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DECOMPOSITION THEOREMS FOR q^*-RINGS

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Let R be a ring with identity. The study of rings in which every left (right) ideal is quasi-injective was begun by Jain, Mohamed, and Singh (3). They called these rings left (right) q-rings. A number of structure theorems have been proved for q-rings. See, for example, (1), (2), and (5). A ring with the dual property (rings in which every homomorphic image of R as a left (right) R-module is quasi-projective) is called left (right) q^*. These rings were first studied by Koehler (4), where some results connecting q^*-rings with q-rings were obtained.

The main object of this paper is to obtain a structure theorem for semi-perfect q^*-rings. Many of the results of (4) connecting q-rings with q^*-rings follow as natural consequences of this theorem. One important consequence shows that any semi-perfect right q-ring is both left and right q^*.

In this paper all modules are unital, and homomorphisms are R-homomorphisms. The Jacobson radical will be denoted by J. For the radical J, the right annhilator of J, $r(J)=\{x\in R \mid Jx=0\}$ is called the *left socle* of R, and is the largest semi-simple left R-module contained in R. In a similar way, one defines the left annhilator of J as $l(J)=\{x\in R \mid xJ=0\}$ which is called the *right socle* of R, and is the largest semi-simple right R-module contained in R. If R is *semi-local* (i.e. R/J is artinian semi-simple) and M is an R-module, the semi-simple module M/JM, called the top of M, will be denoted by $T(M)$. A ring R is *semi-perfect* if and only if R/J is semi-local and idempotents modulo J can be lifted. We shall say that a module K is *large* in M in case $K\cap L\neq 0$ for every non-zero submodule L of M. The *injective hull* of M, denoted by $E(M)$, is an injective module such that there exists a monomorphism $i:M\to E(M)$ with the property that $i(M)$ is large in $E(M)$.

In order to prove the main theorem of this paper, the following facts, definitions, and lemmas are needed. A ring R is said to be *left* (*right*) *duo* in case for each $x\in R$, $Rx=RxR(xR=RxR)$. A module M is *quasi-injective* in case the natural homomorphism $\mathrm{Hom}_R(M,M)\to\mathrm{Hom}_R(K,M)$ is epic for all submodules K of M. A module M is said to be *projective relative* to N if for each factor module P of N the natural homomorphism $\mathrm{Hom}_R(M,N)\to\mathrm{Hom}_R(M,P)$ is epic. when $N=M$, M is said to be *quasi-projective*. The class of

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modules to which M is projective is closed under taking submodules, factors, and finite direct sums (6). From this it is easily seen that $P_1 \oplus P_2$ is quasi-projective if and only if P_i is projective relative to P_j for $i,j = 1,2$.

The following proposition is due to Koehler and first appears in (4).

PROPOSITION 1. *Let R be a semi-perfect ring. Then R is a left q^*-ring if and only if every left ideal in the radical J of R is an ideal.*

LEMMA 1. *Let R be a semi-perfect left q^*-ring and e and f primitive idempotents such that $Re \cap Rf = 0$. Then $eRf \subseteq r(J)$.*

Proof. Suppose $eRf \neq 0$. Consider $\alpha \in R$ such that $e\alpha f \neq 0$. Then there is a factor module of Re say Re/Ie such that Re/Ie is isomorphic to a submodule of Rf. This isomorphism is given by right multiplication of the element αf with kernal Ie. Consider the cyclic left R-module $Re/Je \oplus Rf$ which is a factor module of $Re \oplus Rf$. Since $Re/Je \oplus Rf$ is quasi-projective, Re/Je is projective relative to Rf by (6, Proposition 1). As Re/Ie is isomorphic to a submodule of Rf, we have that Re/Je is projective to Re/Ie. Since $Re/Ie \neq 0$, and Je is maximal in Re, the natural epimorphism $Re/Ie \to Re/Je \to 0$ splits. But e is primitive so $Ie = Je$. Thus $Je\alpha f = 0$. As α is arbitrary, $JeRf = 0$ yielding the result.

One interesting consequence of Lemma 1 is the following corollary.

COROLLARY 1. *Let R be a semi-perfect left q-ring. Then R is a q^*-ring.*

Proof. By (4, Theorem 3.2), R is a right q^*-ring. To show that R is a left q^*-ring, we need only show by proposition 1 that each left ideal I contained in J is two sided. By (3, Theorem 2.3), $I = Ke$ where K is a two sided ideal and e is an idempotent. Thus,

$$I.R = KeR = KeRe \oplus KeR(1-e) = Ke = I.$$

Here $KeR(1-e) = 0$ follows from lemma 1 and $I \subseteq J$.

Let R be a semi-perfect ring. Then as a left R-module, R can be expressed as a direct sum

$$R = Re_1 \oplus \cdots \oplus Re_n$$

where $e_1, \ldots, e_n$ represent a complete set of orthogonal idempotents and each Re_i is an indecomposable projective left ideal. Using the above decomposition, we shall define the semi-simple left ideal $K \subseteq r(J) \cap J$ as follows: The left ideal K consists of the direct sum of all simple left ideals $\{T_\alpha\}_{\alpha \in A}$ such that $T_\alpha \subseteq Je_i$ for some $i, 1 \leq i \leq n$, and each T_α has the property that $T_\alpha \not\cong T(Re_i)$. In other words K is the sum of all simple left ideals in $r(J) \cap J$ which are not isomorphic to the top of the indecomposable projective left ideal containing them. Of course it is possible that $K = 0$, as in the case of local rings and semi-simple rings.

We note in passing that the left ideal K depends on the decomposition

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$Re_1, \ldots, Re_n$. That is to say, for another set of indecomposable projective left ideals $Rf_1, \ldots, Rf_n$ such that $R = Rf_1 \oplus \cdots \oplus Rf_n$ we may have, using the above definition, a different value for K. However, throughout this paper, only one decomposition of R into indecomposable projective left ideals will be specified making our definition of K unambiguous.

The following lemma shows that modulo K, left q^*-rings have a nice decomposition.

LEMMA 2. *Let R be a semi-perfect left q^*-ring. Then R/K is a ring direct sum of left duo local rings and a semi-simple ring T where T is the direct sum of all Re_i such that $Je_i = 0$.*

Proof. Let $\{e_i\}_{i=1}^n$ be a set of primitive orthogonal idempotents such that $e_1 + \cdots + e_n = 1$. First consider e_i such that $Je_i \neq 0$. Define $\bar{R} = R/K$ and $\bar{e}_i = e_i + K$.

By lemma 1,

$$(1-e_i)Re_i \subseteq r(J)$$
$$e_iR(1-e_i) \subseteq r(J).$$

Also $(1-e_i)Re_i \subseteq Je_i$ since Je_i is the unique maximal left ideal contained in Re_i. Thus $(1-e_i)Re_i \subseteq r(J) \cap J$. Now if $Re_i \cong Re_j$ for some $j \neq i$, then $e_jRe_i \subseteq r(J)$. This implies that Re_i is simple, a contradiction. So we actually have that $(1-e_i)Re_i \subseteq K$. Therefore, $\overline{(1-e_i)}\bar{R}\bar{e}_i = 0$.

Now suppose that $e_iR(1-e_i) \not\subseteq J$. Then there exists $e_j (j \neq i)$ such that Re_j is simple and $e_iRe_j \neq 0$. This implies that $T(Re_i) \cong Re_j$. Since R is semi-perfect, we have $Re_i \cong Re_j$ a contradiction to $Je_i \neq 0$. Thus $e_iR(1-e_i) \subseteq K$, so that $\bar{e}_i\bar{R}\overline{(1-e_i)} = 0$. Therefore, $\bar{R}\bar{e}_i = \bar{R}\bar{e}_i\bar{R}$ and $\bar{R}\overline{(1-e_i)} = \bar{R}\overline{(1-e_i)}\bar{R}$. Thus $\bar{R}\bar{e}_i$ is a local ring direct summand of $\bar{R}$.

Now let T be the sum of all Re_k, $(1 \le k \le n)$ such that Re_k is simple. For each e_i such that $Je_i \neq 0$, e_kRe_i, $e_iRe_k \subseteq K$ by the preceding remarks. This implies that $\bar{T}$ is an ideal direct summand of $\bar{R}$.

Finally note that for each Re_i such that $Je_i \neq 0$, we have for $x \in Je_i$, $Rx = RxR$ by proposition 1. Thus $\bar{R}\bar{x} = \bar{R}\bar{x}\bar{R}$ so $\bar{R}\bar{e}_i$ is left duo.

LEMMA 3. *Let R be a semi-perfect left q^*-ring and f a primitive idempotent such that $fJ(1-f) \neq 0$. Then $J^2f = 0$ and $Jf = (1-f)Jf = r(J)f$.*

Proof. Suppose Jf has a composition factor isomorphic to $T(Rf)$. This means that there exists an $x_1 \in Jf$ such that $fx_1f = x_1f \neq 0$. By hypothesis there exists an idempotent e orthogonal to f and an element $x_2 \neq 0$, $x_2 \in Je$ such that $fx_2e \neq 0$, $(1-f)x_2e = 0$. Consider $x = x_1 + x_2$. Thus,

$$xf = fxf, \qquad (1-f)xe = 0, \qquad x = fx.$$

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By proposition 1, $Rx(e+f)=Rx(e+f)R$. Thus $x(e+f)e=\alpha x(e+f)(\alpha\in R)$. Therefore, $xe=\alpha xe+\alpha xf$, which implies $xe=\alpha xe$ and $\alpha xf=f\alpha fxf=0$. By lemma 1, $xe\in r(J)$. thus $\alpha f\notin Jf$, otherwise $xe=\alpha xe=f\alpha fxe=0$, a contradiction. Thus $f\alpha f$ is a unit. So $\alpha xf=0$ implies that $xf=fx_1f=x_1f=0$, a contradiction. Therefore Jf has no composition factors isomorphic to $T(Rf)$. Thus $fJf=0$. Hence applying lemma 1, we have $Jf=(1-f)Jf=r(J)f$. Likewise, $J^2f=J.r(J)f=0$.

For the following theorem, we shall assume that $e_1,\ldots,e_n$ is a set of primitive orthogonal idempotents such that $e_1+\cdots+e_n=1$, and K as previously defined with respect to the decomposition $R=Re_1+\cdots+Re_n$.

THEOREM 1. *Let R be a semi-perfect ring. Then R is left q^* if and only if the left ideal K is two sided and such that*

(1) *R/K is the ring direct sum of left duo local rings and a semi-simple ring T where T is the direct sum of all Re_i such that $Je_i=0$.*
(2) *If for some e_i, $(1\le i\le n)$, we have $e_iJ(1-e_i)\ne 0$ then $Je_i=Ke_i=r(J)e_i$ and $J^2e_i=0$.*
(3) *$e_iR(1-e_i)\subseteq r(J)$, $(1\le i\le n)$ and $xR\subseteq Rx$ for all $x\in K$.*

Proof. Assume R is left q^*. Conditions (1), (2), and (3) follow from lemmas 1, 2, 3, proposition 1, and the definition of K.

Assume the above conditions are satisfied by R. Let L be a left ideal of R such that $L\subseteq J$. By proposition 1 we need only show that L is two sided. Assume the e_i are ordered so that $e_1,\ldots,e_k$ satisfy condition (2), that is to say $e_iJ(1-e_i)\ne 0$, $1\le i\le k$ with $k\le n$. Also let Re_i, $k+1\le i\le m$ be the Re_i such that $Je_i\ne 0$ and $e_iJ(1-e_i)=0$, $m\le n$.

Clearly $L\subseteq J$ implies that $L\subseteq\sum Le_i$, $1\le i\le m$. We first show that $L=\sum Le_i$. Let $x\in L$, then $x=xe_1+\cdots+xe_m$. Using condition 2, $xe_i\in K$ for all $i\le k$. For $k+1\le i\le m$, we have $e_ix(1-e_i)=0$, so that $e_ix=e_ixe_i\in L$. Now consider the following equation:

$$x-\sum_{i=k+1}^{m} e_ixe_i=\sum_{i=k+1}^{m}(1-e_i)xe_i+\sum_{i=1}^{k} xe_i. \tag{1}$$

An easy consequence of condition 3 shows that $\sum_{i=k+1}^{m}(1-e_i)xe_i\in K$. Hence the right side of the equation is contained in K and the left side of the equation is contained in L. Setting $z=\sum_{i=1}^{k}xe_i+\sum_{i=k+1}^{m}(1-e_i)xe_i$ and using condition 3, we have $Rz=RzR\subseteq K\cap L\subseteq L$. Thus $ze_i=xe_i\in L$, for $i\leqslant k$, and $ze_i=(1-e_i)xe_i\in L$, $k+1\le i\le m$. Using these observations, we see that $Le_i\subseteq L$. Whence $L=\sum Le_i$.

Now we show that L is two sided. Consider $x\in L$, $r\in R$. Since $xe_i\in L$ for each $i\le m$, it suffices to show that $xe_ir\in L$ for all $i\le m$. By condition 3, we have that $e_iR(1-e_i)\subseteq r(J)$, for all $i\le m$. As $x\in J$, we have $xe_ir(1-e_i)=0$. Thus

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$xe_ir = xe_ire_i$. To show that $xe_ire_i \in L$, we first observe that

$$xe_ire_i = (1-e_i)xe_ire_i + e_ixe_ire_i.$$

Using condition 3 we have that $(1-e_i)xe_i \in K$. Thus applying condition 3 again,

(2) $$(1-e_i)xe_i \cdot re_i = \beta(1-e_i)xe_i \quad \text{for some} \quad \beta \in R.$$

By condition 1,

$$e_ixe_i \cdot e_ire_i = e_i\alpha e_i \cdot e_ixe_i + ke_i \quad \text{where} \quad ke_i \in K.$$

Noting that $e_ike_i = 0$ and multiplying the above equation by e_i on the left we obtain,

(3) $$e_ixe_i \cdot e_ire_i = e_i\alpha e_i \cdot e_ixe_i.$$

Combining (2) and (3), we obtain

$$xe_i \cdot re_i = (e_i\alpha e_i + \beta(1-e_i))xe_i.$$

Thus $xe_i \cdot re_i \in Le_i \subseteq L$ for all $i \leq m$ as desired. So L is an ideal which completes the proof.

We make the following observation: The left ideal K was defined to determine when R is left q^*. Clearly we may define the right analogue of K which we shall call K'. Now suppose R is a q^*-ring. Then $K = K'$ as the following argument shows: Let $T \subseteq K$ be a simple left ideal. Then $Te_i \neq 0$ for some $i \leq n$. By theorem 1, we have $T \subseteq Je_i$. Hence $T \not\cong T(Re_i)$. Thus $e_iT = 0$. So $T \subseteq (1-e_i)Je_i \subseteq K'$ applying the right handed version of theorem 1. Hence $K \subseteq K'$. By a symmetric argument $K' \subseteq K$.

We have the following structure theorem for q^*-rings.

Proposition 2. *Let R be a semi-perfect q^*-ring. Then R is a ring direct sum of the following three types of rings:*

(1) *A semi-simple ring.*
(2) *A semi-primary ring S with $J(S)^2 = 0$ and $J(S) = K$.*
(3) *Local duo rings.*

Proof. We shall assume that the e_i are ordered so that

$$e_iJ(1-e_i) \neq 0, \quad \text{or} \quad (1-e_i)Je_i \neq 0 \quad \text{for} \quad 1 \leq i \leq k, \qquad (k \leq n)$$

whenever some of the e_i satisfy the above equation.

Set $e = e_1 + \cdots + e_k$. Then it is easy to show that $eR(1-e) = 0$ and $(1-e)Re = 0$. Thus $Re = ReR$ and $R(1-e) = R(1-e)R$. Hence Re is an ideal direct summand of R. By theorem 1 we have that $e_iJe_i = 0$ for $i \leq k$. Hence,

(1) $$eJe = \sum_{i \neq j} e_iJe_j \subseteq K.$$

As $K \subseteq r(J)$, it is clear that $JeJe = 0$. Hence $J^2(Re) = 0$ (as a ring).

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Now suppose T is a left simple direct summand of K. Then $Te_i \neq 0$ for some $i \leq n$. Since T is two sided, $T \subseteq Je_i$ and $T \cong T(Re_j)$ for some $j \neq i$. Thus $e_jJe_i \neq 0$. Therefore, $T \subseteq Je_i \subseteq Re$. As T was arbitrary we have that $K \subseteq Je$. This statement together with (1) imply that $Je = K$. Using this result and theorem 1, we see that $R(1-e)$ is a direct sum of local duo rings and a semi-simple ring.

COROLLARY 2. *Let R be a semi-perfect left q-ring. Then R is a ring direct sum of the following three types of rings*:

(1) *A semi-simple ring.*
(2) *An artinian q-ring S with $J(S)^2 = 0$.*
(3) *Local duo left q-rings.*

Proof. By corollary 1, R is q^*. So using proposition 2, we need only show that the semi-primary ring S with $J(S)^2 = 0$ is artinian. This follows from the left injectivity of R and the fact that the socle of R must then be finitely generated. That S is a right q-ring follows from (4, Theorem 3.5).

REFERENCES

1. D. A. Hill, *Semi-perfect q-rings.* Math. Ann. **200,** 113–121 (1973).
2. G. Ivanov, *Non-local rings whose ideals are all quasi-injective.* Bull. Austral. Math. Soc. **6,** 45–52 (1972).
3. S. K. Jain, S. H. Mohamed, S. Singh, *Rings in which every right ideal is quasi-injective.* Pacific J. Math. **31,** 73–79 (1969).
4. A. Koehler, *Rings for which every cyclic module is quasi-projective.* Math. Ann. **189,** 311–316 (1970).
5. S. H. Mohamed, *q-Rings with chain conditions.* J. London Math. Soc. (2), **2,** 455–460 (1970).
6. E. Robert, *Projectifs et injectifs relatifs.* C.R. Acad. Sci. Paris Ser. A., **268,** 361–364 (1969).

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