APPROXIMATE DIFFERENTIATION.

To the Editors of the Journal of the Institute of Actuaries.

SIRS,—On p. 210 of the *Journal* for July 1924, in the note appended to my letter of 28 March 1924, you point out that I have used the symbol L_x to denote $\int_0^1 l_{x+t}dt$ instead of $\frac{1}{2}(l_x+l_{x+1})$, and that in consequence my formulas involving L do not hold good for the Life Table.

2. While this comment is quite sound as regards the L column usually, though not invariably, published, it may be worth considering whether the ordinary usage should be continued. On p. 2 of his "Life Contingencies" Spurgeon shows that the Census $L_x=\int_0^1 l_{x+t}dt$; but in discussing the matter on p. 3, he concludes that as the life table l_x does not follow any definite mathematical law, it is necessary in the life table to assume that $L_x=l_{x+\frac{1}{2}}=\frac{1}{2}(l_x+l_{x+1})$. This amounts to saying that because l_x does not follow any definite mathematical law throughout its whole range, it is necessary to assume that between successive integral values of x, the progression of l_x is linear. Such an assumption is, however, not essential, and better results could certainly be obtained by determining the values of L_x and $l_{x+\frac{1}{2}}$ from a wider range of values of l.

3. For this purpose all that is necessary is the assumption that l_{x-1}, l_x, l_{x+1} and l_{x+2} may, as amongst themselves, be regarded as values of a function of the third degree corresponding to the successive values of x. For the short range between $x-1$ and $x+2$ such an assumption would be very near the truth, and in any case would be much nearer than the assumption of linear progression. On this basis convenient and approximately accurate values of L_x and $l_{x+\frac{1}{2}}$ can be readily determined by the method of weighting exemplified in my letters in the *Journal* of July 1923 and 1924. Thus in general terms it may be shown that if $F(x)=\int_0^1 f(x+t)dt$ and $f(x)=a+bx+cx^2+dx^3$

then
$$F(x)=\frac{1}{24}\{-f(x-1)+13f(x)+13f(x+1)-f(x+2)\}$$
$$=\frac{1}{2}\{f(x)+f(x+1)\}+\frac{1}{24}\{\Delta f(x-1)-\Delta f(x+1)\}$$

and
$$f\left(x+\frac{1}{2}\right)=\frac{1}{16}\{-f(x-1)+9f(x)+9f(x+1)-f(x+2)\}$$
$$=\frac{1}{2}\{f(x)+f(x+1)\}+\frac{1}{16}\{\Delta f(x-1)-\Delta f(x+1)\}$$

Substituting l_x for $f(x)$ and L_x for $F(x)$, and remembering that $\Delta l_x=-d_x$ we have

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$$L_x = \frac{1}{2}(l_x + l_{x+1}) - \frac{1}{24}(d_{x-1} - d_{x+1})$$

and $$l_{x+\frac{1}{2}} = \frac{1}{2}(l_x + l_{x+1}) - \frac{1}{16}(d_{x-1} - d_{x+1})$$

and consequently $$l_{x+\frac{1}{2}} = L_x - \frac{1}{48}(d_{x-1} - d_{x+1})$$

4. The weightings $\frac{1}{24}\{-1, 13, 13, -1\}$ and $\frac{1}{16}\{-1, 9, 9, -1\}$ may, if desired, be applied by means of the summation method, since the former is equivalent to $\frac{1}{24}[2]\{15-[3]\}$ and the latter to $\frac{1}{16}[2]\{11-[3]\}$. In either form the latter could readily be applied to the halving of an interval in existing tables, as, for example, the computation of rates for successive half ages. For halving the interval in a table of squares or cubes the method is strictly correct, and approximately so for most actuarial tables.

5. It may be of interest to mention that in compiling the Australian Life Tables prepared in connection with the Census of 1911, I used the expression $\frac{1}{24}\{-1, 13, 13, -1\}$ in two cases. In those tables the initial rate obtained was μ_x, the method followed being that suggested by J. M. Allen (*J.I.A.*, vol. xli, p. 320). From the graphically graduated values of μ_x, values of colog p_x were obtained by means of the formula colog $p_x = M\int_0^1 \mu_{x+t}dt$ where $M = \cdot 43429$ . . . . . and the approximate integration was effected by means of the weightings $\frac{1}{24}\{-1, 13, 13, -1\}$. Later, when l_x had been obtained from colog p_x, the values of L_x were similarly computed by means of the formula $L_x = \int_0^1 l_{x+t}dt$. In both cases special computations were made for $x=0$ and for $x=\omega$. In these tables it is consequently not true that $L_x = \frac{1}{2}(l_x + l_{x+1})$, nor is it true with the L_0 in the English Life Table, No. 8, Males, as given in Spurgeon's "Life Contingencies," p. 396.

Yours faithfully,

CHAS. WICKENS,

Melbourne,

10 *October* 1924.