. We observe that $F_t(s)=F_0(s-t)$ and thus $\overline{\text{span}}\, g$ is the closed span of all the translations of F_0. Consider the 2π-periodic function defined by $G_0(t)=1-|2t/\pi|$, $t\in[-\pi,\pi]$ which is the image of F_0 under the natural isometry of $C_\sigma(T)$ onto a space of 2π-periodic functions. $G_0(t+\pi)=-G_0(t)$, and so the Fourier transform of G_0 is given by

$$\hat{G}_0(n)=\int_{-\pi}^{\pi} G_0(t)e^{-int}\,dt=\int_0^{\pi} G_0(t)[-e^{-in(t+\pi)}+e^{-int}]\,dt \tag{8}$$
$$=\begin{cases}0 & n \text{ even}\\ \dfrac{4}{\pi n^2} & n \text{ odd}\end{cases}$$

It is known from Harmonic Analysis: if V is a translation-invariant subspace of the complex Banach space of the continuous 2π-periodic functions (with the sup-norm) then the function e^{int} (as a function of t) belongs to V if and only if $\hat{f}(n)=\int_{-\pi}^{\pi} f(t)e^{-int}\,dt\neq 0$ for some $f\in V$ (if V is the closed span of all the translations of one function F, this is equivalent to $\hat{F}(n)\neq 0$) and V is the closed span of the functions e^{int} which belong to V. (See [6] Chapter 1.)

In our case, returning to the real space $\overline{\text{span}}\, g$, it follows that the functions $\sin(\pi nt/2)$, $\cos(\pi nt/2)$ belong to $\overline{\text{span}}\, g$ if and only if n is odd, and $\overline{\text{span}}\, g$ is the closed span of this family. On the other hand, it follows, as in (8), that the Fourier transform of each $f\in C_\sigma(T)$ vanishes for even n and since $C_\sigma(T)$ is also translation invariant we conclude that $C_\sigma(T)\subset\overline{\text{span}}\, g$.

COROLLARY 7. *The following properties are equivalent for a completely flat Banach space* $X=\overline{\text{span}}\ g$ *where g is a girth curve and* $\{f_t\}$ *are the corresponding functionals*:

(a) $\{f_t; t\in[0,4)\}$ are *all* the extreme points of $B(X^*)$.

(b) $\|x\|=\sup_{t\in[0,2)}|f_t(x)|$ for each $x\in X$

(c) $B(X^*)=\overline{\text{conv}}^{w^*}\{f_t; t\in[0,4)\}$

(d) X is isometric to $C_\sigma(T)$.

Proof. $\{f_t\}$ is a w^*-compact set (by Lemma 1) of extreme points of $B(X^*)$, and thus the equivalence between (a), (b), (c) follows from known theorems on extreme points (see [1], Chap. V §1). (b) implies that the operator P from Theorem 6 is a surjective isometry, therefore (b) implies (d).

To show that (d) implies (a), it is sufficient to prove that (a) holds for $C_\sigma(T)$, with respect to *any* spanning girth curve and its corresponding functionals. For $t\in R$, let $e_t\in C_\sigma^*(T)$ be the evaluation functional $e_t(f)=f(t)$, $f\in C_\sigma(T)$. It is known that $\text{ext}\,B(C_\sigma^*(T))=\{e_t; t\in R\}$, and this set is easily shown to be w^*-homemorphic to the circle. (This also follows from the previous results: $\{e_t\}$ are obviously the corresponding functionals for the girth curve g from Theorem 6, and by Proposition 4, $\{e_t\}\subset\text{ext}\,B(C_\sigma^*(T))$. Thus the equivalent properties (a), (b), (c) are satisfied, for (b) is merely the definition of the norm in $C_\sigma(T)$. The argument is completed by using Lemma 1). Since a proper subset of the circle is not homeomorphic to it, Lemma 1 and Prop. 4 imply that the set of the corresponding functionals for any spanning girth curve in $C_\sigma(T)$ must be equal to $\text{ext}\,B(C^*(T))$, thus (a) is satisfied.

REMARKS. (a) The geometry of flat Banach spaces in which the semi-norm $|x|=\sup_t|f_t(x)|$ is a norm equivalent to the original one, was studied by Harrel and Karlovitz in [3] without characterizing these spaces. In fact it is enough to assume there equivalence on $\overline{\text{span}}\ g$. Now we have that this condition is satisfied if and only if $P: \overline{\text{span}}\ g\to C_\sigma(T)$ from Theorem 6 is an isomorphism (necessarily surjective).

(b) The operator P from Theorem 6 is not necessarily surjective. Consider the completely flat Banach space $L^1[0,1]$ with the spanning girth curve (introduced in [4]) $g_1:[0,2]\to L^1[0,1]$ defined by $g_1(t)=-\chi_{[0,(t/2))}+\chi_{[(t/2),1]}$ where χ denotes the characteristic function. The corresponding functionals, considered as elements of $L^\infty[0,1]$, are obviously $f_t=-\chi_{[0,(t/2))}+\chi_{[(t/2),1]}$ for $t\in[0,2]$. Now, for $x\in L^1[0,1]$, $s\in[0,2]$

$$(Px)(s)=\int_0^1 f_s(t)x(t)\,dt=-\int_0^{s/2}x(t)\,dt+\int_{s/2}^1 x(t)\,dt$$

and it is easily seen that the function Px is Lipschitz continuous with constant $\|x\|$. On the other hand, not every element of $C_\sigma(T)$ is Lipschitz continuous.

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