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MORE ON EXTENDING CONTINUOUS PSEUDOMETRICS

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1. Introduction and definitions. The concept of extending to a topological space X a continuous pseudometric defined on a subspace S of X has been shown to be very useful. This problem was first studied by Hausdorff for the metric case in 1930 [**9**]. Hausdorff showed that a continuous metric on a closed subset of a metric space can be extended to a continuous metric on the whole space. Bing [**4**] and Arens [**3**] rediscovered this result independently. Recently, Shapiro [**15**] and Alo and Shapiro [**1**] studied various embeddings. It has been shown that extending pseudometrics can be characterized in terms of extending refinements of various types of open covers. In this paper we continue our study of extending pseudometrics. First we show that extending pseudometrics can be characterized in terms of σ-locally finite and σ-discrete covers. We then investigate when can certain types of covers be extended. We show that P-embedding can be used to characterize concepts introduced by Slaughter [**18**] and Aull [**2**]. Finally we give new proofs of the known facts that a normal M-space is countably paracompact and that an F_σ-subset of a collectionwise normal space is collectionwise normal.

Definitions. We will follow the notation and terminology of [**8**]. For the definition of a normal cover, consult [**20**]. If $\mathscr{U} = (U_\alpha)_{\alpha \in I}$ is a family of subsets of a space X, then by $\mathscr{U}|S$ we mean the family $(U_\alpha \cap S)_{\alpha \in I}$. We say that $\mathscr{U}$ has *cardinality at most* γ (γ an infinite cardinal number) if $|I| \leqq \gamma$. We say that the family $(U_\alpha)_{\alpha \in I}$ is *locally finite* (*discrete*) if for each $x \in X$ there exist a neighbourhood G of x and a finite subset J of I (J of I such that $|J| \leqq 1$) such that $G \cap U_\alpha = \emptyset$ for every $\alpha \notin J$. We say that $\mathscr{U}$ is *σ-locally finite* (*σ-discrete*) in case $\mathscr{U}$ can be expressed as a countable union of locally finite (discrete) families. We say that S is an *F_σ-set* if S can be expressed as a countable union of closed subsets of X. When a family of subsets of a subspace S of X is said to be *open*, *locally finite*, *etc.*, this refers to the topology of S.

Next let X be a topological space, let $S \subset X$, let $\mathscr{U} = (U_\alpha)_{\alpha \in I}$ be a cover of X, and let $\mathscr{V} = (V_\beta)_{\beta \in J}$ be a cover of S. Then $\mathscr{U}$ is an *extension of* $\mathscr{V}$ if $I = J$ and if $U_\alpha \cap S = V_\alpha$ for all $\alpha \in I$.

If γ is an infinite cardinal number, a subset S of a topological space X is said to be *P^γ-embedded in X* if every γ-separable continuous pseudometric on S can be extended to a γ-separable continuous pseudometric on X. (A pseudometric d on X is *γ-separable* if there exists a subset G of X such that $|G| \leqq \gamma$

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and such that G is dense in X relative to the pseudometric topology $\mathscr{T}_d$. A pseudometric d is *continuous* if d is continuous relative to the product topology on $X \times X$.) We say that S is *P-embedded* in X if every continuous pseudometric on S can be extended to a continuous pseudometric on X. The subspace S is *T-embedded* in X if every totally bounded continuous pseudometric on S can be extended to a totally bounded continuous pseudometric on X. We say that S is z-embedded in X if for every zero set Z in S there is a zero set Z' in X such that $Z' \cap S = Z$.

We say that X is *γ-collectionwise normal* if for every discrete family $(F_\alpha)_{\alpha \in I}$ of closed subsets of X of cardinality at most γ there exists a family $(G_\alpha)_{\alpha \in I}$ of pairwise disjoint open subsets of X such that $F_\alpha \subset G_\alpha$ for each $\alpha \in I$. We say that X is *collectionwise normal* if X is γ-collectionwise normal for every cardinal number γ.

It is known that P-embedding implies P^γ-embedding which implies $P^{\aleph_0}$-embedding which is equivalent to C-embedding which implies T-embedding which is equivalent to C^*-embedding which implies z-embedding. See [**1**; **5**; **6**; **7**; **10**; **15**] for some of the inter-relationships between these concepts.

2. Characterizations for extending pseudometrics. From [**15**] we have the following result.

2.1. THEOREM. *Suppose that S is a subspace of a topological space X and that γ is an infinite cardinal number. Then the following statements are equivalent:*

(1) *S is P^γ-embedded in X;*

(2) *Every normal locally finite cozero-set cover of S of cardinality at most γ has a refinement that can be extended to a normal open cover of X;*

(3) *Every normal open cover of S of cardinality at most γ has a refinement that can be extended to a normal locally finite cozero-set cover of cardinality at most γ.*

We will now show that either or both of the covers in (2) and (3) being σ-locally finite or σ-discrete give necessary and sufficient conditions for P^γ-embedding. We will then investigate when can we characterize P^γ-embedding in terms of extending covers rather than extending a refinement. First we need some preliminary results. The proof of Proposition 2.2 is immediate, Proposition 2.3 is part of the folklore in this area, and Theorem 2.4 is due to Morita and can be found in [**12**, Theorem 1.2].

2.2. PROPOSITION. *Suppose that X is a topological space and that $(U_\alpha)_{\alpha \in I}$ is a locally finite family of cozero-sets of X. Then $U = \bigcup_{\alpha \in I} U_\alpha$ is a cozero-set of X.*

2.3. PROPOSITION. *If $(U_n)_{n \in \mathbf{N}}$ is a countable cozero-set cover of a topological space X, then there exists a locally finite cozero-set cover $(V_n)_{n \in \mathbf{N}}$ of X such that $V_n \subset U_n$ for all $n \in \mathbf{N}$.*

2.4. THEOREM. *If $\mathscr{U}$ is an open cover of a topological space X, then the following statements are equivalent:*

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(1) *$\mathscr{U}$ is normal;*

(2) *There exist a metric space Y, a continuous map f from X into Y, and an open cover $\mathscr{V}$ of Y such that $f^{-1}(\mathscr{V})$ is a refinement of $\mathscr{U}$;*

(3) *$\mathscr{U}$ has a locally finite cozero-set refinement.*

2.5. THEOREM. *Suppose that X is a topological space and that γ is an infinite cardinal number. If $\mathscr{U}$ is a σ-locally finite cozero-set cover of cardinality at most γ, then $\mathscr{U}$ admits a locally finite cozero-set refinement of cardinality at most γ.*

Proof. Suppose that $\mathscr{U}$ is a σ-locally finite cozero-set cover of X. Thus $\mathscr{U} = \bigcup_{n \in \mathbf{N}} \mathscr{U}_n$ where each $\mathscr{U}_n = (U_\alpha^n)_{\alpha \in I_n}$ is a family of locally finite cozero-sets of X. For each $n \in \mathbf{N}$, let

$$U_n = \bigcup_{\alpha \in I_n} U_\alpha^n$$

and note that by Proposition 2.2, each U_n is a cozero-set in X. Therefore $(U_n)_{n \in \mathbf{N}}$ is a countable cozero-set cover of X and hence, by Proposition 2.3, there exists a locally finite cozero-set cover $(V_n)_{n \in \mathbf{N}}$ of X such that $V_n \subset U_n$ for each $n \in \mathbf{N}$. Now let $M = \{(n, \alpha): n \in \mathbf{N}$ and $\alpha \in I_n\}$ and for each $(n, \alpha) \in M$, let

$$W_{n\alpha} = V_n \cap U_\alpha^n.$$

We assert that $\mathscr{N} = (W_{n\alpha})_{(n,\alpha) \in M}$ is a locally finite cozero-set cover of X that refines $\mathscr{U}$. To see that $\mathscr{N}$ is locally finite, let $x \in X$. Since $(V_n)_{n \in \mathbf{N}}$ is locally finite, there exist a neighbourhood G_0 of x and a finite subset F of $\mathbf{N}$ such that $G_0 \cap V_n = \emptyset$ if $n \notin F$. Moreover, since $\mathscr{U}_n$ is locally finite, for each $n \in F$, there exist a neighbourhood G_n of x and a finite subset K_n of I_n such that $G_n \cap U_\alpha^n = \emptyset$ if $\alpha \notin K_n$. Let $G = G_0 \cap (\bigcap_{n \in F} G_n)$ and let

$$N = \{(n, \alpha) \in M: n \in F \quad \text{and} \quad \alpha \in K_n\}.$$

Clearly G is a neighbourhood of x and N is a finite subset of M and one easily verifies that $G \cap W_{n\alpha} = \emptyset$ if $(n, \alpha) \notin N$. Note that the cover $\mathscr{N}$ has cardinality at most $\aleph_0 \cdot \gamma = \gamma$. Since the other assertions are obvious, the proof is now complete.

2.6. LEMMA. *Suppose that f is a continuous function from a topological space X into a topological space Y. If $\mathscr{U}$ is a discrete collection of cozero-subsets of Y, then $f^{-1}(\mathscr{U}) = \{f^{-1}(U): U \in \mathscr{U}\}$ is a discrete collection of cozero-subsets of X.*

The proof, which is straightforward, is omitted.

2.7. THEOREM. *Suppose that X is a topological space and that γ is an infinite cardinal number. If $\mathscr{U}$ is a locally finite cozero-set cover of cardinality at most γ, then $\mathscr{U}$ admits a σ-discrete cozero-set refinement of cardinality at most γ.*

Proof. Suppose that $\mathscr{U}$ is a locally finite cozero-set cover of X of cardinality at most γ. By Theorem 2.4, there exist a metric space (Y, d), a continuous function f from X into (Y, d), and an open cover $\mathscr{V}$ of cardinality at most

γ of (Y, d) such that $f^{-1}(\mathscr{V})$ refines $\mathscr{U}$. Now an open cover in a metric space has a σ-discrete cozero-set refinement and this refinement has cardinality at most γ (see for example [**11**, p. 129]). Therefore, there exists a cover $\mathscr{N}$ of Y such that $\mathscr{N} = \bigcup_{n \in \mathbf{N}} \mathscr{N}_n$, each $\mathscr{N}_n$ is a collection of discrete cozero-sets in Y, and $\mathscr{N}$ refines $\mathscr{V}$. Note that $f^{-1}(\mathscr{N}) = \bigcup_{n \in \mathbf{N}} f^{-1}(\mathscr{N}_n)$, and by Lemma 2.6, each $f^{-1}(\mathscr{N}_n)$ is a discrete collection of cozero-sets of X. Since $f^{-1}(\mathscr{N})$ clearly refines $\mathscr{U}$, the proof is now complete.

We can now state and prove the main result.

2.8. THEOREM. *Suppose that S is a subset of a topological space X and that γ is an infinite cardinal number. Then the following statements are equivalent:*

(1) *S is P^γ-embedded in X;*

(2) *Every normal locally finite cozero-set cover of S of cardinality at most γ has a refinement that can be extended to a σ-discrete cozero-set cover of X;*

(3) *Every σ-locally finite cozero-set cover of S of cardinality at most γ has a refinement that can be extended to a locally finite cozero-set cover of X of cardinality at most γ;*

(4) *Every σ-locally finite cozero-set cover of S of cardinality at most γ has a refinement that can be extended to a σ-discrete cozero-set cover of X;*

(5) *Every σ-discrete cozero-set cover of S of cardinality at most γ has a refinement that can be extended to a σ-locally finite cozero-set cover of X of cardinality at most γ.*

Proof. (1) *implies* (2). If $\mathscr{U}$ is a normal locally finite cozero-set cover of S of cardinality at most γ, then, by Theorem 2.1, $\mathscr{U}$ has a refinement that extends to a normal open cover $\mathscr{V}$. By Theorem 2.4, every normal open cover has a locally finite cozero-set refinement $\mathscr{N}$ and by Theorem 2.7, $\mathscr{N}$ has a σ-discrete refinement $\mathscr{A}$. Clearly $\mathscr{A}|S$ refines $\mathscr{U}$, hence (2) holds.

(2) *implies* (4). This implication follows from the facts that every σ-locally finite cozero-set cover of cardinality at most γ has a locally finite cozero-set refinement of cardinality at most γ and that every locally finite cozero-set cover is normal. (See Theorems 2.5 and 2.4.)

(4) *implies* (1). If $\mathscr{U}$ is a normal locally finite cozero-set cover of S of cardinality at most γ, then clearly $\mathscr{U}$ is σ-locally finite and therefore by (4) there exists a σ-discrete cozero-set cover $\mathscr{V}$ of S such that $\mathscr{V}|S$ refines $\mathscr{U}$. By Theorem 2.5, $\mathscr{V}$ has a locally finite cozero-set refinement $\mathscr{N}$ and clearly $\mathscr{N}|S$ refines $\mathscr{U}$. Therefore by Theorem 2.1, S is P^γ-embedded in X. The proof that (1), (3), and (5) are equivalent proceeds in a similar manner and we omit it.

Since a subspace S of a topological space X is P-embedded in X if and only if it is P^γ-embedded in X for all infinite cardinal numbers γ, we obtain the following as an immediate corollary.

2.9. COROLLARY. *If S is a subset of a topological space X, then the following statements are equivalent:*

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(1) *S is P-embedded in X;*

(2) *Every normal locally finite cozero-set cover of S has a refinement that can be extended to a σ-discrete cozero-set cover of X;*

(3) *Every σ-locally finite cozero-set cover of S has a refinement that can be extended to a locally finite cozero-set cover of X;*

(4) *Every σ-locally finite cozero-set cover of S has a refinement that can be extended to a σ-discrete cozero-set cover of X;*

(5) *Every σ-discrete cozero-set cover of S has a refinement that can be extended to a σ-locally finite cozero-set cover of X.*

Of course, as with most cardinality definitions, the most interesting case of P^γ-embedding is when $\gamma = \aleph_0$. As a corollary of Theorem 2.8 we obtain that a subspace S of X is $P^{\aleph_0}$-embedded in X if and only if every countable cozero-set cover of S has a refinement that can be extended to a countable cozero-set cover of X. Actually we proved this in [**15**] as well as showed that S is $P^{\aleph_0}$-embedded in X if and only if every countable star-finite cozero-set cover of S has a refinement that can be extended to a star-finite cozero-set cover of X. In [**6**], Gantner has shown that S is $P^{\aleph_0}$-embedded in X if and only if S is C-embedded in X. Our next result shows that in the case of $P^{\aleph_0}$-embedding we can extend the cover rather than require a refinement.

2.10. Proposition. *If S is a subset of a topological space X, then the following statements are equivalent:*

(1) *S is $P^{\aleph_0}$-embedded in X;*

(2) *Every countable cozero-set cover of S extends to a cozero-set cover of X.*

Proof. (1) *implies* (2). Suppose that $\mathscr{U} = (U_n)_{n\in\mathbf{N}}$ is a countable cozero-set cover of S. By hypothesis, S is C-embedded in X and hence for each $n \in \mathbf{N}$, there exists a cozero-set V_n in X such that $V_n \cap S = U_n$. Let $V = \bigcup_{n\in\mathbf{N}} V_n$ and note that V is a cozero-set of X (Proposition 2.2), hence $X - V$ is a zero-set in X disjoint from S. Again, since S is C-embedded in X, there exists a cozero-set G such that $(X - V) \subset G$ and $G \cap S = \emptyset$. Define $\mathscr{W} = (W_n)_{n\in\mathbf{N}}$ as follows: let $W_1 = V_1 \cup G$; and let $W_n = V_n$ for $n \neq 1$. Then $\mathscr{W}$ is a countable cozero-set cover of X such that $\mathscr{W}|S = \mathscr{U}$.

Clearly (2) *implies* (1).

In [**1**], Alo and Shapiro defined T-embedding. Similar to Theorem 2.1, T-embedding is characterized in terms of finite normal cozero-set covers. We now show that T-embedding can be characterized in terms of extending finite covers.

2.11. Proposition. *If S is a subspace of a topological space X, then the following statements are equivalent:*

(1) *S is T-embedded in X;*

(2) *Every finite cozero-set cover of S extends to a finite cozero-set cover of X.*

Proof. (1) *implies* (2). Let $\mathscr{U} = (U_1, \ldots, U_n)$ be a finite cozero-set cover of S. By (1), $\mathscr{U}$ has a refinement that can be extended to a finite cozero-set cover of X and one can choose the indices so that they agree. We thus have a finite cozero-set cover $\mathscr{V} = (V_1, \ldots, V_n)$ of X such that $V_i \cap S = U_i$ for $i = 1, \ldots, n$. Also, since a T-embedded subset is z-embedded, for each $i = 1, \ldots, n$, there exists a cozero-set W_i in X such that $W_i \cap S = U_i$. For each $i = 1, \ldots, n$, let $A_i = W_i \cap V_i$. Then $(A_1, \ldots, A_n)$ is a finite cozero-set cover of X such that $A_i \cap S = U_i$ for $i = 1, \ldots, n$.

From [**1**, Theorem 2.7] and Theorem 2.4, it follows that S is T-embedded in X if and only if every finite cozero-set cover of S has a refinement that can be extended to a finite cozero-set cover of X. The implication (2) *implies* (1) is now immediate.

Remark. Imler [**10**] has recently shown that *a subspace S of a topological space X is P^γ-embedded in X if and only if every normal cozero-set cover of S of cardinality at most γ can be extended to a normal open cover of X*. It seems unlikely that if $\alpha > \aleph_0$ and if S is P^γ-embedded in X, then every locally finite cozero-set cover of S of cardinality at most γ can be extended to a locally finite cozero-set cover of X.

3. Applications of P-embedding. In [**2**] Aull studied five types of collectionwise normal subsets of a topological space. He defined a subspace S of a topological space X to be α-collectionwise normal if for every discrete (in X) family $(F_\alpha)_{\alpha \in I}$ of closed subsets of S, there exists a family $(G_\alpha)_{\alpha \in I}$ of pairwise disjoint open subsets of X such that $F_\alpha \subset G_\alpha$ for all $\alpha \in I$. In [**18**], Slaughter defined a closed set S of a topological space X to satisfy condition γ_m (m a cardinal number) if for any discrete family $(F_\alpha)_{\alpha \in I}$ of at most m closed subsets of S there exists a pairwise disjoint open family $(G_\alpha)_{\alpha \in I}$ of X such that $F_\alpha \subset U_\alpha$ for all $\alpha \in I$. (Note that in the above definition, $(F_\alpha)_{\alpha \in I}$ is discrete in S if and only if $(F_\alpha)_{\alpha \in I}$ is discrete in X.) If S satisfies condition γ_m for all cardinal numbers m, we say that S *satisfies condition* γ. It is clear that a closed subspace S of a topological space X is α-collectionwise normal if and only if S satisfies condition γ. Slaughter proved that if f is a closed continuous function from a topological space X onto a paracompact space Y, then X is collectionwise normal if and only if X satisfies condition γ at $f^{-1}(y)$ for all $y \in Y$. We will show that a closed subset S of a normal space X satisfies condition γ at S if and only if S is collectionwise normal and P-embedded in X. We can then show that every normal M-space is countably paracompact. Finally we give a new proof of the fact that an F_σ-subset of a collectionwise normal space is collectionwise normal.

3.1. LEMMA. *Let X be a normal space and suppose that $(F_\alpha)_{\alpha \in I}$ is a family of closed discrete subsets in X. If there exists a pairwise disjoint family $(U_\alpha)_{\alpha \in I}$ of open subsets of X such that $F_\alpha \subset U_\alpha$ for every $\alpha \in I$, then there exists a discrete family $(G_\alpha)_{\alpha \in I}$ of cozero-subsets of X such that $F_\alpha \subset G_\alpha$ for all $\alpha \in I$.*

Proof. Let $V = \{x \in X\colon (U_\alpha)_{\alpha \in I}$ is discrete at $x\}$ and observe that V is an open subset of X. If $x \in \bigcup_{\alpha\in I} F_\alpha$, then $x \in F_\beta \subset U_\beta$ for some $\beta \in I$. Set $K = \{\beta\}$ and note that U_β is a neighbourhood of x such that $U_\beta \cap U_\alpha = \emptyset$ if $\alpha \notin K$. Thus $x \in V$ and we conclude that $\bigcup_{\alpha\in I} F_\alpha \subset V$. Since X is normal and $\bigcup_{\alpha\in I} F_\alpha$ is closed, there exists a cozero-set W such that

$$\bigcup_{\alpha\in I} F_\alpha \subset W \subset \text{cl } W \subset V.$$

Also by the normality of X, for each $\alpha \in I$, there exists a cozero-set V_α in X such that $F_\alpha \subset V_\alpha \subset U_\alpha$. For each $\alpha \in I$, set $G_\alpha = V_\alpha \cap W$ and note that G_α is a cozero-set containing F_α. Consider any $x \in X$. If $x \in V$, then $(U_\alpha)_{\alpha\in I}$ is discrete at x and therefore $(G_\alpha)_{\alpha\in I}$ is discrete at x. And if $x \notin V$, then $X - \text{cl } W$ is a neighbourhood of x such that $(X - \text{cl } W) \cap G_\alpha = \emptyset$ for all $\alpha \in I$, and so again $(G_\alpha)_{\alpha\in I}$ is discrete at x.

3.2. Theorem. *Suppose that S is a closed subspace of a normal topological space X. If S satisfies condition γ, then S is P-embedded in X.*

Proof. Suppose that $\mathscr{U}$ is a σ-discrete cozero-set cover of S so that $\mathscr{U} = \bigcup_{n\in\mathbf{N}} \mathscr{U}_n$ and each $\mathscr{U}_n = (U_\alpha{}^n)_{\alpha\in I_n}$ is a discrete family of cozero-sets in S. For each natural number n note that $(\text{cl}_s\, U_\alpha{}^n)_{\alpha\in I_n}$ is discrete. For if $x \in S$, then there exist a neighbourhood G of x and a subset K of I_n with $|K| \leqq 1$ such that $G \cap U_\alpha{}^n = \emptyset$ if $\alpha \notin K$. Now if $y \in G \cap \text{cl}_s\, U_\alpha{}^n$, then G is a neighbourhood of y, and y is in $\text{cl}_s\, U_\alpha{}^n$; hence $G \cap U_\alpha{}^n \neq \emptyset$, a contradiction. Furthermore, since S is closed, $\text{cl}_s\, U_\alpha{}^n = \text{cl}_X\, U_\alpha{}^n$. Therefore by condition γ, there exists a pairwise disjoint open (in X) family $(V_\alpha{}^n)_{\alpha\in I_n}$ such that $U_\alpha{}^n \subset \text{cl}_s\, U_\alpha{}^n \subset V_\alpha{}^n$. By Lemma 3.1 there exists a discrete family $(H_\alpha{}^n)_{\alpha\in I_n}$ of cozero-subsets of X such that $U_\alpha{}^n \subset H_\alpha{}^n$ for every $\alpha \in I_n$. Furthermore, since S is a closed subset of a normal space, it is C-embedded, and hence for each $\alpha \in I_n$ there exists a cozero-set $W_\alpha{}^n$ in X such that $W_\alpha{}^n \cap S = U_\alpha{}^n$. For each $\alpha \in I_n$, set

$$G_\alpha{}^n = W_\alpha{}^n \cap H_\alpha{}^n.$$

Then $\mathscr{G}_n = (G_\alpha{}^n)_{\alpha\in I_n}$ is still discrete. Now $G = \bigcup_{n\in\mathbf{N}} \bigcup_{\alpha\in I_n} G_\alpha{}^n$ is open and hence there exists a cozero-set G_0 in X such that $(X - G) \subset G_0$ and $G_0 \cap S = \emptyset$. Let $\mathscr{N}_1 = \{G_0\}$ and let $\mathscr{N}_n = \mathscr{G}_{n-1}$ for $n \in \mathbf{N}$, $n > 1$. Then one easily shows that $\mathscr{N} = \bigcup_{n\in\mathbf{N}} \mathscr{N}_n$ is a σ-discrete cozero-set cover of X such that $\mathscr{N}|S$ refines $\mathscr{U}$. To see that $\mathscr{N}|S$ is indeed a cover of S, observe that $(\bigcup \mathscr{N}_n) \cap S = \bigcup \mathscr{U}_{n-1}$ for $n > 1$. The proof is now complete.

3.3. Theorem. *Suppose that S is a closed subspace of a normal space X. If S is collectionwise normal and P-embedded in X, then S satisfies condition γ.*

Proof. Suppose that $(F_\alpha)_{\alpha\in I}$ is a family of closed discrete subsets of S. Since S is collectionwise normal, there exists a discrete family $(G_\alpha)_{\alpha\in I}$ of open subsets of S such that $F_\alpha \subset G_\alpha$ for every $\alpha \in I$. In [**12**, Theorem 2.7], it is shown that S is P-embedded in X if and only if for every locally finite family $(G_\alpha)_{\alpha\in I}$ of open subsets of S and every closed family $(F_\alpha)_{\alpha\in I}$ of subsets of S

such that $F_\alpha \subset G_\alpha$ for each $\alpha \in I$ we have a locally finite family $(H_\alpha)_{\alpha \in I}$ of open subsets of X such that $F_\alpha \subset H_\alpha \cap S \subset G_\alpha$ for each $\alpha \in I$. Let $(H_\alpha)_{\alpha \in I}$ be such a family and note that for each $\alpha \in I$ there exist an open set U_α and a closed set V_α such that $F_\alpha \subset U_\alpha \subset V_\alpha \subset H_\alpha$.

For each $(\alpha, \beta) \in I \times I$ with $\alpha \neq \beta$, let $W_{\alpha\beta} = V_\alpha \cap V_\beta$. Let

$$\mathscr{N} = \{W_{\alpha\beta}\colon (\alpha, \beta) \in I \times I, \alpha \neq \beta\}$$

and note that if $x \in X$, then there exist a neighbourhood G of x and a finite subset F of I such that $G \cap V_\alpha = \emptyset$ if $\alpha \notin F$. Clearly $F \times F$ is finite and $G \cap W_{\alpha\beta} = \emptyset$ if $(\alpha, \beta) \notin F \times F$. Thus $\mathscr{N}$ is a closed locally finite family in X and therefore $\mathscr{N}$ is closure-preserving, whence $W = \bigcup \mathscr{N}$ is a closed subset of X. Note that $W \cap S = \emptyset$ for if $x \in W \cap S$, then $x \in W_{\alpha\beta}$ for some $(\alpha, \beta) \in I \times I$ with $\alpha \neq \beta$. Therefore $x \in V_\alpha \cap V_\beta \cap S$; thus $x \in G_\alpha \cap G_\beta$, a contradiction. Therefore there exist disjoint open sets A_1 and A_2 such that $W \subset A_1$ and $S \subset A_2$.

For each $\alpha \in I$, let $B_\alpha = A_2 \cap U_\alpha$ and let $\mathscr{B} = (B_\alpha)_{\alpha \in I}$. Clearly $\mathscr{B}$ is a family of open subsets of X such that $F_\alpha \subset B_\alpha$. To see that $\mathscr{B}$ is pairwise disjoint, note that $x \in B_\alpha \cap B_\beta$ implies that $x \in A_2 \cap U_\alpha \cap U_\beta$ so that $x \in A_2$ and $x \in U_\alpha \cap U_\beta \subset V_\alpha \cap V_\beta = W_{\alpha\beta} \subset W \subset A_1$, a contradiction. Therefore S satisfies condition γ.

From Theorems 3.2 and 3.3 and the observation that if S satisfies condition γ then clearly S is collectionwise normal, we have the following.

3.4. THEOREM. *If S is a closed subspace of a normal space X, then the following statements are equivalent:*

(1) *The subspace S satisfies condition γ;*

(2) *The subspace S is collectionwise normal and P-embedded in X.*

Imler also observed this result and has an entirely different proof in [**10**, Theorem 4.12].

In [**16**] we defined a map f from a topological space X onto a topological space Y to be *paraproper* in case f is closed continuous and $f^{-1}(y)$ is paracompact and P-embedded in X for every $y \in Y$. It was then shown that if X is a regular topological space and if $f\colon X \twoheadrightarrow Y$ is paraproper, then Y paracompact implies X is paracompact. With minor modification in the proof of the aforementioned theorem one can prove (see [**17**, Theorem 12.18]) the following.

3.5. THEOREM. *Suppose that X and Y are topological spaces, that γ is an infinite cardinal number, and that $f\colon X \to Y$ is a closed continuous map such that $f^{-1}(y)$ is γ-paracompact and P^γ-embedded in X for each $y \in Y$. If L is a paracompact subset of Y, then $f^{-1}(L)$ is γ-paracompact.*

As an application of the above we obtain the following.

3.6. THEOREM [**13**, Theorems 6.3 and 3.10]. *Every normal M-space is countably paracompact.*

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Proof. By [**13**, Theorem 6.1] there exists a metric space Y and a closed continuous function f from X onto Y such that $f^{-1}(y)$ is countably compact for all $y \in Y$. Since every discrete collection in $f^{-1}(y)$ is finite, $f^{-1}(y)$ satisfies condition γ for all $y \in Y$ and therefore, by Theorem 3.4, $f^{-1}(y)$ is P-embedded in X. Since $f^{-1}(y)$ is countably paracompact for all $y \in Y$, Theorem 3.5 now yields the result.

In [**19**], Smirnov has shown that every F_σ-set in a normal space is normal. In [**5**], Blair showed that X is normal if and only if every closed subset of X is z-embedded in X. Blair also proved that every F_σ-set in a normal space is z-embedded. Since X is collectionwise normal if and only if every closed subset is P-embedded in X, it is natural to wonder about the analogies to the above concerning collectionwise normality and P-embedding. It turns out that an F_σ-set in a metric space need not be P-embedded (in fact, it need not even be C-embedded). For if X is the real line and if S is the open unit interval, then S is an F_σ-set that is not C-embedded in X. On the other hand, it is known (see [**14**, p. 53]) that an F_σ-subset of a collectionwise normal space is collectionwise normal. We now prove that an F_σ-subset of a γ-collectionwise normal space is γ-collectionwise normal.

3.7. THEOREM. *Suppose that γ is an infinite cardinal number and that X is a γ-collectionwise normal topological space. If S is an F_σ-set in X, then S is γ-collectionwise normal.*

Proof. Suppose that $S = \bigcup_{n \in \mathbf{N}} F_n$ where each F_n is a closed subset of X. Note that S is normal and z-embedded in X. To prove that S is γ-collectionwise normal we show that every closed subset of S is P^γ-embedded in S. Let K be a closed subset of S and let $\mathscr{U} = (U_\alpha)_{\alpha \in I}$ be a locally finite cozero-set cover of K of cardinality at most γ. Let n be a natural number and note that $K \cap F_n$ is a closed subset of the γ-collectionwise normal space X and is therefore P^γ-embedded in X. Therefore there exists a locally finite cozero-set cover $\mathscr{V}_n = (V_\alpha{}^n)_{\alpha \in I_n}$ of X such that $V_\alpha{}^n \cap (K \cap F_n) \subset U_\alpha$. Moreover, since S is z-embedded in X, for all $\alpha \in I$ there exists a cozero-set $U_\alpha{}^*$ in X such that $U_\alpha{}^* \cap S = U_\alpha$. For each $\alpha \in I_n$ let $W_\alpha{}^n = V_\alpha{}^n \cap U_\alpha{}^*$ and note that $\mathscr{N}_n = (W_\alpha{}^n)_{\alpha \in I_n}$ is a locally finite cozero-set family in X such that $W_\alpha{}^n \cap S \subset U_\alpha$. For each $\alpha \in I_n$, let $A_\alpha{}^n = W_\alpha{}^n \cap S$, let $\mathscr{A}_{n+1} = (A_\alpha{}^n)_{\alpha \in I_n}$ and note that $\mathscr{A}_{n+1}$ is a locally finite cozero-set family in S such that $A_\alpha{}^n \cap K \subset U_\alpha$.

Let $G = \bigcup_{n \in \mathbf{N}} \bigcup_{\alpha \in I_n} A_\alpha{}^n$ and note that K and $S - G$ are disjoint closed sets in S such that $(S - G) \subset H$ and $H \cap K = \emptyset$. Set $\mathscr{A}_1 = \{H\}$ and observe that $\mathscr{A} = \bigcup_{n \in \mathbf{N}} \mathscr{A}_n$ is a σ-locally finite cozero-set cover of S such that $\mathscr{A}|K$ refines $\mathscr{U}$. Therefore by Theorem 2.8, K is P^γ-embedded in S and therefore S is γ-collectionwise normal.

3.8. COROLLARY [**14**]. *If X is collectionwise normal and if S is an F_σ-subspace of X, then S is collectionwise normal.*

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