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UNIFORM CONVERGENCE OF REGULARIZATION METHODS FOR FREDHOLM EQUATIONS OF THE FIRST KIND

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Abstract

For Fredholm equations of the first kind with continuous kernels we investigate the uniform convergence of a general class of regularization methods. Applications are made to Tikhonov regularization and Landweber's iteration method.

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1. Introduction

It is well known that a Fredholm integral equation of the first kind, that is, an equation of the form

$$\int_a^b k(s,t)x(t)\,dt = g(s), \qquad a \leqslant s \leqslant b, \tag{1}$$

where $k(\cdot,\cdot)$ is a square integrable kernel is ill-posed [5], i.e., the solution x does not depend continuously (in the L^2-sense) on the data g. Equation (1) may be written abstractly as the operator equation

$$Kx = g \tag{2}$$

where K is the compact linear integral operator on the Hilbert space $L^2[a,b]$ generated by the kernel $k(\cdot,\cdot)$.

In practical situations the data g results from measurement and consequently only an approximate version g^δ satisfying

$$\|g - g^\delta\| \leqslant \delta, \tag{3}$$

where δ is a known error level, is available ($\|\cdot\|$ refers to the L^2-norm). By a regularization method for (2) is meant a family $\{\mathscr{R}_\alpha\}_{\alpha>0}$ of continuous operators and a parameter choice $\alpha = \alpha(\delta)$ such that

$$\alpha(\delta) \to 0, \qquad \mathscr{R}_{\alpha(\delta)} g^\delta \to x \quad \text{as } \delta \to 0$$

where x is the (unique) minimal norm solution of (2) (actually our results hold true if (2) is least squares soluble and x is the minimal norm least squares solution). A general class of regularization methods for operator equations in Hilbert space may be constructed by setting $\mathscr{R}_\alpha = R_\alpha(\tilde{K})K^*$ where K^* is the adjoint of K, $\tilde{K} = K^*K$ and $\{R_\alpha\}$ is a family of continuous real valued functions on $[0, \|K\|^2]$ satisfying

$$tR_\alpha(t) \to 1 \qquad \text{as } \alpha \to 0 \text{ for each } t > 0 \tag{4}$$

and

$$|tR_\alpha(t)| \leqslant C \qquad \text{for all } \alpha > 0, t > 0. \tag{5}$$

Some general results, framed entirely within the context of Hilbert space, on the convergence of such methods are presented in [1]. In this note we consider the case of a continuous kernel and establish some corresponding results on uniform, rather than mean, convergence.

2. General results

For notational convenience, we set

$$x_\alpha = R_\alpha(\tilde{K})K^*g \quad \text{and} \quad x_\alpha^\delta = R_\alpha(\tilde{K})K^*g^\delta.$$

The convergence traits of these approximations depend upon the functions

$$r(\alpha) = \sup\left\{|R_\alpha(t)|: t \in \left[0, \|K\|^2\right]\right\}$$

and

$$\omega(\alpha, \nu) = \sup\left\{t^\nu|1 - tR_\alpha(t)|: t \in \left[0, \|K\|^2\right]\right\}, \qquad \nu > 0.$$

We state two basic results; proofs may be found in [1] (or [2]). Below, $R(T)$ and $N(T)$ will designate the range and nullspace, respectively, of the operator T.

THEOREM 1. *$\|x - x_\alpha\| \to 0$ as $\alpha \to 0$ and if $x \in R(\tilde{K}^\nu)$, then $\|x - x_\alpha\| = O(\omega(\alpha, \nu))$.*

THEOREM 2. *$\|x_\alpha - x_\alpha^\delta\| < \delta\sqrt{Cr(\alpha)}$.*

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In each of the theorems above the norm is that induced by the Hilbert space inner product, e.g., the L^2-norm. In what follows we will denote the uniform norm by $\|\cdot\|_\infty$, but continue to view K as an operator on $L^2[a, b]$.

THEOREM 3. *Suppose the kernel k is continuous on $[a, b] \times [a, b]$. If $x \in R(K^*)$, then $\|x - x_\alpha\|_\infty \to 0$ as $\alpha \to 0$. Moreover, if $x \in R(\tilde{K}^\nu K^*)$ for some $\nu > 0$, then $\|x - x_\alpha\|_\infty = O(\omega(\alpha, \nu))$.*

PROOF. Let $k_s(t) = k(t, s)$ and suppose $x = K^*w$, where $w \in N(K^*)^\perp$.

Note that since the functions R_α are continuous, and hence are uniform limits of polynomials, we have $R_\alpha(\tilde{K})K^* = K^*R_\alpha(\hat{K})$, where $\hat{K} = KK^*$. For any $s \in [a, b]$, we then have

$$\begin{aligned} x_\alpha(s) - x(s) &= R_\alpha(\tilde{K})K^*g(s) - K^*w(s) \\ &= K^*[R_\alpha(\hat{K})Kx - w](s) = (k_s, w_\alpha - w) \end{aligned}$$

where $(\cdot, \cdot)$ is the L^2 inner product and $w_\alpha = R_\alpha(\hat{K})Kx$.

By Theorem 1 (applied to K^* rather than K), w_α converges in mean to w, the minimal norm solution of $K^*w = x$. Remembering that k is continuous, we find that $\|x_\alpha - x\|_\infty \leqslant M\|w_\alpha - w\| \to 0$ as $\alpha \to 0$ for a suitable constant M.

If $x \in R(\tilde{K}^\nu K^*)$, then $x = K^*w$ where $w \in R(\hat{K}^\nu)$. Therefore by Theorem 1 $\|w_\alpha - w\| = O(\omega(\alpha, \nu))$ and hence $\|x_\alpha - x\|_\infty = O(\omega(\alpha, \nu))$ as above.

We now deal with the case in which only approximate data g^δ satisfying (3) are available and we suppose that the regularization parameter is a function of the error level, say $\alpha = \alpha(\delta) \to 0$ as $\delta \to 0$.

THEOREM 4. *If k is continuous and $x \in R(K^*)$, then $\|x_\alpha - x_\alpha^\delta\|_\infty = O(r(\alpha)\delta)$.*

PROOF. As in the previous proof, $(x_\alpha - x_\alpha^\delta)(s) = (k_s, z_\alpha - z_\alpha^\delta)$ where $z_\alpha = R_\alpha(\hat{K})g$ and $z_\alpha^\delta = R_\alpha(\hat{K})g^\delta$. Therefore by (3) and the definition of $r(\alpha)$, we have $\|x_\alpha - x_\alpha^\delta\|_\infty \leqslant Mr(\alpha)\delta$.

THEOREM 5. *If k is continuous, $x \in R(K^*)$ and $\delta = O(1/r(\alpha))$, then $\|x - x_\alpha^\delta\|_\infty \to 0$ as $\delta \to 0$.*

PROOF. By Theorem 3 we know that $\|x - x_\alpha\|_\infty \to 0$ as $\alpha = \alpha(\delta) \to 0$. Therefore it suffices to consider $x_\alpha - x_\alpha^\delta$. By Theorem 4, $\|x_\alpha - x_\alpha^\delta\|_\infty$ is bounded. Also, for $s, s' \in [a, b]$

$$\begin{aligned} \left|x_\alpha(s) - x_\alpha^\delta(s) - x_\alpha(s') + x_\alpha^\delta(s')\right| &= \left|\left(k_s - k_{s'}, R_\alpha(\hat{K})(g - g^\delta)\right)\right| \\ &\leqslant \|k_s - k_{s'}\| r(\alpha)\delta \end{aligned}$$

and, as $k(\cdot,\cdot)$ is uniformly continuous, $\{x_\alpha - x_\alpha^\delta\}$ is a uniformly bounded equicontinuous family.

Moreover, $\delta\sqrt{r(\alpha)} = \delta r(\alpha)/\sqrt{r(\alpha)} \to 0$ as $\delta \to 0$, since $r(\alpha) \to \infty$ as $\alpha \to 0$ (see [1], [2]). Therefore by Theorem 2, $\|x_\alpha - x_\alpha^\delta\| \to 0$ as $\delta \to 0$, and it follows that $\|x_\alpha - x_\alpha^\delta\|_\infty \to 0$ as $\delta \to 0$.

It should be stressed that the theorems above require that the approximate data g^δ lies near g only in the L^2-sense, but not in uniform norm.

3. Two examples

The most familiar example of a regularization method is Tikhonov regularization in which $R_\alpha(t) = (\alpha + t)^{-1}$, i.e., $x_\alpha = (\alpha I + \tilde{K})^{-1}K^*g$. In this case Theorem 5 specializes to give a result of Khudak [3].

COROLLARY 1. *If k is continuous, $x \in R(K^*)$ and $\alpha = O(\delta)$, then $\|x - x_\alpha^\delta\|_\infty \to 0$ as $\delta \to 0$.*

For this method we have $\omega(\alpha, \nu) = \alpha^\nu$ for $0 < \nu \leqslant 1$, and $r(\alpha) = 1/\alpha$ (see [1], [2]). Using Theorems 3 and 4 we obtain

COROLLARY 2. *If k is continuous and $x \in R(\tilde{K}^\nu K^*)$ for some ν with $0 < \nu \leqslant 1$, then $\|x - x_\alpha\|_\infty = O(\alpha^\nu)$. Moreover, if $\alpha = C\delta^{1/(\nu+1)}$, then $\|x - x_\alpha^\delta\|_\infty = O(\delta^{\nu/(\nu+1)})$.*

As a second instance of the theory we consider Landweber's iteration method [4]. In this method we assume $\|K\| \leqslant \sqrt{2}$, which is no restriction as (2) may be multiplied by a constant to make it so. The iteration is given by

$$x_0 = K^*g, \qquad x_{n+1} = (I - \tilde{K})x_n + K^*g$$

and the role of α is assumed by the iteration number n (more precisely $1/n$). In this case we have

$$R_n(t) = \sum_{j=0}^{n} (1-t)^j, \qquad r(n) = n+1, \qquad \omega(n,\nu) = (n+1)^{-\nu} \qquad (\nu \geqslant 1).$$

We then obtain the following generalization of a result of Landweber.

COROLLARY 3. *If k is continuous and $x \in R(K^*)$, then $\|x - x_n\|_\infty \to 0$ as $n \to \infty$. Moreover, if $x \in R(\tilde{K}^\nu K)$ for some $\nu \geqslant 1$, then $\|x - x_n\|_\infty = O(n^{-\nu})$.*

For the case of imprecise data we have

COROLLARY 4. *If k is continuous, $x \in R(K^*)$ and $n = n(\delta)$ and satisfies $\delta = O(1/n)$, then $\|x - x_n^\delta\|_\infty \to 0$ as $\delta \to 0$. Moreover, if $x \in R(\tilde{K}^\nu K^*)$ for some $\nu \geqslant 1$ and $n = [\delta^{-1/(\nu+1)}]$, then $\|x - x_n^\delta\|_\infty = O(\delta^{\nu/(\nu+1)})$.*

In particular we see that if x is "regular" enough, i.e., $x \in R(\tilde{K}^\nu K^*)$ for ν large enough, then a uniform order of accuracy arbitrarily near to the optimal order $O(\delta)$ can be attained by Landweber's iteration. Such regularity generally implies a certain order of smoothness for x and, in the case of Volterra kernels, satisfaction of certain boundary conditions. Finally we note that saturation results for Tikhonov regularization show that nearly optimal orders of convergence are not possible for that method (see [1]).

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