

# SOME REGULARITY ESTIMATES FOR CONVOLUTION SEMIGROUPS ON A GROUP OF POLYNOMIAL GROWTH

NICK DUNGEY

(Received 20 December 2002; revised 25 June 2003)

Communicated by A. H. Dooley

## Abstract

We study a convolution semigroup satisfying Gaussian estimates on a group  $G$  of polynomial volume growth. If  $Q$  is a subgroup satisfying a certain geometric condition, we obtain high order regularity estimates for the semigroup in the direction of  $Q$ . Applications to heat kernels and convolution powers are given.

2000 *Mathematics subject classification*: primary 22E30, 35B40.

## 1. Introduction

The heat kernel and its regularity properties play an important role in harmonic analysis on a Lie group  $G$ , and have been intensively studied (see [21, 17] and references therein). Let us mention some relevant results in the case that  $G$  is a Lie group of polynomial growth. Varopoulos showed (see [20, 21, 17]) that the heat kernel  $K_t$  corresponding to a right-invariant sublaplacian on  $G$  satisfies a global Gaussian estimate for all times  $t > 0$ . Then Saloff-Coste [18] proved that derivatives of order one of  $K_t$  satisfy a similar estimate with an extra factor  $t^{-1/2}$ . In case  $G$  is nilpotent then  $m$ -th order derivatives of  $K_t$  satisfy similar estimates with an extra factor  $t^{-m/2}$  (see for example [21, 19]), but an example of Alexopoulos [1] showed that this is not true for solvable groups when  $m \geq 2$ . A precise characterization of the groups  $G$  for which the  $m \geq 2$  estimates are valid was given in [13]. It is worth noting that the failure of the  $m \geq 2$  estimates can only occur for large times  $t \geq 1$ .

Recently the author [7] proved precise estimates for multiple derivatives of  $K_t$  for large times  $t \geq 1$  when the derivatives are taken in the direction of the nilradical, that is,

the largest nilpotent, analytic normal subgroup of  $G$ . This result is of interest because it reveals a further connection between the algebraic structure of  $G$  and regularity properties of  $K_t$ . Another proof of this result was given in a subsequent paper of ter Elst [10]. Note also that some closely related estimates (for derivatives of order at most 2) were used in the proof of Alexopoulos of the boundedness of first order Riesz transforms on  $G$  (see [1, Theorem 7.7]).

The proofs of [7, 10] need the rather intricate structure theory for Lie groups of polynomial growth, and also use in an essential way the structure of the subelliptic differential operator corresponding to  $K_t$ . This paper is partly motivated by the question of extending results of [7, 10] to other classes of groups  $G$  and convolution kernels  $K_t$ , for example, convolution powers of probability densities on a discrete group studied in [15, 2]. Our main conclusion is that higher order regularity estimates for a suitable convolution semigroup, in the direction of a subgroup  $Q$  of  $G$ , follow from a simple geometric condition on  $Q$  in  $G$  (see (4) below).

Thus we can extend estimates of [7] to a much wider context. We will describe specific examples in which our results yield new regularity estimates.

The regularity estimates we obtain are closely related to the question of boundedness of Riesz transform operators. For Lie groups, we refer the reader to [8, 7] for recent results for certain Riesz transforms connected with this paper, while first order transforms were considered in [1, 3]. For discrete groups of polynomial growth, the boundedness of first order Riesz transforms was obtained in [2]; we will show elsewhere that our estimates lead to a simpler proof, which also extends to certain higher order transforms.

Part of our argument is no doubt related to a convolution trick which has been used in estimating heat kernel derivatives on Lie groups: see for example [10, Lemma 3.5]. Our argument, however, requires much less initial regularity for the convolution semigroup in question, and (because we use difference operators instead of derivatives) applies to a wide class of locally compact groups. Even in the well-studied case where  $G$  is a nilpotent Lie group, our method is of interest, since it shows that one can derive regularity without using scaling techniques.

Our general setting is the following. Let  $G$  be a unimodular, second countable, locally compact group, and fix a Haar measure  $dg$ . Suppose  $G$  is compactly generated, so that there is a compact neighbourhood  $U$  of the identity  $e$  of  $G$  such that  $G = \bigcup_{n=1}^{\infty} U^n$ , where  $U^n = \{u_1 \cdots u_n : u_i \in U\}$ . We can always assume that  $U$  is symmetric, that is,  $U = U^{-1}$ . Let  $G$  have polynomial growth of order  $D \geq 1$ :

$$(1) \quad c^{-1}n^D \leq dg(U^n) \leq cn^D$$

for some  $c > 0$  and all  $n \in \mathbb{N}$  (the condition  $D \geq 1$  excludes compact groups). It is well known that condition (1) does not depend on the choice of  $U$ ; for if  $V$  is another compact generating neighbourhood, there exists  $k \in \mathbb{N}$  with  $U^n \subseteq V^{kn}$  and  $V^n \subseteq U^{kn}$

for all  $n$ .

The modulus  $\rho : G \rightarrow \mathbb{N}$  defined (as in [15] or [3]) by

$$\rho(g) = \inf\{n \in \mathbb{N} : g \in U^n\}$$

satisfies

$$\rho(g) \geq 1, \quad \rho(gh) \leq \rho(g) + \rho(h), \quad \rho(g) = \rho(g^{-1})$$

for all  $g, h \in G$ . Observe that  $U^n = \{g \in G : \rho(g) \leq n\}$ , and that  $\rho$  is bounded over any compact subset of  $G$ .

Let  $L = L_G$  denote the left regular representation of  $G$ , so that  $(L(h)\varphi)(g) = \varphi(h^{-1}g)$  for a function  $\varphi : G \rightarrow \mathbb{C}$  and  $g, h \in G$ . Then  $L_G$  acts in the function spaces  $L_p = L_p(G; dg)$ ,  $1 \leq p \leq \infty$ . The convolution of two functions  $\varphi, \psi$  is defined by  $(\varphi * \psi)(g) = \int_G dh \varphi(h) \psi(h^{-1}g)$ ,  $g \in G$ . In general,  $c, c', b, b'$  and so on, denote positive constants whose value we allow to change from line to line when convenient.

Throughout, let  $\mathcal{T}$  denote one of the sets  $[1, \infty)$  or  $\mathbb{N} = \{1, 2, 3, \dots\}$  (the set  $\mathcal{T}$  will be fixed in any particular discussion). Suppose we are given a family  $\{K_t\}_{t \in \mathcal{T}}$  of functions  $K_t : G \rightarrow \mathbb{C}$ , with  $K_t \in L_1 \cap L_\infty$ , which is a convolution semigroup in the sense that  $K_{t_1+t_2} = K_{t_1} * K_{t_2}$  for all  $t_1, t_2 \in \mathcal{T}$ . In particular, if  $\mathcal{T} = \mathbb{N}$  then  $K_n = K_1 * K_1 * \dots * K_1$  is just the  $n$ -th convolution power of  $K_1$ .

For  $b, t > 0$ , define the Gaussian  $G_{b,t} : G \rightarrow \mathbb{R}$  by  $G_{b,t}(g) = t^{-D/2} e^{-bp(g)^2/t}$ . We assume that  $K_t$  satisfies an estimate

$$(2) \quad |K_t| \leq cG_{b,t}$$

for all  $t \in \mathcal{T}$  (for real-valued functions  $F_1, F_2$  over  $G$  we write  $F_1 \leq F_2$  to abbreviate  $F_1(g) \leq F_2(g)$  for all  $g \in G$ ). We also require a ‘Hölder’ estimate, namely, for some  $\nu \in (0, 1]$  one has

$$(3) \quad |(I - L(h))K_t| \leq c(\rho(h)t^{-1/2})^\nu G_{b,t}$$

for all  $t \in \mathcal{T}$  and  $h \in G$  such that  $\rho(h) \leq t^{1/2}$ , where  $I$  denotes the identity operator on functions.

Finally, let there be given a subgroup  $Q$  of  $G$  and  $\alpha \in (0, 1]$  such that

$$(4) \quad \rho(glg^{-1}) \leq c\rho(l) + c\rho(l)^\alpha \rho(g)^{1-\alpha}$$

for all  $g \in G$  and  $l \in Q$ . Under these assumptions our basic theorem is the following. The formulation of the theorem is partly inspired by [10, Lemma 3.5].

**THEOREM 1.1.** *Let  $R$  be a densely defined operator in  $L_2$  which is right-invariant (that is,  $R$  commutes with right translations). Suppose there are  $a > 0, b > 0, \delta \geq 0$ ,*

such that  $|RK_t| \leq at^{-\delta}G_{b,t}$  for all  $t \in \mathcal{T}$ . Then there are  $c > 0, b' > 0$ , which depend on  $b, \delta$  and on the constants in (1)–(4), such that

$$|(I - L(l))RK_t| \leq ca(\rho(l)t^{-1/2})t^{-\delta}G_{b',t}$$

for all  $t \in \mathcal{T}$  and  $l \in Q$  with  $\rho(l) \leq t^{1/2}$ . (In particular,  $c$  and  $b'$  are independent of  $a$ .)

Note that the Hölder exponent  $\nu$  from (3) does not occur in the conclusion of Theorem 1.1; roughly speaking, the Hölder estimate self-improves into a Lipschitz estimate in the direction of  $Q$ .

The operators  $L(h), h \in G$ , are obviously right-invariant. Thus by beginning with the estimates (2), (3), and applying Theorem 1.1 repeatedly, we arrive at the following result for higher order oscillations of  $K_t$ .

**THEOREM 1.2.** *For any  $k \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ , one has estimates*

$$\begin{aligned} & |(I - L(l_1))(I - L(l_2)) \cdots (I - L(l_k))K_t| \\ & \leq c(\rho(l_1)t^{-1/2}) \cdots (\rho(l_k)t^{-1/2})G_{b,t}, \\ & |(I - L(l_1))(I - L(l_2)) \cdots (I - L(l_k))(I - L(h))K_t| \\ & \leq c(\rho(l_1)t^{-1/2}) \cdots (\rho(l_k)t^{-1/2})(\rho(h)t^{-1/2})^\nu G_{b,t} \end{aligned}$$

for all  $t \in \mathcal{T}, l_1, \dots, l_k \in Q$  and  $h \in G$  such that  $\rho(l_j) \leq t^{1/2}$  for all  $j$  and  $\rho(h) \leq t^{1/2}$ .

The next theorem shows that the estimate of Theorem 1.1 can sometimes be improved in specific directions within the subgroup  $Q$ .

**THEOREM 1.3.** *Let  $w \geq 1, l_0 \in Q$  and suppose that  $\rho(l_0^n) \leq c_0n^{1/w}$  for some  $c_0 > 0$  and all  $n \in \mathbb{N}$ . Assume the hypotheses of Theorem 1.1. Then there exist  $c, b' > 0$  such that  $|(I - L(l_0))RK_t| \leq cat^{-w/2}t^{-\delta}G_{b',t}$  for all  $t \in \mathcal{T}$ .*

Since any  $h \in G$  satisfies  $\rho(h^n) \leq n\rho(h), n \in \mathbb{N}$ , we can always take  $w = 1$  in Theorem 1.3. When  $w > 1$ , the theorem gives a stronger conclusion. (For a trivial example, if  $l_0$  falls within some compact subgroup of  $G$ , then  $\sup\{\rho(l_0^n) : n \in \mathbb{N}\} < \infty$  and we could choose  $w$  arbitrarily large).

We will apply the above results to two main classes of groups  $G$ : Lie groups of polynomial growth, and discrete, finitely generated groups of polynomial growth. Let us briefly discuss these cases: further details and generalizations will be given in Section 4.

**1.1. Lie groups of polynomial growth** Let  $G$  be a connected Lie group of polynomial growth of order  $D$ , with Lie algebra  $\mathfrak{g}$  and exponential map  $\exp : \mathfrak{g} \rightarrow G$ . To each element  $x \in \mathfrak{g}$  we associate a right-invariant vector field

$$X = dL_G(x) = -\lim_{t \rightarrow 0} t^{-1}(I - L(\exp tx)).$$

In the Lie setting, it is natural to try to rewrite our results in terms of right-invariant derivatives of  $K_t$ . Let  $\mathcal{R}(G)$  denote the algebra of all right-invariant differential operators on  $G$  (it is the complex linear span of the identity  $I$  and all monomials  $X_1 \dots X_k$ , where  $k \geq 1$  and  $X_i = dL_G(x_i)$  for some  $x_i \in \mathfrak{g}$ ). We introduce an assumption of smoothness for small times in  $\mathcal{T}$ : suppose that for any  $P \in \mathcal{R}(G)$  there exist  $c, b > 0$  (depending on  $P$ ) with

$$(5) \quad |PK_t| \leq cG_{b,1}$$

for all  $t \in \mathcal{T}$  with  $1 \leq t < 2$ . Then we have

**THEOREM 1.4.** *Suppose  $Q$  is a Lie subgroup of  $G$ , with Lie algebra  $\mathfrak{q} \subseteq \mathfrak{g}$ , such that (4) holds. Suppose  $y \in \mathfrak{q}$  and  $w \geq 1, c_0 > 0$  with  $\rho(\exp sy) \leq c_0|s|^{1/w}$  for all  $s \in \mathbb{R}$  with  $|s| \geq 1$ , and set  $Y = dL_G(y)$ . Let  $R \in \mathcal{R}(G)$ , adopt the hypotheses of Theorem 1.1, and assume (5).*

*Then there are  $c, b' > 0$  such that  $|YRK_t| \leq cat^{-w/2}t^{-\delta}G_{b',t}$  for  $t \in \mathcal{T}$ . Therefore, if  $y_1, \dots, y_k \in \mathfrak{q}$  and  $w_1, \dots, w_k \geq 1$  satisfy  $\rho(\exp sy_j) \leq c_0|s|^{1/w_j}, |s| \geq 1$ , then with  $Y_i = dL_G(y_i), |Y_1 \dots Y_k K_t| \leq ct^{-(w_1+\dots+w_k)/2}G_{b,t}$  for all  $t \in \mathcal{T}$ .*

Note that any  $x \in \mathfrak{g}$  satisfies  $\rho(\exp sx) = O(|s|)$  for  $|s| \geq 1$ . Thus one can always choose  $w = 1$  in Theorem 1.4. If  $w > 1$  then the theorem gives a sharper estimate.

The main example of a Lie subgroup satisfying (4) is the nilradical  $Q$ , that is, the largest nilpotent normal analytic subgroup of  $G$ . We will verify (4) for this case in Section 3 below. In particular, note that if  $G$  is nilpotent then  $G = Q$ .

The most important examples of convolution semigroups satisfying our assumptions are (i)  $\{K_t\}_{t>0}$  is the heat kernel of a right-invariant sublaplacian  $H$  on  $G$  (or of a more general second-order, subelliptic operator), and (ii)  $\{K_n\}_{n \in \mathbb{N}}$  are the  $n$ -th convolution powers of a smooth probability density  $K_1$  on  $G$  satisfying certain conditions. We will explain these examples fully in Section 4.

Note that kernel estimates on Lie groups are often expressed in terms of a subelliptic modulus  $\rho_A$  associated with a generating basis  $A_1, \dots, A_d$  of (say) right-invariant vector fields (see [17, 21]). It is well known (see, for example, [21, Section III.4]) that  $\rho$  and  $\rho_A$  are equivalent at infinity: there is  $c > 0$  with  $c^{-1}\rho \leq 1 + \rho_A \leq c\rho$ . The local behaviour of  $\rho_A$  is irrelevant to our results (which deal only with large times), and we prefer always to work with  $\rho$ .

It is interesting to observe that, because  $1 + \rho_A \leq c\rho$ , condition (3) is weaker than the analogous condition with  $\rho_A$  replacing  $\rho$ .

**1.2. Discrete groups of polynomial growth** Let  $G$  be a finitely generated, discrete group of polynomial growth of order  $D$ . By a theorem of Gromov [14],  $G$  has a finitely generated, nilpotent, normal subgroup  $Q$  such that  $G/Q$  is finite. We will verify in Section 3 that  $Q$  satisfies assumption (4). Observe that if  $G$  is nilpotent then we can take  $G = Q$ .

Let  $\{K_n\}_{n \in \mathbb{N}}$  be the  $n$ -th convolution powers of a probability density  $K_1$  on  $G$  satisfying certain conditions (namely,  $K_1$  is symmetric, and the support of  $K_1$  is finite and generates  $G$ ). Then results of Hebisch and Saloff-Coste [15] yield (2) and (3) with  $\nu = 1$ , so that our Theorems 1.1–1.3 apply. We explain this example further in Section 4.

It is worth noting that though we only consider second-order Gaussian bounds, our results readily extend to a setting of  $m$ -th order Gaussian estimates for general  $m > 0$ . Here the  $m$ -th order Gaussian is given by  $G_{b,t}^{(m)}(g) = t^{-D/m} \exp(-b(\rho(g)^m/t)^{1/(m-1)})$ . To prove this generalization, the main observation is that analogues of (6) and (7) below are valid for  $m$ -th order Gaussians. We omit further details.

## 2. Proof of the main theorems

In this section we prove Theorems 1.1, 1.3 and 1.4. First, we use (4) to prove a weak version of Theorem 1.1 (Proposition 2.2 below). Then we obtain Theorem 1.1 by iterating the weak result, applying at each step an interpolation lemma for Hölder estimates. Theorems 1.3 and 1.4 are proved by a similar iteration and interpolation process.

Without loss of generality, in the proofs we always assume that the constant  $a$  in the hypotheses of these theorems equals 1 (otherwise, simply replace  $R$  by  $a^{-1}R$ ).

To avoid excessive bracketing, let us abbreviate  $(L(h)\varphi)(g)$  to  $L(h)\varphi(g)$  when there is no risk of confusion.

We need some well-known estimates for Gaussians (see for example [9, Section 2]). For any  $b > 0$ , there is  $c > 0$  such that

$$(6) \quad \int_G dg G_{b,t}(g) \leq c, \quad \int_{\rho(g) \geq \lambda t^{1/2}} dg G_{b,t}(g) \leq ce^{-b\lambda^2/2}$$

for all  $t \geq 1$  and  $\lambda > 0$ , and moreover there is  $b' > 0$  such that

$$(7) \quad G_{b,s} * G_{b,t} \leq cG_{b',s+t}$$

for all  $s \geq 1$  and  $t \geq 1$ . The following elementary remark will also be useful.

**LEMMA 2.1.** *Suppose  $a > 0$ ,  $b > 0$ ,  $t \geq 1$ , and  $\varphi : G \rightarrow \mathbb{C}$  is a function with  $|\varphi| \leq aG_{b,t}$ . Then for any  $\kappa > 0$  and any  $\delta \in [0, 1)$ , there are  $c', b' > 0$  which*

depend only on  $b, \kappa, \delta$ , such that  $|L(h)\varphi(g)| \leq c' a G_{b,t}(g)$  for all  $g, h \in G$  such that  $\rho(h) \leq \kappa t^{1/2} + \delta\rho(g)$ .

PROOF. It follows from  $|\varphi(h^{-1}g)| \leq at^{-D/2}e^{-b\rho(h^{-1}g)^2/t}$  by noting that  $\rho(h^{-1}g) \geq \rho(g) - \rho(h) \geq (1 - \delta)\rho(g) - \kappa t^{1/2}$ . □

PROPOSITION 2.2. Suppose  $R$  a densely defined, right-invariant operator in  $L_2$ , and that for some  $b > 0, \delta \geq 0, |RK_t| \leq t^{-\delta}G_{b,t}$  for all  $t \in \mathcal{T}$ . Then there are  $c > 0, b' > 0$ , with

$$(8) \quad |(I - L(l))RK_t| \leq c(\rho(l)t^{-1/2})^{\alpha\nu} t^{-\delta}G_{b',t}$$

for all  $t \in \mathcal{T}$  and  $l \in Q$  with  $\rho(l) \leq t^{1/2}$ .

PROOF. Let  $g \in G, l \in Q$  and  $t \in \mathcal{T}$  with  $\rho(l) \leq t^{1/2}$  (constants below will be independent of  $g, l, t$ ). Since  $R$  is right-invariant then  $RK_{2t} = RK_t * K_t$ , so that

$$(9) \quad \begin{aligned} (I - L(l))RK_{2t}(g) &= \int dh RK_t(h)[K_t(h^{-1}g) - K_t(h^{-1}l^{-1}g)] \\ &= \int dh RK_t(h)(I - L(h^{-1}lh))K_t(h^{-1}g). \end{aligned}$$

Split the integral in (9) into two regions  $\rho(h) \leq 2^{-1}\rho(g), \rho(h) > 2^{-1}\rho(g)$ . In case  $\rho(h) \leq 2^{-1}\rho(g)$  then  $\rho(h^{-1}g) \geq \rho(g) - \rho(h) \geq \rho(h)$ , and using (4) we obtain

$$\rho(h^{-1}lh) \leq c(\rho(l) + \rho(l)^\alpha \rho(h)^{1-\alpha}) \leq c'\rho(l) + 2^{-1}\rho(h) \leq c't^{1/2} + 2^{-1}\rho(h^{-1}g).$$

Therefore, from (2), (3), and Lemma 2.1, when  $\rho(h) \leq 2^{-1}\rho(g)$  we get an estimate

$$\begin{aligned} &|(I - L(h^{-1}lh))K_t(h^{-1}g)| \\ &\leq c(\rho(h^{-1}lh)t^{-1/2})^\nu t^{-D/2}e^{-b\rho(h^{-1}g)^2/t} \\ &\leq c' \left\{ (\rho(l)t^{-1/2})^\nu + (\rho(l)t^{-1/2})^{\alpha\nu} (\rho(h)t^{-1/2})^{(1-\alpha)\nu} \right\} t^{-D/2}e^{-b\rho(h^{-1}g)^2/t}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\int_{\rho(h) \leq 2^{-1}\rho(g)} dh |RK_t(h)| |(I - L(h^{-1}lh))K_t(h^{-1}g)| \\ &\leq c(\rho(l)t^{-1/2})^\nu t^{-\delta} \int dh G_{b,t}(h)G_{b,t}(h^{-1}g) \\ &\quad + c(\rho(l)t^{-1/2})^{\alpha\nu} t^{-\delta} \int dh G_{b,t}(h) (\rho(h)t^{-1/2})^{(1-\alpha)\nu} G_{b,t}(h^{-1}g) \\ &\leq c(\rho(l)t^{-1/2})^{\alpha\nu} t^{-\delta}G_{b',t}(g), \end{aligned}$$

where in the last step we absorbed  $(\rho(h)t^{-1/2})^{(1-\alpha)v}$  into the Gaussian and applied (7).

Next, it follows from (3) and (2) that  $\|(I - L(x))K_t\|_\infty \leq c(\rho(x)t^{-1/2})^v t^{-D/2}$  for all  $x \in G$ . Therefore (4) gives an estimate, when  $\rho(h) \geq 2^{-1}\rho(g)$ ,

$$\|(I - L(h^{-1}lh))K_t\|_\infty \leq c \left\{ (\rho(l)t^{-1/2})^v + (\rho(l)t^{-1/2})^{\alpha v} (\rho(h)t^{-1/2})^{(1-\alpha)v} \right\} t^{-D/2}.$$

Thus

$$\begin{aligned} & \int_{\rho(h) \geq 2^{-1}\rho(g)} dh |RK_t(h)| |(I - L(h^{-1}lh))K_t(h^{-1}g)| \\ & \leq ct^{-\delta} t^{-D/2} \int_{\rho(h) \geq 2^{-1}\rho(g)} dh G_{b,t}(h) \left\{ \left(\frac{\rho(l)}{t^{1/2}}\right)^v + \left(\frac{\rho(l)}{t^{1/2}}\right)^{\alpha v} \left(\frac{\rho(h)}{t^{1/2}}\right)^{(1-\alpha)v} \right\} \\ & \leq c (\rho(l)t^{-1/2})^{\alpha v} t^{-\delta} t^{-D/2} e^{-b\rho(g)^2/t}, \end{aligned}$$

where in the last step we applied (6). From the above estimates and (9), the desired estimate (8) follows for all  $t \in 2\mathcal{I}$ .

Finally, consider  $t \in \mathcal{I}$  such that  $t \notin 2\mathcal{I}$ . When  $\mathcal{I} = [1, \infty)$  then Lemma 2.1 yields (8) when  $t \in [1, 2)$ . When  $\mathcal{I} = \mathbb{N}$  then Lemma 2.1 yields (8) for  $t = 1$ , while for  $t$  of form  $t = 2n + 1$ ,  $n \in \mathbb{N}$ , then (8) follows by writing  $(I - L(l))RK_t = ((I - L(l))RK_{2n}) * K_1$  and applying (7). Thus (after adjusting constants) we obtain (8) for all  $t \in \mathcal{I}$ , and the proposition is proved.  $\square$

The following interpolation lemma for Hölder estimates will allow us to improve the estimate of Proposition 2.2. To prove the lemma we rely on a general identity for an operator  $A$ ,

$$(10) \quad I - A = \sum_{j=0}^k 2^{-j-1} [I - A^{2^j}]^2 + 2^{-k-1} (I - A^{2^{k+1}})$$

which is valid for all  $k \in \mathbb{N}_0$  (for the use of this identity in interpolation theory see [6, Section 3.4.2]).

**LEMMA 2.3.** *Let  $G_1$  be a group,  $\psi : G_1 \rightarrow G$  a homomorphism and  $\rho_1 : G_1 \rightarrow [0, \infty)$  a submultiplicative function:  $\rho_1(s_1s_2) \leq \rho_1(s_1) + \rho_1(s_2)$  for  $s_1, s_2 \in G_1$ . Suppose  $w \geq 1$  and that  $\rho(\psi(s)) \leq c_0(1 + \rho_1(s)^{1/w})$  for all  $s \in G_1$ . Let  $t \geq 1$ ,  $a > 0$ ,  $0 < \gamma < \beta$ ,  $\gamma \leq 1$ , and suppose  $\varphi : G \rightarrow \mathbb{C}$  is a function with*

$$|\varphi| \leq aG_{b,t}, \quad |(I - L(\psi(s)))^2\varphi| \leq a (\rho_1(s)t^{-w/2})^\beta G_{b,t}$$

for all  $s \in G_1$  with  $\rho_1(s) \leq t^{w/2}$ . Then there exist  $c', b' > 0$ , which depend on  $w, c_0, \gamma, \beta, b$  but not on  $t$  or  $a$ , such that

$$|(I - L(\psi(s)))\varphi| \leq c'a (\rho_1(s)t^{-w/2})^\gamma G_{b',t}$$

for all  $s \in G_1$  with  $\rho_1(s) \leq t^{w/2}$ .



PROOF. Without loss of generality we may suppose that  $a = 1$ . Let  $s \in G_1$  with  $0 < \rho_1(s) \leq t^{w/2}$  (the case  $\rho_1(s) = 0$  is similar but easier and is left to the reader). Choose  $k \in \mathbb{N}_0$  with  $2^k \leq (\rho_1(s)t^{-w/2})^{-1} \leq 2^{k+1}$ ; constants in this proof will be independent of  $t, s$  and  $k$ . Applying (10) with  $A = L(\psi(s))$  we have

$$[I - L(\psi(s))]\varphi = \sum_{j=0}^k 2^{-j-1} [I - L(\psi(s^{2^j}))]^2 \varphi + 2^{-k-1} [I - L(\psi(s^{2^{k+1}}))]\varphi.$$

To estimate the last term, first observe that

$$\rho(\psi(s^{2^{k+1}})) \leq c_0(1 + (2^{k+1}\rho_1(s))^{1/w}) \leq c_0(1 + 2t^{1/2}) \leq 3c_0t^{1/2}.$$

Therefore, the hypothesis and Lemma 2.1 yield an estimate

$$2^{-k-1} |[I - L(\psi(s^{2^{k+1}}))]\varphi| \leq c2^{-k} G_{b,t} \leq c(2^{-k})^\gamma G_{b,t} \leq c'(\rho_1(s)t^{-w/2})^\gamma G_{b,t},$$

where we used  $\gamma \leq 1$ . For  $j \in \{0, 1, \dots, k\}$ , one has

$$\rho_1(s^{2^j}) \leq 2^j \rho_1(s) \leq 2^k \rho_1(s) \leq t^{w/2},$$

so applying the hypothesis we find that

$$\begin{aligned} \sum_{j=0}^k 2^{-j-1} |[I - L(\psi(s^{2^j}))]^2 \varphi| &\leq c \sum_{j=0}^k 2^{-j} (2^j \rho_1(s)t^{-w/2})^\beta G_{b,t} \\ &= c(\rho_1(s)t^{-w/2})^\beta \sum_{j=0}^k (2^{\beta-1})^j G_{b,t}. \end{aligned}$$

Since  $\gamma \leq 1$  and  $\beta - \gamma > 0$  we have

$$\sum_{j=0}^k (2^{\beta-1})^j \leq \sum_{j=0}^k (2^{\beta-\gamma})^j \leq c(2^k)^{\beta-\gamma} \leq c(\rho_1(s)t^{-w/2})^{\gamma-\beta},$$

where  $c$  depends only on  $\beta$  and  $\gamma$ . Lemma 2.3 follows by collecting these estimates. □

COROLLARY 2.4. *Suppose  $G_1$  is a subgroup of  $G, t \geq 1, a > 0, 0 < \gamma < \beta, \gamma \leq 1$ , and  $\varphi : G \rightarrow \mathbb{C}$  satisfies  $|\varphi| \leq aG_{b,t}$  and  $|(I - L(l))^2 \varphi| \leq a(\rho(l)t^{-1/2})^\beta G_{b,t}$  for all  $l \in G_1$  with  $\rho(l) \leq t^{1/2}$ . Then there exist  $c', b' > 0$ , which only depend on  $\gamma, \beta, b$ , such that  $|(I - L(l))\varphi| \leq c'a(\rho(l)t^{-1/2})^\gamma G_{b',t}$  for all  $l \in G_1$  with  $\rho(l) \leq t^{1/2}$ .*

PROOF. Apply Lemma 2.3 with  $\psi : G_1 \rightarrow G$  the inclusion map,  $\rho_1(s) = \rho(s)$  for  $s \in G_1$ , and  $w = 1$ . □

**PROOF OF THEOREM 1.1.** Let  $R$  be as in the statement of the theorem, and suppose  $\alpha\nu < 1$  (if  $\alpha\nu = 1$  then by Proposition 2.2 there is nothing to prove). Choose  $N \in \mathbb{N}$  and  $\mu_1, \dots, \mu_N$  satisfying  $\alpha\nu = \mu_1 < \mu_2 < \dots < \mu_N = 1$  and  $\mu_{i+1} - \mu_i < \alpha\nu$  for  $i \in \{1, 2, \dots, N - 1\}$ . We show that for each  $i$  there is an estimate

$$(11) \quad |(I - L(l))RK_t| \leq c (\rho(l)t^{-1/2})^{\mu_i} t^{-\delta} G_{b,t}$$

for all  $t \in \mathcal{T}$  and  $l \in Q$  with  $\rho(l) \leq t^{1/2}$ . The case  $i = 1$  is just Proposition 2.2. Suppose  $j \in \{1, \dots, N - 1\}$  and (11) holds when  $i = j$ . Then applying Proposition 2.2 for the right-invariant operators  $(I - L(l))R$  yields an estimate

$$|(I - L(l_1))(I - L(l))RK_t| \leq c (\rho(l_1)t^{-1/2})^{\alpha\nu} (\rho(l)t^{-1/2})^{\mu_j} t^{-\delta} G_{b,t}$$

for all  $l_1, l \in Q$  with  $\rho(l_1) \leq t^{1/2}$ ,  $\rho(l) \leq t^{1/2}$ . Taking  $l_1 = l$  in this estimate, we apply Corollary 2.4 with  $\varphi = RK_t$  and with  $\beta = \mu_j + \alpha\nu$ ,  $\gamma = \mu_{j+1} < \beta$ . Thus we obtain (11) with  $i = j + 1$ , and induction yields (11) for all  $i$ . With  $i = N$  we get Theorem 1.1. □

**PROOF OF THEOREM 1.3.** The proof is a variation of the iteration-reduction argument just given. Let  $R, w$  and  $l_0 \in Q$  be as in the statement of the theorem.

Choose  $N \in \mathbb{N}$ ,  $N > w$  and  $\mu_1, \dots, \mu_N$  so that  $1/w = \mu_1 < \dots < \mu_N = 1$  and  $\mu_{i+1} - \mu_i < 1/w$  for  $i \in \{1, \dots, N - 1\}$ . We show that for each  $i \in \{1, \dots, N\}$  one has an estimate

$$(12) \quad |(I - L(l_0^n))RK_t| \leq c (|n|t^{-w/2})^{\mu_i} t^{-\delta} G_{b,t}$$

for all  $t \in \mathcal{T}$  and  $n \in \mathbb{Z}$  with  $|n| \leq t^{w/2}$ . For  $i = 1$ , the estimate is valid by Theorem 1.1 and because  $\rho(l_0^n) \leq c|n|^{1/w}$  for  $|n| \geq 1$ .

Then argue inductively as in the proof of Theorem 1.1: if (12) is valid for  $i$ , apply Theorem 1.1 to get a bound for  $(I - L(l_0^n))^2 RK_t$ . Then use Lemma 2.3 with  $\beta = \mu_i + 1/w$ ,  $\gamma = \mu_{i+1}$ , to obtain (12) with  $\mu_{i+1}$  replacing  $\mu_i$ . Here, to apply Lemma 2.3 we take  $G_1 = \mathbb{Z}$  and the homomorphism  $\psi : \mathbb{Z} \rightarrow G$ ,  $\psi(n) = l_0^n$ , and set  $\rho_1(n) = |n|$  for  $n \in \mathbb{Z}$ .

With  $i = N$  and  $n = 1$  in (12), we get Theorem 1.3. □

**PROOF OF THEOREM 1.4.** This is slightly more delicate than the previous proofs, because we now also interpolate with the local assumption (5).

First we extend (5) to large times. Let  $P \in \mathcal{A}(G)$  be any right-invariant differential operator. If  $t \in \mathcal{T}$  and  $t \geq 2$ , then  $t - 1 \in \mathcal{T}$  and we can write  $PK_t = PK_1 * K_{t-1}$ . Then from (5) and (2), by applying (7) we obtain an estimate of form

$$(13) \quad |PK_t| \leq cG_{b,t}$$

for all  $t \in \mathcal{I}$ .

Let  $R, \delta, \gamma, w$  be as in the hypothesis of Theorem 1.4. Let us fix  $0 < \varepsilon < 1$  sufficiently small so that

$$\tau := 2^{-1}(1 - \varepsilon) - \varepsilon(w/2 + \delta) > 0,$$

and fix  $\rho > 0$  with  $\rho < \varepsilon$  and  $(w/2)\rho < \tau$ . Choose  $N \in \mathbb{N}$  and  $\mu_0, \dots, \mu_N$  such that  $0 = \mu_0 < \mu_1 < \dots < \mu_N = 1$  and  $\mu_{i+1} - \mu_i < \rho$ . We show that for each  $i \in \{0, 1, \dots, N\}$  there is an estimate

$$(14) \quad |(I - L(\exp sy))RK_t| \leq c(|s|t^{-w/2})^{\mu_i} t^{-\delta} G_{b,t}$$

for all  $t \in \mathcal{I}$  and  $s \in \mathbb{R}$  with  $|s| \leq t^{w/2}$ . Note that Theorem 1.4 follows from the case  $i = N$ , because  $YRK_t(g) = -\lim_{s \rightarrow 0} s^{-1}(I - L(\exp sy))RK_t(g)$ .

Notice that an estimate

$$(15) \quad \rho(\exp sy) \leq c(1 + |s|^{1/w}) \leq c't^{1/2}$$

is valid whenever  $|s| \leq t^{w/2}$  and  $t \geq 1$ . In case  $i = 0, \mu_i = 0$ , then (14) follows from  $|RK_t| \leq t^{-\delta} G_{b,t}$  by applying Lemma 2.1 and (15).

Assume inductively (14) for some  $i \in \{0, 1, \dots, N - 1\}$ . Then it follows from Theorem 1.1 and (15) that

$$(16) \quad |(I - L(\exp sy))^2 RK_t| \leq c(1 + |s|^{1/w}) t^{-1/2} (|s|t^{-w/2})^{\mu_i} t^{-\delta} G_{b,t}$$

when  $|s| \leq t^{w/2}$ . We need to improve this estimate for small  $|s|$ . By writing  $I - L(\exp sy) = -\int_0^s du L(\exp uy)Y$ , and applying (13) with  $P = Y^2R$ , one gets a bound

$$\|(I - L(\exp sy))^2 RK_t\|_\infty \leq |s|^2 \|Y^2 RK_t\|_\infty \leq c|s|^2 t^{-D/2}$$

whenever  $t \in \mathcal{I}$  and  $|s| \leq 1$ . Interpolating this with (16), we obtain for  $|s| \leq 1$  and  $t \in \mathcal{I}$  that

$$(17) \quad \begin{aligned} & |(I - L(\exp sy))^2 RK_t| \\ & \leq |(I - L(\exp sy))^2 RK_t|^{1-\varepsilon} (\|(I - L(\exp sy))^2 RK_t\|_\infty)^\varepsilon \\ & \leq c(t^{-(1/2)-(w/2)\mu_i-\delta})^{1-\varepsilon} |s|^{\mu_i(1-\varepsilon)} |s|^{2\varepsilon} G_{b,t} \\ & \leq ct^{-\delta} t^{-(w/2)\mu_i} t^{-\tau} |s|^{\mu_i+\varepsilon} G_{b,t} \leq ct^{-\delta} (|s|t^{-w/2})^{\mu_i+\rho} G_{b,t}, \end{aligned}$$

where the last steps follow from the choice of  $\varepsilon$  and  $\rho$ . Observe that (17) also holds for  $1 \leq |s| \leq t^{w/2}$ , by (16) and because  $\rho < 2\tau/w < 1/w$ .

Apply Lemma 2.3 with  $\beta = \mu_i + \rho, \gamma = \mu_{i+1}$ , for the homomorphism  $\psi : \mathbb{R} \rightarrow G$  such that  $\psi(s) = \exp sy, s \in \mathbb{R}$ , and with  $\rho_i(s) = |s|$ . Then from (17) we get (14) with  $\mu_{i+1}$  replacing  $\mu_i$ . This ends the proof of (14) and of the theorem.  $\square$

### 3. The geometric condition

Given  $\alpha \in (0, 1]$  and a subgroup  $Q$  of  $G$ , let us say that condition  $(G, Q, \alpha)$  holds if an estimate (4) holds. In this section we study this condition and, in particular, we identify important examples of  $G$  and  $Q$  for which it holds.

We begin with a few general remarks.

(a) An elementary estimate shows that condition  $(G, Q, \alpha)$  implies condition  $(G, Q, \alpha')$  whenever  $0 < \alpha' < \alpha \leq 1$ .

(b) Let  $K$  be a compact normal subgroup of  $G$ , let  $G' = G/K$ , and suppose that condition  $(G', G', \alpha)$  holds. We claim that condition  $(G, G, \alpha)$  holds.

Note that we may choose the compact generating set  $U$  so that  $K \subseteq U$ . If  $\pi : G \rightarrow G'$  is the canonical homomorphism, then  $U' = \pi(U)$  is a compact generating set for  $G'$ , such that  $\pi(U^n) = (U')^n$  and  $\pi^{-1}((U')^n) \subseteq KU^n \subseteq U^{n+1}$  for  $n \in \mathbb{N}$ . If  $\rho, \rho'$  denote the moduli on  $G, G'$  respectively associated with  $U$  and  $U'$ , we have (see [21, Lemma IV.5.5])  $c^{-1}\rho'(\pi g) \leq \rho(g) \leq c\rho'(\pi g)$  for all  $g \in G$ . Then the claim follows easily.

(c) If condition  $(G, Q, \alpha)$  holds, and  $K$  is a compact normal subgroup of  $G$ , then  $KQ = QK$  is a subgroup of  $G$ , and  $(G, KQ, \alpha)$  holds. Indeed, setting  $c_0 = \sup\{\rho(k) : k \in K\} < \infty$ , it suffices to observe that  $\rho(q) \leq \rho(kq) + c_0$  and  $\rho(gkqg^{-1}) \leq \rho(gkg^{-1}) + \rho(gqg^{-1}) \leq c_0 + \rho(gqg^{-1})$  for all  $g \in G, k \in K$  and  $q \in Q$ .

In particular, note that condition  $(G, K, 1)$  holds for any compact normal subgroup of  $G$ . This is not true for an arbitrary compact subgroup of  $G$  (see Example 3.4 below).

The next lemma is basic for the development of this section.

**LEMMA 3.1.** *Let  $G$  be a simply connected, nilpotent Lie group with Lie algebra  $\mathfrak{g}$  of nilpotent step  $r$ . Then condition  $(G, G, r^{-1})$  holds.*

**PROOF.** Let  $\mathfrak{g}_1 \supseteq \mathfrak{g}_2 \supseteq \dots$  be the lower central series of  $\mathfrak{g}$ , so  $\mathfrak{g}_1 = \mathfrak{g}$  and  $\mathfrak{g}_{j+1} = [\mathfrak{g}, \mathfrak{g}_j]$  for all  $j$ . Then  $\mathfrak{g}_j = \{0\}$  when  $j \geq r + 1$ . Choose linear subspaces  $V_j$  of  $\mathfrak{g}$  with  $\mathfrak{g}_j = V_j \oplus \mathfrak{g}_{j+1}$ ,  $1 \leq j \leq r$ , and let  $b_1, \dots, b_d$  be a vector space basis for  $\mathfrak{g}$  such that  $b_i \in V_{w(i)}$  for some  $w(i) \in \{1, \dots, r\}$ . If  $x = \sum_i \xi_i b_i \in \mathfrak{g}$  set

$$\|x\| = \sum_{i=1}^d |\xi_i|^{1/w(i)}.$$

Note the inequalities  $\|x + y\| \leq c\|x\| + c\|y\|$  and  $\|\lambda x\| \leq (1 + |\lambda|)\|x\|$  for  $x, y \in \mathfrak{g}$  and  $\lambda \in \mathbb{R}$ .

Since  $G$  is simply connected then  $\exp : \mathfrak{g} \rightarrow G$  is a diffeomorphism, and there is  $c > 0$  with  $c^{-1}\rho(g) \leq \|\exp^{-1}(g)\| + 1 \leq c\rho(g)$  for all  $g \in G$  (see [21, Sections III.4 and IV.5]). If  $g = \exp x$ ,  $l = \exp y$ , then  $glg^{-1} = \exp(e^{\text{ad}x}y)$ , and we see that to prove the lemma it suffices to show that

$$(18) \quad \|e^{\text{ad}x}y\| \leq c(1 + \|y\| + \|y\|^{1/r}\|x\|^{1-(1/r)})$$

for all  $x, y \in \mathfrak{g}$ . Observe that

$$\|e^{\text{ad}x}y\| = \left\| y + \sum_{k=1}^{r-1} (k!)^{-1}(\text{ad}x)^k y \right\| \leq c\|y\| + c \sum_{k=1}^{r-1} \|(\text{ad}x)^k y\|.$$

Write  $x = \sum \xi_i b_i$ ,  $y = \sum \eta_j b_j$ . Because  $[\mathfrak{g}_p, \mathfrak{g}_q] \subseteq \mathfrak{g}_{p+q}$  for all  $p, q$ , we can expand  $(\text{ad}x)^k y$  as a finite linear combination, with constant coefficients, of terms of the form  $(\xi_{i_1} \dots \xi_{i_k} \eta_j) b_s$ , where  $w(i_1) + \dots + w(i_k) + w(j) \leq w(s) \leq r$ . Fix such a term and set  $\sigma = w(i_1) + \dots + w(i_k) + w(j) \in \{1, \dots, r\}$ . Then

$$\begin{aligned} \|(\xi_{i_1} \dots \xi_{i_k} \eta_j) b_s\| &= |\xi_{i_1} \dots \xi_{i_k} \eta_j|^{1/w(s)} \\ &\leq 1 + |\xi_{i_1} \dots \xi_{i_k} \eta_j|^{1/\sigma} \\ &\leq 1 + (\|x\|^{w(i_1)+\dots+w(i_k)} \|y\|^{w(j)})^{1/\sigma} \\ &= 1 + \|x\|^{1-(w(j)/\sigma)} \|y\|^{w(j)/\sigma} \\ &\leq 1 + c(\|y\| + \|y\|^{1/r}\|x\|^{1-(1/r)}), \end{aligned}$$

where the last line follows by an elementary inequality because  $w(j)/\sigma \in [1/r, 1]$ . By collecting these estimates we obtain (18) and the lemma. □

**COROLLARY 3.2.** *Condition  $(G, G, r^{-1})$  holds whenever  $G$  is a connected nilpotent Lie group with Lie algebra of nilpotent step  $r$ .*

**PROOF.** There exists a compact normal subgroup  $K$  of  $G$  such that the group  $G' = G/K$  is simply connected (see [21, Section IV.1]). Since  $G'$  is nilpotent of step  $r'$  with  $r' \leq r$ , the corollary follows from Lemma 3.1 together with Remarks (a) and (b) above. □

In connection with the above results, let us recall that any connected nilpotent Lie group has polynomial growth (see for example [21, Chapter IV]).

**REMARK.** For  $G$  a connected Lie group of polynomial growth, one has the following precise criterion. Condition  $(G, G, \alpha)$  holds for some  $\alpha \in (0, 1]$  if and only if  $G$  is the local direct product of a compact Lie group and a nilpotent Lie group, in other words,

$G = K \cdot Q$  where  $K, Q$  are closed, mutually commuting, analytic normal subgroups of  $G$  with  $K$  compact,  $Q$  nilpotent and  $K \cap Q$  discrete.

Since  $G/K \cong Q/(K \cap Q)$  is nilpotent, the ‘if’ direction follows from Corollary 3.2 and Remark (b). To prove the converse we could use structure theory, but one can alternatively argue as follows. If  $(G, G, \alpha)$  holds and  $\{K_t\}_{t>0}$  is the heat kernel for a right-invariant sublaplacian on  $G$ , then Theorem 1.4 would imply an estimate  $\|X_1 X_2 K_t\|_\infty \leq ct^{-1}t^{-D/2}$ ,  $t \geq 1$ , for any given right-invariant vector fields  $X_1, X_2$  on  $G$ . By results of [13] (see also [7]) this can only occur if  $G$  is a local direct product as above.

To obtain a result for  $G$  an arbitrary connected Lie group of polynomial growth, we need some structure theory (for detailed descriptions of the structure theory see [1, 3, 8, 7] and references therein). Denote by  $\mathfrak{g}$  the Lie algebra of  $G$ . One has  $G = MS$  where  $S$  is the radical (the largest solvable normal analytic subgroup of  $G$ ) and  $M$  is any Levi subgroup of  $G$ :  $M$  is necessarily compact. If  $Q$  denotes the nilradical (the largest nilpotent normal analytic subgroup) of  $G$ , then  $Q \subseteq S$ .

To the solvable Lie group  $S$  one can associate its nilshadow  $S_N$  which is a nilpotent Lie group. We can identify  $S = S_N$  as manifolds, such that the corresponding Lie algebras  $\mathfrak{s}$  and  $\mathfrak{s}_N$  are identified as vector spaces. The group structures of  $S$  and  $S_N$  are related by

$$(19) \quad s_1 s_2 = (T(s_2^{-1N})s_1) *_N s_2, \quad s_1^{-1} = T(s_1)(s_1^{-1N})$$

for all  $s_1, s_2 \in S$  (see [8, 7]). Here  $*_N$  denotes the group product of  $S_N$ ,  $s^{-1N}$  is the  $S_N$ -inverse of  $s$ , and  $T$  is a certain homomorphism from  $S_N$  to the group of smooth automorphisms of  $S_N$ , such that

$$(20) \quad T(T(s_1)s_2) = T(s_2), \quad T(q)s_1 = s_1$$

for all  $s_1, s_2 \in S$  and  $q \in Q$ . Let  $\rho, \rho_S, \rho_N$  denote moduli associated with compact generating neighbourhoods for the respective groups  $G, S, S_N$ . Then

$$(21) \quad \rho(ms) \leq c_1\rho(s) \leq c_2\rho_S(s) \leq c_3\rho_N(s) \leq c_4\rho(ms)$$

for all  $m \in M$  and  $s \in S$ . The automorphisms  $T(s)$  are uniformly bounded, in the sense that there is  $c > 0$  with  $\rho(T(s_1)s_2) \leq c\rho(s_2)$  for all  $s_1, s_2 \in S$ .

**PROPOSITION 3.3.** *Let  $G$  be a connected Lie group of polynomial growth, with radical  $S$  and nilradical  $Q$ . Then condition  $(G, Q, r^{-1})$  is satisfied, where  $r$  is the nilpotent step of the Lie algebra  $\mathfrak{s}_N$  of the nilshadow  $S_N$ .*

**PROOF.** A straightforward calculation using (19) and (20) shows that

$$sqs^{-1} = T(s)(s *_N q *_N s^{-1N})$$

for all  $s \in S$  and  $q \in Q$ . Let  $\alpha = r^{-1}$ . Applying the uniform boundedness of the  $T(s)$ , Corollary 3.2, and the equivalences (21), we obtain

$$\rho(sq s^{-1}) \leq c\rho(s *_N q *_N s^{-1N}) \leq c' (\rho(q) + \rho(q)^\alpha \rho(s)^{1-\alpha})$$

for all  $s \in S$  and  $q \in Q$ . Therefore condition  $(S, Q, \alpha)$  holds.

Finally, for any  $g \in G$  we write  $g = ms$ ,  $m \in M$ ,  $s \in S$ , and using compactness of  $M$  get  $\rho(gqg^{-1}) \leq 2\rho(m) + \rho(sq s^{-1}) \leq c + \rho(sq s^{-1})$ . Then condition  $(G, Q, \alpha)$  follows from  $(S, Q, \alpha)$  and the bound  $\rho(s) \leq c\rho(ms)$ . □

REMARK. The above proposition can be slightly improved as follows. It is shown in [7] that  $G$  has a (possibly trivial) compact normal subgroup  $K$ , whose Lie algebra is the largest semisimple ideal of the Lie algebra of  $G$ . By Remark (c) above, we conclude that condition  $(G, Q', \alpha)$  holds where  $Q' = KQ$ .

EXAMPLE 3.4. Let us verify directly that  $(G, G, \alpha)$  does not hold for the group  $G$  of Euclidean motions of the plane (this fact is a special case of the Remark after Corollary 3.2).

We identify  $G = \mathbb{T} \times \mathbb{R}^2$  as a manifold, where  $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ , and the group multiplication is given by

$$(e^{it_1}, x_1, y_1)(e^{it_2}, x_2, y_2) = (e^{i(t_1+t_2)}, (\cos t_2)x_1 + (\sin t_2)y_1 + x_2, -(\sin t_2)x_1 + (\cos t_2)y_1 + y_2)$$

for  $t_j, x_j, y_j \in \mathbb{R}$ . Then  $G$  is a three-dimensional solvable Lie group of polynomial growth of order  $D = 2$ . It is a semidirect product of the compact subgroup  $\mathbb{T}_1 = \{(z, 0, 0) : z \in \mathbb{T}\}$  with the normal subgroup  $\mathbb{R}_1^2 = \{(1, x, y) : x, y \in \mathbb{R}\}$ . If  $\rho$  is a modulus (associated with a compact generating neighbourhood of  $G$ ), then

$$c^{-1}(1 + |x| + |y|) \leq \rho((z, x, y)) \leq c(1 + |x| + |y|)$$

for all  $g = (z, x, y) \in G$ . A calculation yields  $(1, x, 0)(-1, 0, 0)(1, x, 0)^{-1} = (-1, -2x, 0)$ , so that  $\rho((1, x, 0)(-1, 0, 0)(1, x, 0)^{-1}) \geq c(1 + |x|)$  for all  $x \in \mathbb{R}$ . Thus  $(G, \mathbb{T}_1, \alpha)$  fails, and hence  $(G, G, \alpha)$  fails, for any  $\alpha \in (0, 1]$ . On the other hand,  $(G, \mathbb{R}_1^2, \alpha)$  holds (with  $\alpha = 1$ ) because  $\mathbb{R}_1^2$  is the nilradical of  $G$ .

We have the following version of Theorem 1.4. Let  $\mathfrak{s}_N = \mathfrak{s}_{N,1} \supseteq \mathfrak{s}_{N,2} \supseteq \dots$  denote the lower central series of  $\mathfrak{s}_N$ .

COROLLARY 3.5. *Let  $G$  be a Lie group of polynomial growth,  $Q$  the nilradical of  $G$ , and  $\mathfrak{q}$  the Lie algebra of  $Q$ . Let  $R \in \mathcal{R}(G)$ , adopt the hypotheses of Theorem 1.1, and assume (5). If  $y \in \mathfrak{q}$ ,  $y \in \mathfrak{s}_{N,w}$ , and  $Y = dL_G(y)$ , then there are  $c, b' > 0$  such that  $|YRK_t| \leq cat^{-w/2}t^{-\delta}G_{b',t}$  for all  $t \in \mathcal{T}$ . Therefore if  $y_1, \dots, y_k \in \mathfrak{q}$  with  $y_i \in \mathfrak{s}_{N,w_i}$  for  $i = 1, \dots, k$ , then  $|Y_1 \dots Y_k K_t| \leq ct^{-(w_1+\dots+w_k)/2}G_{b',t}$  for all  $t \in \mathcal{T}$ .*

PROOF. Let  $y \in \mathfrak{q} \cap \mathfrak{s}_{N;w}$ . Since  $S_N$  is nilpotent and  $y \in \mathfrak{s}_{N;w}$  one has an estimate  $\rho_N(\exp_N sy) \leq c(1 + |s|^{1/w})$  for all  $s \in \mathbb{R}$  (see [21, Section IV.5]), where  $\exp_N$  is the exponential map of  $S_N$ . But  $\exp(x) = \exp_N(x)$  for all  $x \in \mathfrak{q}$  by, for example, [7, Section 10]. Then from (21),  $\rho(\exp sy) \leq c'(1 + |s|^{1/w})$  for all  $s \in \mathbb{R}$ . The corollary now follows from Theorem 1.4 and Proposition 3.3.  $\square$

Next we consider discrete groups. Note that a theorem of Bass [4] states that any finitely generated, discrete nilpotent group has polynomial growth of some order  $D$ .

LEMMA 3.6. *If  $G$  is a finitely generated, discrete nilpotent group then condition  $(G, G, \alpha)$  holds for some  $\alpha \in (0, 1]$ .*

PROOF. Let  $\tau(G)$  be the torsion subgroup of  $G$ , consisting of all elements of  $G$  of finite order. Then  $\tau(G)$  is a finite normal subgroup of  $G$  and the quotient  $G' = G/\tau(G)$  is finitely generated, discrete, nilpotent and torsion-free (see [5, Chapter 0]).

Since  $G'$  is torsion-free then  $G'$  is isomorphic to, hence can be identified with, a discrete, cocompact lattice subgroup of a simply connected nilpotent Lie group  $N$  (see [16]). Let  $\rho', \rho_N$  be moduli associated with compact generating neighborhoods for the respective groups  $G'$  and  $N$ . One has (as in [2, Section 1]) an inequality

$$c^{-1} \rho_N(g') \leq \rho'(g') \leq c \rho_N(g')$$

for all  $g' \in G'$ . Thus it follows from Lemma 3.1 that  $(G', G', r^{-1})$  holds, with  $r$  the nilpotent step of the Lie algebra of  $N$ . By Remark (b) at the start of this section, condition  $(G, G, r^{-1})$  holds.  $\square$

Recall that a theorem of Gromov [14] implies that any finitely generated discrete group of polynomial growth has a finitely generated, nilpotent normal subgroup of finite index.

COROLLARY 3.7. *Let  $G$  be a finitely generated, discrete group of polynomial growth and  $Q$  a finitely generated, nilpotent subgroup such that  $G/Q$  is finite. Then condition  $(G, Q, \alpha)$  holds for some  $\alpha \in (0, 1]$ .*

PROOF. Choose elements  $g_1, \dots, g_d \in G$  such that  $G = g_1 Q \cup \dots \cup g_d Q$ . If  $\rho, \rho_Q$  are moduli for  $G$  and  $Q$  respectively, it is straightforward to see that

$$\rho(g_i q) \leq c_1 \rho(q) \leq c_2 \rho_Q(q) \leq c_3 \rho(g_i q)$$

for all  $q \in Q$  and  $i \in \{1, \dots, d\}$ . Then since  $\rho(g_i q q_1 q^{-1} g_i^{-1}) \leq c + \rho(q q_1 q^{-1})$  for all  $q, q_1 \in Q$ , the corollary follows by means of condition  $(Q, Q, \alpha)$ .  $\square$



EXAMPLE 3.8. Let  $G$  be the group of Example 3.4 and consider the discrete subgroup  $G_0 = \{(\pm 1, n_1, n_2) : n_1, n_2 \in \mathbb{Z}\}$  of  $G$ . If  $\rho, \rho_0$  are moduli for  $G$  and  $G_0$  respectively, then  $c^{-1}\rho(\gamma) \leq \rho_0(\gamma) \leq c\rho(\gamma)$  for all  $\gamma \in G_0$ . Therefore, the calculation of Example 3.4 shows that condition  $(G_0, G_0, \alpha)$  fails. But it is easy to see that  $(G_0, Q_0, 1)$  holds, where  $Q_0 = \{(1, n_1, n_2) : n_1, n_2 \in \mathbb{Z}\}$  is an abelian subgroup of index 2 in  $G_0$ .

### 4. Applications

This section describes the main examples of convolution semigroups to which our results apply.

In examples (i) to (iv),  $G$  will denote a connected Lie group of polynomial growth with Lie algebra  $\mathfrak{g}$ , and  $Q$  denotes the nilradical of  $G$ , with Lie algebra  $\mathfrak{q}$ .

(i) On  $G$  consider a right-invariant sublaplacian  $H = -\sum_{i=1}^d A_i^2$ , where  $A_i = dL_G(a_i)$  and  $a_1, \dots, a_d \in \mathfrak{g}$  are a list of elements which generate the Lie algebra  $\mathfrak{g}$ . The theory of these operators is well-developed: see [17, 21, 1, 3, 13] and references therein. In particular  $H$  generates a semigroup  $S_t = e^{-tH}$ ,  $t > 0$ , in  $L_p$  ( $1 \leq p \leq \infty$ ), and  $S_t$  acts via a smooth convolution kernel  $\{K_t\}_{t>0}$ :  $S_t\varphi = K_t * \varphi$  for  $\varphi \in L_p$ .

Given any right-invariant vector field  $X = dL_G(x)$  one has estimates

$$(22) \quad |K_t| \leq cG_{b,t}, \quad |XK_t| \leq ct^{-1/2}G_{b,t}$$

for all  $t \geq 1$ . Note that for  $X \in \{A_1, \dots, A_d\}$  the estimate for  $|XK_t|$  was proved in [18], while for general  $X$  it is contained in results of [1, 3]. It is easy to obtain (3) with  $\nu = 1$  as a consequence of (22). Finally, the local estimate (5) is well-known (see [21, Chapter V] or [11]).

Thus our results in Theorems 1.1-1.4 and Corollary 3.5 apply immediately, with  $\mathcal{I} = [1, \infty)$ . In particular, suppose given  $k \in \mathbb{N}_0$  and elements  $y_1, \dots, y_k \in \mathfrak{q}$  with  $y_i \in \mathfrak{s}_{N_i;w_i}$ ,  $w_i \geq 1$ . Then Corollary 3.5 and (22) yield an estimate

$$(23) \quad |Y_1 \dots Y_k K_t| + t^{1/2} |Y_1 \dots Y_k X K_t| \leq ct^{-(w_1 + \dots + w_k)/2} G_{b,t}$$

for  $t \geq 1$ . In this way we recover some results of [7, 10]. Observe that if  $G$  is nilpotent, that is,  $G = Q$ , then the bounds (23) give high order regularity in arbitrary directions on  $G$ .

(ii) We can generalize example (i) by considering, as in [8], a second-order operator  $H = -\sum_{k,l=1}^d c_{k,l} A_k A_l$ , where  $c_{k,l}$  are complex constants satisfying the ellipticity condition

$$\operatorname{Re} \sum_{k,l} c_{k,l} \xi_k \bar{\xi}_l \geq \mu |\xi|^2$$

for some  $\mu > 0$  and all  $\xi \in \mathbb{C}^d$ . Then the convolution kernel  $K_t$  for  $S_t = e^{-tH}$  satisfies (22) (see [12, 8, 10]). The local estimate (5) is contained in results of [11].

Then as in example (i), using our results we deduce estimates (23). Thus we recover some results of [7, 10] for complex operators.

(iii) We can also generalize example (i) by considering a sublaplacian with drift

$$H = - \sum_{i=1}^d A_i^2 + A_0,$$

where the drift term  $A_0 = dL_G(a_0)$  is ‘centered’ in the sense of Alexopoulos [3]. This means that  $a_0$  belongs to a certain ideal  $\mathfrak{h}$  of  $\mathfrak{g}$  with  $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{h}$ . For such  $H$ , the estimates (22) and (5) are contained in the results of [3].

Applying Corollary 3.5 we obtain the estimates (23) for  $H$ . These estimates seem to be new. We remark that these estimates could probably also be obtained from the asymptotic expansion theorems of [3], though our method is more direct.

(iv) Let  $K_1 \in L_1 \cap L_\infty$  be a bounded probability density on  $G$ : thus  $K_1 \geq 0$  and  $\int_G dg K_1(g) = 1$ . In addition, suppose  $K_1$  is symmetric ( $K(g^{-1}) = K(g)$ ), compactly supported, and that there exists an open neighborhood  $U_0$  of the identity  $e$  such that  $\inf\{K_1(g) : g \in U_0\} > 0$ .

For  $n \in \mathbb{N} = \mathcal{I}$  let  $K_n = K_1 * K_1 * \dots * K_1$  be the  $n$ -th convolution power of  $K_1$ . Then the results of [15], in particular Theorem 5.1 of that paper, yield (2) and (3) with  $\nu = 1$ . Then Theorems 1.1–1.3 apply in this setting.

If we also assume  $K_1$  is  $C^\infty$ -smooth, then (5) obviously holds. Thus Corollary 3.5 applies, and we obtain, for example, the estimate

$$|Y_1 \dots Y_k K_t| \leq ct^{-(w_1 + \dots + w_k)/2} G_{b,t}$$

when  $y_i \in \mathfrak{q} \cap \mathfrak{s}_{N;w_i}$ .

In the remaining examples, we suppose that  $G$  is a finitely generated, discrete group of polynomial growth. Fix a finitely generated nilpotent normal subgroup  $Q$  of  $G$ , such that  $G/Q$  is finite.

(v) Let  $K_1 : G \rightarrow \mathbb{R}$  be a symmetric probability density on  $G$ , such that the support  $\{g \in G : K_1(g) > 0\}$  of  $K_1$  is finite and generates  $G$ .

Let  $K_n$  be the  $n$ -th convolution power of  $K_1$  for  $n \in \mathbb{N}$ . Then (2) and (3) hold with  $\nu = 1$ , again by [15, Theorem 5.1]. Then Theorems 1.1–1.3 apply in this setting. The estimates thus obtained from Theorem 1.2 seem to be new (in the case that  $G = Q$  is nilpotent then they are obtained in [2]).

(vi) Generalizing example (v), one can consider a possibly non-symmetric probability density  $K_1$  on  $G$ , such that the support of  $K_1$  is finite and generates  $G$ . If  $K_1$  is ‘centered’ in the sense of Alexopoulos [2], then (2) and (3) with  $\nu = 1$  are contained in results of [2]. Therefore Theorems 1.1–1.3 also apply to centered probability densities.

### Acknowledgements

During this work the author was supported by an Australian Research Council (ARC) Postdoctoral Fellowship. I would like to thank Derek Robinson and Tom ter Elst for some valuable discussions, and especially Tom ter Elst for explaining his paper [10].

### References

- [1] G. Alexopoulos, 'An application of homogenization theory to harmonic analysis: Harnack inequalities and Riesz transforms on Lie groups of polynomial growth', *Canad. J. Math.* **44** (1992), 691–727.
- [2] ———, 'Random walks on discrete groups of polynomial volume growth', *Ann. Probab.* **30** (2002), 723–801.
- [3] ———, 'Sub-Laplacians with drift on Lie groups of polynomial volume growth', *Mem. Amer. Math. Soc.* **155** (2002), number 739.
- [4] H. Bass, 'The degree of polynomial growth of finitely generated nilpotent groups', *Proc. London Math. Soc.* **25** (1972), 603–614.
- [5] G. Baumslag, *Lecture notes on nilpotent groups*, CBMS Regional Conference Series in Mathematics No. 2 (Amer. Math. Soc., Providence, R.I., 1971).
- [6] P. L. Butzer and H. Berens, *Semi-groups of operators and approximation*, Die Grundlehren der mathematischen Wissenschaften Band 145 (Springer, New York, 1967).
- [7] N. Dungey, 'High order regularity for subelliptic operators on Lie groups of polynomial growth', *Rev. Mat. Iberoamericana*, to appear.
- [8] N. Dungey, A. F. M. ter Elst and D. W. Robinson, *Analysis on Lie groups with polynomial growth*, Progress in Mathematics 214 (Birkhäuser, Boston, MA, 2003).
- [9] X. T. Duong and D. W. Robinson, 'Semigroup kernels, Poisson bounds, and holomorphic functional calculus', *J. Funct. Anal.* **142** (1996), 89–129.
- [10] A. F. M. ter Elst, 'Derivatives of kernels associated to complex subelliptic operators', *Bull. Austral. Math. Soc.* **67** (2003), 393–406.
- [11] A. F. M. ter Elst and D. W. Robinson, 'Weighted subcoercive operators on Lie groups', *J. Funct. Anal.* **157** (1998), 88–163.
- [12] ———, 'Gaussian bounds for complex subelliptic operators on Lie groups of polynomial growth', *Bull. Austral. Math. Soc.* **67** (2003), 201–218.
- [13] A. F. M. ter Elst, D. W. Robinson and A. Sikora, 'Riesz transforms and Lie groups of polynomial growth', *J. Funct. Anal.* **162** (1999), 14–51.
- [14] M. Gromov, 'Groups of polynomial growth and expanding maps', *Inst. Hautes Études Sci. Publ. Math.* **53** (1981), 53–78.
- [15] W. Hebisch and L. Saloff-Coste, 'Gaussian estimates for Markov chains and random walks on groups', *Ann. Probab.* **21** (1993), 673–709.
- [16] M. S. Raghunathan, *Discrete subgroups of Lie groups* (Springer, New York, 1972).
- [17] D. W. Robinson, *Elliptic operators and Lie groups*, Oxford Math. Monographs (Clarendon Press, Oxford, 1991).
- [18] L. Saloff-Coste, 'Analyse sur les groupes de Lie à croissance polynômiale', *Ark. Mat.* **28** (1990), 315–331.
- [19] N. T. Varopoulos, 'Analysis on nilpotent groups', *J. Funct. Anal.* **66** (1986), 406–431.
- [20] ———, 'Analysis on Lie groups', *J. Funct. Anal.* **76** (1988), 346–410.

- [21] N. T. Varopoulos, L. Saloff-Coste and T. Coulhon, *Analysis and geometry on groups*, Cambridge Tracts in Math. 100 (Cambridge University Press, Cambridge, 1992).

School of Mathematics  
University of New South Wales  
Sydney 2052  
Australia  
e-mail: [dungey@maths.unsw.edu.au](mailto:dungey@maths.unsw.edu.au)