THE AGGREGATE CLAIMS DISTRIBUTION IN THE INDIVIDUAL MODEL WITH ARBITRARY POSITIVE CLAIMS

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ABSTRACT

In an earlier paper the author derived a recursion formula which permits the exact computation of the aggregate claims distribution in the individual life model. To save computing time he also proposed an approximative procedure based on the exact recursion.

In the present contribution the exact recursion formula and the related approximations are generalized to the individual risk theory model with arbitrary positive claims. Error bounds for the approximations are given and it is shown that they are smaller than those of the Kornya-type approximations.

KEYWORDS

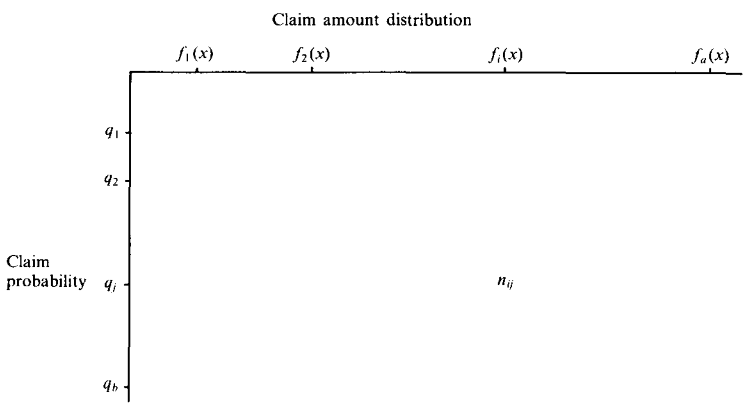
Individual model; recursion formula; aggregate claims distribution; error bound.

1. INTRODUCTION

Consider a portfolio of independent policies which produce at most one claim during a certain exposure period. The claim amounts are supposed to be integral multiples of some convenient monetary unit.

Let the portfolio be classified into $a \times b$ classes, as displayed in table 1.

TABLE 1
CLASSIFICATION OF THE PORTFOLIO



In table 1 the following notation is used

$f_i(x)$: conditional distribution of the claim amounts for a policy in column i given that a claim has occurred, $i = 1, 2, \ldots, a$ and $x = 1, 2, \ldots, m_i$;
q_j : probability that a policy of row j produces a claim, $j = 1, 2, \ldots, b$;
n_{ij} : number of policies in column i and row j.

Further set

$p_j = 1 - q_j$: probability that a policy of row j produces no claim;

$n_j = \sum_{i=1}^{a} n_{ij}$: number of policies with claim probability equal to q_j;

$n = \sum_{j=1}^{b} n_j$: total number of policies;

$m = \sum_{i=1}^{a} \sum_{j=1}^{b} n_{ij} m_i$: maximum possible amount of aggregate claims.

Let S denote the total amount of claims in the exposure period and $p_S(s)$ the probability that S will be precisely s units.

In this paper two recursive procedures are proposed to compute $p_S(s)$ exactly. They are generalizations of the recursion formula of De Pril (1986) for the individual life model.

From a practical point of view a disadvantage of these exact procedures is that they require a lot of computing time when applied to large portfolios, as illustrated by Kuon et al. (1987) for the individual life case. For this reason it is shown how the exact algorithm can be used in an approximative way. The resulting approximations permit to calculate the aggregate claims distribution up to a prescribed accuracy. The error bounds are easy to calculate and it is shown that they are smaller than those of the approximations developed by Kornya (1983) and Hipp (1986). This generalizes the results of De Pril (1988) for the individual life model.

2. EXACT RECURSIVE PROCEDURES

Theorems 1 and 2 contain two versions of a recursive procedure which permits the exact computation of the aggregate claims distribution. The first is based on a two stage recursion formula; the second contains higher order convolutions (and is thus in general also a two stage formula).

THEOREM 1. A two stage recursion formula for $p_S(s)$ is

$$p_S(0) = \prod_{j=1}^{b} (p_j)^{n_j} \tag{1.a}$$

$$s\, p_S(s) = \sum_{i=1}^{a} \sum_{j=1}^{b} n_{ij} \sum_{x=1}^{s} w_{ij}(x)\, p_S(s-x) \qquad s = 1,2,\ldots,m \tag{1.b}$$

where the auxiliary functions $w_{ij}(x)$ are given by

$$w_{ij}(1) = \frac{q_j}{p_j} f_i(1) \tag{2.a}$$

$$w_{ij}(x) = \frac{q_j}{p_j}\left[x f_i(x) - \sum_{k=1}^{x-1} f_i(k)\, w_{ij}(x-k)\right] \qquad x = 2,3,\ldots,m_i \tag{2.b}$$

$$w_{ij}(x) = -\frac{q_j}{p_j} \sum_{k=1}^{m_i} f_i(k)\, w_{ij}(x-k) \qquad x = m_i+1,\ldots \tag{2.c}$$

PROOF. The probability generating function of S is

$$P_S(u) = \sum_{s=0}^{m} p_S(s)\, u^s \tag{3.a}$$

$$= \prod_{i=1}^{a} \prod_{j=1}^{b} [p_j + q_j\, G_i(u)]^{n_{ij}} \tag{3.b}$$

with $G_i(u)$ the generating function of the $f_i(x)$

$$G_i(u) = \sum_{x=1}^{m_i} f_i(x)\, u^x \tag{4}$$

Putting $u = 0$ in (3.b) gives immediately the starting value (1.a). To prove (1.b), take the derivative of (3.b)

$$P_S'(u) = P_S(u) \sum_{i=1}^{a} \sum_{j=1}^{b} n_{ij}\, W_{ij}(u) \tag{5}$$

where $W_{ij}(u)$ denotes an auxiliary function defined as

$$W_{ij}(u) = \sum_{x=0}^{\infty} w_{ij}(x+1)\, u^x \tag{6.a}$$

$$= \frac{q_j\, G_i'(u)}{p_j + q_j\, G_i(u)} \tag{6.b}$$

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Taking, according to Leibniz's formula, the derivative of order $s-1$ of (5) and putting $u = 0$ yields formula (1.b).

The recursion formula (2) for the $w_{ij}(x)$ is obtained by taking the derivative of order $x-1$ of

$$q_j\, G_i'(u) = [p_j + q_j\, G_i(u)]\, W_{ij}(u)$$

and putting $u = 0$. This completes the proof.

THEOREM 2. A recursion formula for $p_S(s)$ is

$$\text{(7.a)} \qquad p_S(0) = \prod_{j=1}^{b} (p_j)^{n_j}$$

$$\text{(7.b)} \quad s\, p_S(s) = \sum_{i=1}^{a} \sum_{k=1}^{s} A(i,k) \sum_{x=k}^{\min(s,\, km_i)} x f_i^{*k}(x)\, p_S(s-x) \quad s = 1, 2, \ldots, m$$

where the coefficients $A(i,k)$ are given by

$$\text{(8)} \qquad A(i,k) = \frac{(-1)^{k+1}}{k} \sum_{j=1}^{b} n_{ij} \left(\frac{q_j}{p_j}\right)^k$$

and where $f_i^{*k}(x)$ denotes the k-fold convolution of $f_i(x)$.

PROOF. Expansion of the denominator of (6.b) gives

$$W_{ij}(u) = \frac{q_j}{p_j} G_i'(u) \sum_{k=0}^{\infty} (-1)^k \left(\frac{q_j}{p_j}\right)^k (G_i(u))^k$$
$$= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left(\frac{q_j}{p_j}\right)^k \frac{d(G_i(u))^k}{du}$$

Taking now the derivative of order $x-1$ of both sides leads to an explicit representation of the $w_{ij}(x)$

$$\text{(9)} \qquad w_{ij}(x) = \sum_{k=]x/m_i[}^{x} \frac{(-1)^{k+1}}{k} \left(\frac{q_j}{p_j}\right)^k x f_i^{*k}(x) \qquad x = 1, 2, \ldots$$

where $]x/m_i[$ denotes the smallest integer greater than or equal to x/m_i.

The theorem is proved by inserting (9) in (1.b) and interchanging the order of the summations over x and k.

REMARK 1. The k-fold convolution of the $f_i(x)$ in (7.b) can be computed by the usual recursive definition of convolutions or by the recursion formula described in DE PRIL (1985). In some special cases an explicit expression for this convolution can be given, so that $p_S(s)$ can be computed by a one stage recursion formula.

REMARK 2. Notice that the general results also hold for infinite m_i's. If one takes $m_i = \infty$ for all i several summation limits look simpler and formula (2.c) can be dropped. Nevertheless the formulas are presented for finite m_i's since in practical applications the claim amounts will be bounded and in programming the algorithms it will be necessary to know the exact limits of the summations.

3. APPLICATIONS AND SPECIAL CASES

First consider the following special cases.

SPECIAL CASE 1. The individual life model is obtained by putting $f_i(x) = \delta_{ix}$, where i represents a risk sum and δ_{ix} denotes the Kronecker delta.

Since $f_i^{*k}(x) = \delta_{ki,x}$, the recursion (7) reduces to

$$p_S(0) = \prod_{j=1}^{b} (p_j)^{n_j} \tag{10.a}$$

$$s\, p_S(s) = \sum_{i=1}^{\min(a,s)} \sum_{k=1}^{[s/i]} a(i,k)\, p_S(s-ki) \qquad s = 1, 2, \ldots, m \tag{10.b}$$

with

$$a(i,k) = (-1)^{k+1}\, i \sum_{j=1}^{b} n_{ij} \left(\frac{q_j}{p_j}\right)^k \tag{11}$$

and where $[s/i]$ denotes the greatest integer less than or equal to s/i. This formula is theorem 1 of DE PRIL (1986).

SPECIAL CASE 2. A recursion formula for the number of claims can be obtained by putting $a = 1$ and $f_1(x) = \delta_{1x}$. Then $f_1^{*k}(x) = \delta_{kx}$ and (7) reduces to

$$p_S(0) = \prod_{j=1}^{b} (p_j)^{n_j} \tag{12.a}$$

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(12.b) $$s\,p_S(s) = \sum_{k=1}^{s} a(k)\,p_S(s-k) \qquad s = 1, 2, \ldots, m$$

with

(13) $$a(k) = (-1)^{k+1} \sum_{j=1}^{b} n_j \left(\frac{q_j}{p_j} \right)^k$$

This formula can be found in WHITE and GREVILLE (1959).

More general illustrations of the model considered in this paper, in which the amount of a claim is a random variable in the proper sense, are presented in the following examples. Some inspiration for these applications was found in the textbook of BOWERS et al. (1986).

EXAMPLE 1. A life coverage with a double indemnity provision provides the death benefit to be doubled when death is caused by accidental means. Let the probability of an accidental death, given that there is a death, be constant and denoted by α. Then, the conditional claim amount distribution will be defined by

$$f_i(x) = \begin{cases} 1-\alpha & x = i \\ \alpha & x = 2i \\ 0 & \text{elsewhere} \end{cases}$$

The k-fold convolution of the $f_i(x)$ may be written as

$$f_i^{*k}(x) = \begin{cases} \binom{k}{h-k} (1-\alpha)^{2k-h} \alpha^{h-k} & x = hi \quad \text{with} \quad h = k, k+1, \ldots, 2k \\ 0 & \text{elsewhere} \end{cases}$$

Substituting this expression into (7), and reversing the role of k and h, leads to the following recursion formula

(14.a) $$p_S(0) = \prod_{j=1}^{b} (p_j)^{n_j}$$

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$$(14.b) \qquad s\,p_S(s) = \sum_{i=1}^{\min(a,s)} \sum_{k=1}^{[s/i]} b(i,k)\,p_S(s-ki) \qquad s = 1, 2, \ldots, m$$

with

$$(15) \quad b(i,k) = ik \sum_{h=]k/2[}^{k} \frac{(-1)^{h+1}}{h} \binom{h}{k-h} (1-\alpha)^{2h-k} \alpha^{k-h} \sum_{j=1}^{b} n_{ij} \left(\frac{q_j}{p_j}\right)^h$$

It is clear that in the limiting case $\alpha = 0$ (15) reduces to (11) and thus (14) to (10).

EXAMPLE 2. A hospital insurance provides a flat daily benefit during hospitalization. Let the members of the covered group be classified into $a \times b$ classes according to the following two criteria: the amount i of the benefit per day and the probability q_j to enter a hospital during the reference period. Assume that the distribution of the length of stay in the hospital is the same for each member. Denote by c the maximum number of days for which benefits are paid and by $\beta(t)$ the probability of continuance of a hospital claim for t days, $t = 1, 2, \ldots, c$. In this case the model can be applied by setting

$$f_i(x) = \begin{cases} \beta(t) & x = ti \quad \text{with} \quad t = 1, 2, \ldots, c \\ 0 & \text{elsewhere} \end{cases}$$

A similar application is that of short-term disability insurance. For $c = 2$ this example reduces of course to example 1.

EXAMPLE 3. A fire insurance company covers n structures against fire damage up to an amount stated in the contract. Assume that fires are mutually independent events and that the probability of more than one claim per structure is zero. The contracts are classified into $a \times b$ classes according to the following two criteria: the contract amount m_i and the probability q_j of a fire within a given time period. In fire insurance the claim amount has a wide variability. Therefore, assume that the conditional distribution of the claim amounts, given that a claim has occurred, is uniformly distributed over the interval from 1 to the contract amount, that is

$$f_i(x) = \frac{1}{m_i} \qquad \text{for} \qquad x = 1, 2, \ldots, m_i$$

It can be shown that an explicit expression for the k-fold convolution of the $f_i(x)$ is given by

$$f_i^{*k}(x) = \sum_{y=0}^{\left[\frac{x-k}{m_i}\right]} \frac{(-1)^y}{m_i^k} \binom{k}{y} \binom{x-m_i y-1}{k-1} \qquad x = k, k+1, \ldots, km_i$$

See e.g. problem 18 on page 284 of Feller (1968). The probabilities $p_S(s)$ can thus, in principle, be computed by a one stage recursion formula. It should however be noticed that several consecutive values of $f_i^{*k}(x)$ are needed, so that it would presumably be much faster to use a two stage formula and compute the convolutions recursively.

4. APPROXIMATIONS DERIVED FROM THE EXACT PROCEDURE

It is clear that a rigorous computation of (7) necessistates a lot of computer time, especially if a is large and if the $f_i(x)$ are defined for more than a few values. In practical applications however the q_j will be small, so that the coefficients $A(i, k)$ will tend to zero (very) fast if k increases. The exact formula (7) can thus be used in an approximative way by truncating the summation over k. If the coefficients $A(i, k)$ are neglected for $k > r$, the following r-th order approximations $p_S^{(r)}(s)$ of $p_S(s)$ are obtained

(16.a) $$p_S^{(r)}(0) = \prod_{j=1}^{b} (p_j)^{n_j}$$

(16.b) $$s\, p_S^{(r)}(s) = \prod_{i=1}^{a} \sum_{k=1}^{\min(r,s)} A(i,k) \sum_{x=k}^{\min(s,km_i)} x f_i^{*k}(x)\, p_S^{(r)}(s-x) \qquad s = 1, 2, \ldots, m.$$

It is immediately seen that these approximations are exact for the first values

(17) $$p_S^{(r)}(s) = p_S(s) \qquad \text{for} \qquad s = 0, 1, \ldots, r.$$

For future reference the generating function of the $p_S^{(r)}(s)$ is denoted by

(18) $$P_S^{(r)}(u) = \sum_{s=0}^{m} p_S^{(r)}(s)\, u^s$$

Set also

(19) $$\bar{P}_S^{(r)}(u) = \sum_{s=0}^{\infty} p_S^{(r)}(s)\, u^s$$

where $p_S^{(r)}(s)$ is defined for $s > m$ by extending the range of formula (16.b) to all positive integers.

5. ERROR BOUNDS FOR THE APPROXIMATIONS $p_S^{(r)}(s)$

Our derivation of error bounds will be based on the theory of partial ordering of real power series. This method was also used by KORNYA (1983) and DE PRIL (1988). To render this presentation self-contained, let us first repeat some results.

DEFINITION. Let $A(u) = \sum_{k=0}^{\infty} a_k u^k$ and $B(u) = \sum_{k=0}^{\infty} b_k u^k$ be power series.

Then say that $A(u) \leq_u B(u)$ provided that, for any non-negative integer n, the sum of the first $n+1$ coefficients satisfies

$$\sum_{k=0}^{n} a_k \leq \sum_{k=0}^{n} b_k .$$

NOTATION. Let $A(u) = \sum_{k=0}^{\infty} a_k u^k$. Then, denote by $| A(u) |_u$ the power series

$$| A(u) |_u = \sum_{k=0}^{\infty} | a_k | u^k.$$

LEMMA. Let $A(u)$, $B(u)$ and $C(u)$ be power series, then

i) $| A(u) |_u \leq_u \sum_{k=0}^{\infty} | a_k |$ if $\sum_{k=0}^{\infty} | a_k | < \infty$;

ii) $| A(u) + B(u) |_u \leq_u | A(u) |_u + | B(u) |_u$;

iii) $| A(u) \cdot B(u) |_u \leq_u | A(u) |_u \cdot | B(u) |_u$;

iv) $| 1 - \exp A(u) |_u \leq_u \exp (| A(u) |_u) - 1$;

v) If $| A(u) |_u \leq_u | B(u) |_u$,
then $| A(u) |_u \cdot | C(u) |_u \leq_u | B(u) |_u \cdot | C(u) |_u$;

vi) If $| A(u) |_u \leq_u | B(u) |_u$,
then $\exp (| A(u) |_u) - 1 \leq_u \exp (| B(u) |_u) - 1$.

THEOREM 3. If $q_j < 1/2$, $j = 1, 2, \ldots, b$, then

$$\sum_{s=0}^{m} | p_S(s) - p_S^{(r)}(s) | \leq e^{\varepsilon(r)} - 1 \tag{20}$$

with

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$$\varepsilon(r) = \frac{1}{r+1} \sum_{i=1}^{a} \sum_{j=1}^{b} n_{ij} \frac{p_j}{p_j - q_j} \left(\frac{q_j}{p_j}\right)^{r+1} \tag{21}$$

PROOF. From the lemma it follows that

$$\begin{aligned} \left| P_S(u) - P_S^{(r)}(u) \right|_u &\leq_u \left| P_S(u) - \bar{P}_S^{(r)}(u) \right|_u \\ &\leq_u \left| P_S(u) \right|_u \cdot \left| 1 - \exp\left(\ln \bar{P}_S^{(r)}(u) - \ln P_S(u)\right) \right|_u \\ &\leq_u \exp\left(\left| \ln \bar{P}_S^{(r)}(u) - \ln P_S(u) \right|_u \right) - 1 \end{aligned} \tag{22}$$

Considering (3.b) $\ln P_S(u)$ can be written as

$$\begin{aligned} \ln P_S(u) &= \ln p_S(0) + \sum_{i=1}^{a} \sum_{j=1}^{b} n_{ij} \ln \left(1 + \frac{q_j}{p_j} G_i(u)\right) \\ &= \ln p_S(0) + \sum_{k=1}^{\infty} \sum_{i=1}^{a} A(i,k) \left(G_i(u)\right)^k \end{aligned} \tag{23}$$

To derive an expression for $\ln \bar{P}_S^{(r)}(u)$ consider the derivative

$$\begin{aligned} \bar{P}_S'^{(r)}(u) &= \sum_{s=1}^{\infty} s\, p_S^{(r)}(s)\, u^{s-1} \\ &= \sum_{s=1}^{\infty} \sum_{i=1}^{a} \sum_{k=1}^{\min(r,s)} A(i,k) \sum_{x=k}^{\min(s,\, km_i)} x f_i^{*k}(x)\, p_S^{(r)} \\ &\quad (s-x)\, u^{s-1} \\ &= \sum_{k=1}^{r} \sum_{i=1}^{a} A(i,k) \sum_{x=k}^{km_i} x f_i^{*k}(x) \sum_{s=x}^{\infty} p_S^{(r)}(s-x)\, u^{s-1} \\ &= \bar{P}_S^{(r)}(u) \sum_{k=1}^{r} \sum_{i=1}^{a} A(i,k) \sum_{x=k}^{km_i} x f_i^{*k}(x)\, u^{x-1} \end{aligned}$$

Integration gives

$$\ln \bar{P}_S^{(r)}(u) = \ln p_S(0) + \sum_{k=1}^{r} \sum_{i=1}^{a} A(i,k) \left(G_i(u)\right)^k \tag{24}$$

Now (23) and (24) lead to

$$\begin{aligned} \left| \ln \bar{P}_S^{(r)}(u) - \ln P_S(u) \right|_u &= \left| \sum_{k=r+1}^{\infty} \sum_{i=1}^{a} A(i,k) \left(G_i(u)\right)^k \right|_u \\ &\leq_u \sum_{i=1}^{a} \sum_{k=r+1}^{\infty} \left| A(i,k) \right| \end{aligned}$$

$$\leq_u \frac{1}{r+1} \sum_{i=1}^{a} \sum_{j=1}^{b} n_{ij} \sum_{k=r+1}^{\infty} \left(\frac{q_j}{p_j}\right)^k$$

$$= \varepsilon(r)$$

so that by (22)

$$\left| P_S(u) - P_S^{(r)}(u) \right|_u \leq_u e^{\varepsilon(r)} - 1$$

which proves the theorem.

6. COMPARISON WITH KORNYA'S APPROXIMATIONS

The method of KORNYA (1983) was originally written for a life portfolio and generalized by HIPP (1986) to the individual model with arbitrary positive claims.

To derive the Kornya approximations write the logarithm of (3.b) as

$$\ln P_S(u) = \sum_{i=1}^{a} \sum_{j=1}^{b} n_{ij} \left[\ln\left(1 + \frac{q_j}{p_j} G_i(u)\right) - \ln\left(1 + \frac{q_j}{p_j}\right)\right]$$

(25) $$= \sum_{i=1}^{a} \sum_{j=1}^{b} n_{ij} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left(\frac{q_j}{p_j}\right)^k ((G_i(u))^k - 1)$$

Further, denote the generating function of Kornya's r-th order approximation $k_S^{(r)}(s)$ of $p_S(s)$ by

(26) $$K_S^{(r)}(u) = \sum_{s=0}^{\infty} k_S^{(r)}(s)\, u^s$$

Then, $\ln K_S^{(r)}(u)$, and thus the $k_S^{(r)}(s)$, are defined by neglecting in (25) the terms in $\left(\frac{q_j}{p_j}\right)^k$ for $k > r$

(27) $$\ln K_S^{(r)}(u) = \sum_{i=1}^{a} \sum_{j=1}^{b} n_{ij} \sum_{k=1}^{r} \frac{(-1)^{k+1}}{k} \left(\frac{q_j}{p_j}\right)^k ((G_i(u))^k - 1)$$

The approximations can be calculated recursively as

(28.a) $$k_S^{(r)}(0) = \exp C^{(r)}(0)$$

(28.b) $$s\, k_S^{(r)}(s) = \sum_{x=1}^{\min(s,\, r \max m_i)} x\, C^{(r)}(x)\, k_S^{(r)}(s-x) \qquad s = 1, 2, \ldots$$

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where the $C^{(r)}(x)$ denote the Taylor coefficients of $\ln K_S^{(r)}(u)$

$$\ln K_S^{(r)}(u) = \sum_{x=0}^{r \max m_i} C^{(r)}(x)\, u^x$$

that is

$$C^{(r)}(0) = \sum_{i=1}^{a} \sum_{j=1}^{b} n_{ij} \sum_{k=1}^{r} \frac{(-1)^k}{k} \left(\frac{q_j}{p_j}\right)^k \tag{29.a}$$

$$C^{(r)}(x) = \sum_{i=1}^{a} \sum_{j=1}^{b} n_{ij} \sum_{k=]x/m_i[}^{\min(r,x)} \frac{(-1)^{k+1}}{k} \left(\frac{q_j}{p_j}\right)^k f_i^{*k}(x) \tag{29.b}$$

$$x = 1, 2, \ldots, r \max m_i .$$

Notice that the $k_S^{(r)}(s)$ are—in principle—defined for all non-negative integers s, while the $p_S^{(r)}(s)$ are defined over the correct range $s = 0, 1, \ldots, m$. An error bound for the $k_S^{(r)}(s)$ is given in the following theorem.

THEOREM 4. If $q_j < 1/2$, $j = 1, 2, \ldots, b$, then

$$\sum_{s=0}^{\infty} \left| p_S(s) - k_S^{(r)}(s) \right| \leq e^{\delta(r)} - 1 \tag{30}$$

with

$$\delta(r) = \frac{1}{r+1} \sum_{i=1}^{a} \sum_{j=1}^{b} n_{ij} \left(p_j + \frac{p_j}{p_j - q_j} \right) \left(\frac{q_j}{p_j}\right)^{r+1} \tag{31}$$

PROOF. The proof follows immediately from

$$\left| P_S(u) - K_S^{(r)}(u) \right|_u \leq_u \exp\left(\left| \ln K_S^{(r)}(u) - \ln P_S(u) \right|_u \right) - 1$$

and

$$\left| \ln K_S^{(r)}(u) - \ln P_S(u) \right|_u$$

$$= \left| \sum_{k=r+1}^{\infty} \frac{(-1)^k}{k} \sum_{i=1}^{a} \sum_{j=1}^{b} n_{ij} \left(\frac{q_j}{p_j}\right)^k \right|$$

$$+ \left| \sum_{k=r+1}^{\infty} \frac{(-1)^{k+1}}{k} \sum_{i=1}^{a} \sum_{j=1}^{b} n_{ij} \left(\frac{q_j}{p_j}\right)^k (G_i(u))^k \right|_u$$

$$\leq_u \frac{1}{r+1} \sum_{i=1}^{a} \sum_{j=1}^{b} n_{ij} \left(\frac{q_j}{p_j}\right)^{r+1} \left[\sum_{k=0}^{\infty} \left(-\frac{q_j}{p_j}\right)^k + \sum_{k=0}^{\infty} \left(\frac{q_j}{p_j}\right)^k \right]$$

$$= \delta(r)$$

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An immediate consequence of the theorem is that the approximations are asymptotically correct

$$\lim_{r \to \infty} k_S^{(r)}(s) = \begin{cases} p_S(s) & s = 0, 1, \ldots, m \\ 0 & s = m+1, m+2, \ldots \end{cases} \tag{32}$$

Note that the bound given here is smaller than the one originally given by KORNYA (1983). Further details can be found in DE PRIL (1988).

REMARK 3. From (24) and (27) one has that

$$\ln \bar{P}_S^{(r)}(u) = \ln K_S^{(r)}(u) - \ln k_S^{(r)}(0) + \ln p_S(0)$$

so that the following relationship exists

$$p_S^{(r)}(s) = \frac{p_S(0)}{k_S^{(r)}(0)} k_S^{(r)}(s) \qquad s = 0, 1, \ldots, m. \tag{33}$$

A consequence of (33) is that the $p_S^{(r)}(s)$ can be computed in an alternative way by using the recursion (28), but then starting with the exact value $p_S(0)$ in (28.a). This can also be seen by inserting (9) in (7) and neglecting the terms in $(q_j/p_j)^k$ for $k > r$. Clearly this implies that both approximations can be computed with about the same effort.

REIMERS (1988) made a numerical comparison in the case of life portfolios which seems to indicate that it is faster to use the algorithm (28) than (16). This must be due to the computer language, the programming style and the implementation of the algorithms. Indeed, both algorithms are in fact two versions of the same procedure. They can be obtained from each other by reversing the summations over x and k.

7. COMPARISON WITH HIPP'S APPROXIMATIONS

An alternative to the Kornya approximations has been proposed by HIPP (1986).

His approximations have as interesting property that the first order approximation coincides with the usual compound Poisson approximation in the collective risk theory model.

The starting point is to write $\ln P_S(u)$ as

$$\ln P_S(u) = \sum_{i=1}^{a} \sum_{j=1}^{b} n_{ij} \ln [1 + q_j(G_i(u) - 1)]$$

$$= \sum_{i=1}^{a} \sum_{j=1}^{b} n_{ij} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} q_j^k (G_i(u) - 1)^k \tag{34}$$

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Then, the generating function

$$(35) \qquad H_S^{(r)}(u) = \sum_{s=0}^{\infty} h_S^{(r)}(s)\, u^s$$

of Hipp's r-th order approximations $h_S^{(r)}(s)$ is defined by neglecting in (34) the terms in q_j^k for $k > r$

$$(36) \qquad \ln H_S^{(r)}(u) = \sum_{i=1}^{a} \sum_{j=1}^{b} n_{ij} \sum_{k=1}^{r} \frac{(-1)^{k+1}}{k}\, q_j^k\, (G_i(u)-1)^k$$

The $h_S^{(r)}(s)$ can be calculated by a recursion formula similar to (28). An error bound for these approximations is given in the following theorem.

THEOREM 5. If $q_j < 1/2$, $j = 1, 2, \ldots, b$, then

$$(37) \qquad \sum_{s=0}^{\infty} \left| p_S(s) - h_S^{(r)}(s) \right| \leq e^{\sigma(r)} - 1$$

with

$$(38) \qquad \sigma(r) = \frac{1}{r+1} \sum_{i=1}^{a} \sum_{j=1}^{b} n_{ij} \frac{(2q_j)^{r+1}}{p_j - q_j}$$

PROOF. One has

$$\left| P_S(u) - H_S^{(r)}(u) \right|_u \leq_u \exp\left(\left| \ln H_S^{(r)}(u) - \ln P_S(u) \right|_u \right) - 1$$

with

$$\left| \ln H_S^{(r)}(u) - \ln P_S(u) \right|_u = \left| \sum_{k=r+1}^{\infty} \frac{(-1)^{k+1}}{k} \sum_{i=1}^{a} \sum_{j=1}^{b} n_{ij}\, q_j^k\, (G_i(u)-1)^k \right|_u$$

$$\leq_u \left| \frac{1}{r+1} \sum_{i=1}^{a} \sum_{j=1}^{b} n_{ij} \sum_{k=r+1}^{\infty} q_j^k\, (G_i(u)+1)^k \right|_u$$

$$\leq_u \frac{1}{r+1} \sum_{i=1}^{a} \sum_{j=1}^{b} n_{ij} \sum_{k=r+1}^{\infty} (2q_j)^k$$

$$= \sigma(r)$$

which proves the theorem.

From this theorem one has immediately that

$$(39) \qquad \lim_{r\to\infty} h_S^{(r)}(s) = \begin{cases} p_S(s) & s = 0, 1, \ldots, m \\ 0 & s = m+1, m+2, \ldots \end{cases}$$

The way of convergence of the different approximations can be seen by comparing (17), (32) and (39).

8. CONCLUSION

The paper has focused on two problems: the derivation of an exact recursion formula for the aggregate claims distribution and its numerical evaluation by means of approximations.

The recursions given in the theorems 1 and 2 are exact formulae for the probabilities $p_S(s)$. These formulae are mainly of theoretical interest.

For practical applications a compromise between accuracy and computational effort is found by deriving, in a natural way, approximations $p_S^{(r)}(s)$ from the exact recursion. These approximations are easy to compute and even exact for the first values. From the theorems 3, 4 and 5 it follows that $\varepsilon(r) < \delta(r) < \sigma(r)$, so that the $p_S^{(r)}(s)$ give rise to smaller error bounds than the approximations proposed by KORNYA (1983) and HIPP (1986). Since the computational work is about the same in the three case, preference should be given to the approximations derived here.

In typical applications a value of r equal to 3 or 5 will give very satisfactory results.

In practice the computation of the aggregate claims distribution will proceed as follows

i) Choose a value of r for which the magnitude of error $e^{\varepsilon(r)}-1$, with $\varepsilon(r)$ given bij (21), is sufficiently small.

ii) Compute the convolutions $f_i^{*k}(x)$ for $i = 1, 2, \ldots, a$, $k = 1, 2, \ldots, r$ and $x = k, k+1, \ldots, km_i$

iii) Compute the coefficients $A(i, k)$, defined by (8), for $i = 1, 2, \ldots, a$ and $k = 1, 2, \ldots, r$.

iv) Calculate the approximations $p_S^{(r)}(s)$ recursively by means of (16).

An alternative is to replace steps iii) and iv) by

iii)' Compute the coefficients $C^{(r)}(x)$, defined by (29.b), for $x = 1, 2, \ldots, r \max m_i$

iv)' Calculate the approximations $p_S^{(r)}(s)$ recursively by means of (28), where

(28.a) is replaced by $p_S^{(r)}(0) = \prod_{j=1}^{b} (p_j)^{n_j}$

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