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Existence of Solutions for Abstract Non-Autonomous Neutral Differential Equations

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Abstract. In this paper we discuss the existence of mild and classical solutions for a class of abstract non-autonomous neutral functional differential equations. An application to partial neutral differential equations is considered.

1 Introduction

In this paper we study the existence of mild and classical solutions for a neutral functional differential equation of the form

$$\frac{d}{dt}[x(t) + g(t, x(t - r_1))] = A(t)x(t) + f(t, x_t), \quad t \in [0, a], \tag{1.1}$$

$$x_0 = \varphi \in \mathcal{C} = C([-r, 0]; X), \tag{1.2}$$

where $(X, \|\cdot\|)$ is an abstract Banach space, $(A(t))_{t\in[0,a]}$ is a family of sectorial operators defined on a common domain $\mathcal{D}$ which is dense in X, $0 < r_1 \le \min\{r, a\}$ and $g\colon [0, a] \times X \to X$, $f\colon [0, a] \times \mathcal{C} \to X$ are continuous functions.

This paper is the continuation of our development in [20], where we discussed the existence of solutions for (1.1)–(1.2) when $A(t) = A$ for all $t \in [0, a]$. Similar to [20], our purpose in this paper is to establish the existence of solutions for neutral systems without many of the strong restrictions considered in the literature. To clarify our remarks, we need to make some comments on several papers treating the problem of the existence of solutions for abstract neutral functional differential equations described in the form

$$\frac{d}{dt}\big(x(t) + g(t, x_t)\big) = Ax(t) + f(t, x_t), \quad t \in I = [0, a], \tag{1.3}$$

$$x_0 = \varphi, \tag{1.4}$$

where $A\colon D(A) \subset X \to X$ is a closed linear operator.

In Datko [9] and Adimy and Ezzinbi [1] some linear neutral systems similar to (1.3)–(1.4) were studied under the strong assumption that the range of g is contained in $D(A)$. If A is the generator of a C_0-semigroup of bounded linear operators

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$(T(t))_{t\geq 0}$ (the case studied by Datko), this assumption arises from the treatment of the associated integral equation

$$u(t) = T(t)[\varphi(0) + g(0, u(t-r_1))] - g(t, u(t-r_1)) \\ - \int_0^t AT(t-s)g(s, u(s-r_1))\,ds + \int_0^t T(t-s)f(s, u_s)\,ds,$$

since, except in trivial cases, the operator function $AT(\cdot)$ is not integrable in the operator topology on $[0, b]$, for $b > 0$. The same reason explains the use of a similar assumption in Adimy and Ezzinbi [1], where the case in which A is a Hille–Yosida type operator is studied.

In the papers [19, 22, 23], the system (1.3)–(1.4) was studied under the following more general assumption on g.

($\mathbf{H_g}$) There exists a Banach space $(Y, \|\cdot\|_Y)$, continuously included in X and $H \in L^1([0, a])$ such that $g \in C([0, a] \times \mathcal{B}, Y)$ and $\|AT(t)\|_{\mathcal{L}(Y,X)} \leq H(t)$ for every $t \in [0, a]$.

The condition ($\mathbf{H_g}$) is verified in several situations, for example, in the case when $(T(t))_{t\geq 0}$ is an analytic semigroup and Y is an interpolation space between X and $D(A)$. However, it remains an important restriction on the system.

In [2–5, 10] (among others) an alternative assumption has been used to treat neutral systems. In these works, A is the generator of a compact C_0-semigroup $(T(t))_{t\geq 0}$ and the set $\{AT(t) \colon t \in (0, b]\}$ is bounded in the operator topology. However, as was pointed out in [23], these conditions are valid if and only if A is bounded and $\dim X < \infty$, which restricts the applications to ordinary differential equations. Moreover, if the compactness assumption is removed, it follows that A is bounded which remains a strong restriction.

Abstract non-autonomous neutral differential systems have been studied under similar restrictions, and related to this matter we only cite [13, 18]

Our purpose in this paper is to establish the existence of mild and classical solutions without the above cited restrictions. Briefly, we observe that our results are proved assuming some “temporal” and “spatial” regularity type conditions on the function $t \to g(t, \varphi(t-r_1))$. This simple method permits us to study some neutral systems that have not been considered in the literature.

We now give motivations for the study of abstract neutral differential equations. For related ordinary neutral differential equations we refer the reader to Hale and Lunel [17] and the references therein. Partial neutral differential equations arise, for instance, in the theory of heat conduction in fading memory material. In the classical theory of heat conduction, it is assumed that the internal energy and the heat flux depends linearly on the temperature u and on its gradient ∇u. Under these conditions, the classical heat equation describes sufficiently well the evolution of the temperature in different types of materials. However, this description is not satisfactory in materials with fading memory. In the theory developed in [15, 28], the internal energy and the heat flux are described as functionals of u and u_x. The next system (see [6–8, 26]) has been frequently used to describe this phenomenon,

$$\frac{d}{dt}\Big[u(t,x)+\int_{-\infty}^{t}k_1(t-s)u(s,x)\,ds\Big]=c\triangle u(t,x)+\int_{-\infty}^{t}k_2(t-s)\triangle u(s,x)\,ds,$$
$$u(t,x)=0,\quad x\in\partial\Omega.$$

In this system, $\Omega\subset\mathbb{R}^n$ is open, bounded and has smooth boundary, $(t,x)\in[0,\infty)\times\Omega$, $u(t,x)$ represents the temperature in x at the time t, c is a physical constant, and $k_i\colon\mathbb{R}\to\mathbb{R}$, $i=1,2$, are the internal energy and the heat flux relaxation, respectively. By assuming the solution u is known on $(-\infty,0]$ and $k_2\equiv 0$, we can transform this system into the abstract form (1.3)–(1.4).

We also find abstract neutral systems in the theory of population dynamics; see [11, 12, 24, 25] and their references. Looking at these papers, it is natural to think that the abstract system (2.1)–(2.2) can be used to describe spatial diffusion phenomena, which arise from the natural tendency of biological populations to migrate from high population density regions to regions with lesser density.

There is an extensive literature on ordinary neutral differential equations in the theory of population dynamics; see, for instance, [11, 12, 24, 25]. If in these works we consider the spatial diffusion phenomena which arise in the natural tendency of biological populations to migrate from a high population density region to a region with lesser density, then it is possible to obtain partial neutral differential systems of the form

$$\frac{d}{dt}[u(t,\xi)+g(t,u(t-r_1,\xi))]=\triangle u(t,\xi)+f(t,u(t-r_1,\xi)),$$

which can be described in the abstract form (1.1).

Partial differential neutral systems also appear in transmission line theory. Wu and Xia showed that a ring array of identical resistively coupled lossless transmission lines leads to a system of neutral functional differential equations with discrete diffusive coupling which exhibit various types of discrete waves [29]. By taking a natural limit, they obtained from this system of neutral equations a scalar partial neutral differential equation defined on the unit circle. Hale [16] also investigated such a partial neutral differential equation under the more general form

$$\frac{d}{dt}\mathcal{D}u_t(x)=\frac{\partial^2}{\partial x^2}\mathcal{D}u_t(x)+f(u_t)(x),\quad t\ge 0,$$
$$u_0=\varphi\in C([-r,0];C(S^1;\mathbb{R})),$$

where

$$\mathcal{D}(\psi)(s):=\psi(0)(s)-\int_{-r}^{0}[d\eta(\theta)]\psi(\theta)(s)$$

for $s\in S^1$, $\psi\in C([-r,0];C(S^1;\mathbb{R}))$, and η is a function of bounded variation.

We now consider some notations and technicalities used in the rest of this paper. Let $(W,\|\cdot\|_W)$, $(Z,\|\cdot\|_Z)$ be Banach spaces. In this paper, $\mathcal{C}$ is the space $C([-r,0];X)$

with the sup-norm $\|\psi\|_{\mathcal{C}} = \sup_{\theta\in[-r,0]}\|\psi(\theta)\|$, $\mathcal{L}(W,Z)$ represents the space of bounded linear operators from W into Z endowed with the uniform operator norm $\|\cdot\|_{\mathcal{L}(W,Z)}$, and we write simply $\mathcal{L}(W)$ and $\|\cdot\|_{\mathcal{L}(W)}$ when $Z = W$. The notation $Z \hookrightarrow W$ is used to indicate that $Z \subset W$ and that the inclusion from Z into W is continuous. For a closed linear operator $S\colon D(S) \subset Z \to Z$, we denote by $[S]$ the domain of S endowed with the graph norm $\|\cdot\|_{[S]}$ If S is injective, we use the norm $\|x\|_{[S]} = \|Sx\|$.

As usual, $C([0,b];Z)$ is the space of continuous functions from $[0,b]$ into Z with the sup-norm denoted by $\|\cdot\|_{C([0,b];Z)}$ and $C^{\gamma}([0,b];Z)$, $\gamma \in (0,1)$, represents the space formed for all the γ-Hölder Z-valued continuous functions from $[0,b]$ into Z with the norm $\|\xi\|_{C^{\gamma}([0,b];Z)} = \|\xi\|_{C([0,b];Z)} + [|\xi|]_{C^{\gamma}([0,b];Z)}$ where $[|\xi|]_{C^{\gamma}([0,b];Z)} = \sup_{t,s\in[0,b];t\neq s}\frac{\|\xi(s)-\xi(t)\|_Z}{(t-s)^{\gamma}}$. In addition, $C^{1+\gamma}([0,b];Z)$ is the space formed for all the C^1 functions $\xi \in C^{\gamma}([0,b];Z)$ for which $\xi' \in C^{\gamma}([0,b];Z)$ endowed with the norm $\|\xi\|_{C^{1+\gamma}([0,b];Z)} = \|\xi\|_{C^{\gamma}([0,b];Z)} + \|\xi'\|_{C^{\gamma}([0,b];Z)}$.

2 Existence of Solutions

In this section we discuss the existence of solutions for the abstract system

$$\frac{d}{dt}[x(t) + g(t, x(t-r_1))] = A(t)x(t) + f(t, x_t), \quad t \in [\sigma, \sigma + b], \tag{2.1}$$

$$x_{\sigma} = \varphi \in \mathcal{C} = C([-r,0];X), \tag{2.2}$$

where $A(t)\colon D(A(t)) \subset X \to X$ are closed linear operators and $g\colon [0,a] \times X \to X$, $f : [0,a] \times \mathcal{C} \to X$ are continuous functions.

To establish our results, we always assume that the following conditions are verified.

($\mathbf{H_1}$) There are $C > 0$, $\vartheta \in (\pi/2, \pi)$ and a neighborhood of zero Σ, such that $\rho(A(t)) \supset \Lambda_{\vartheta} = \{\lambda \in \mathbb{C} : |\arg(\lambda)| < \vartheta\} \cup \Sigma$ and $\|R(\lambda, A(t))\| \leq C|\lambda|^{-1}$ for all $(\lambda, t) \in \Lambda_{\vartheta} \times [0,a]$.

($\mathbf{H_2}$) There exists a Banach space $(\mathcal{D}, \|\cdot\|_{\mathcal{D}})$ and $\alpha \in (0,1)$ such that $\mathcal{D}$ is dense in X, $\mathcal{D} = D(A(t))$ for all $t \in [0,a]$ and $A(\cdot) \in C^{\alpha}([0,a]; \mathcal{L}(\mathcal{D}, X))$.

Under condition $\mathbf{H_1}$, each operator $A(t)$ is the infinitesimal generator of an analytic semigroup on X. Moreover, from Lunardi [27, Chapter VI] we know that if $\mathbf{H_1}$ and $\mathbf{H_2}$ are verified, then there exists an evolution operator $U(\cdot)$ associated with the non-autonomous abstract Cauchy problem

$$x'(t) = A(t)x(t), \quad t \geq s, t, s \in [0,a], \tag{2.3}$$

$$x(s) = x. \tag{2.4}$$

Definition 2.1 A family of linear operators $\{U(t,s) : t, s \in [0,a], t \geq s\} \subset \mathcal{L}(X)$ is an evolution operator for (2.3)–(2.4) if $U(s,s)x = x$ for all $(s,x) \in [0,a] \times X$, $U(t,r)U(r,s) = U(t,s)$ for $a \geq t \geq r \geq s \geq 0$, $U(\cdot,s)x \in C^1((s,a];X) \cap C((s,a];\mathcal{D})$ and $\frac{d}{dt}U(t,s)x = A(t)U(t,s)x$ for all $x \in X$ and every $t > s$.

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We now introduce the following concept of mild solution for the system (2.1)–(2.2).

Definition 2.2 A function $u \in C([-r+\sigma, \sigma+b]; X)$, $b > 0$, $\sigma \in \mathbb{R}$, is called a mild solution of (2.1)–(2.2) if $u_\sigma = \varphi$ and

$$u(t) = U(t,\sigma)[\varphi(0) + g(\sigma, \varphi(-r_1))] - g(t, u(t-r_1)) \\ - \int_\sigma^t U(t,s)A(s)g(s, u(s-r_1))\, ds + \int_\sigma^t U(t,s) f(s, u_s)\, ds,$$

for every $t \in [\sigma, \sigma + b]$.

We can now establish our first existence result.

Theorem 2.3 *Assume the following conditions are satisfied.*

(i) *There are Banach spaces $(Y_i, \|\cdot\|_{Y_i})$, $i = 0, \dots, 2n$, $n \geq 2$, such that $Y_{i+1} \hookrightarrow Y_i \hookrightarrow Y_0 = X$, $g \in C([0,a] \times Y_{i+1}, Y_i)$ for all $i = 1, \dots, 2n-1$, and $A(\cdot)g(\cdot) \in C([0,a] \times Y_{j+1}, Y_{j-1})$ for each $j = 2, \dots, 2n-1$.*

(ii) *There are Banach spaces $(Z_i, \|\cdot\|_{Z_i})_{i\in\mathbb{N}}$ and natural numbers $p_1, \dots, p_{2n}$ satisfying $p_{i+1} - p_i \geq 2$ such that*

$$Y_i \hookrightarrow Z_{p_i} \hookrightarrow \cdots \hookrightarrow Z_{p_{(i-1)}+1} \hookrightarrow Y_{i-1} \hookrightarrow \cdots \hookrightarrow Y_1 \hookrightarrow Z_{p_1} \cdots \hookrightarrow Z_1 \hookrightarrow Z_0 = X,$$

for all $i = 1, \dots, 2n$; $U(t,\cdot) \in L^1([0,t), \mathcal{L}(Z_j, Z_{j+2})) \cap L^1([0,t), \mathcal{L}(Z_{p_i-2}, Y_i))$ for all $t \in [0,a]$, $j \in \{0, \dots, p_{2n}-2\}$ and $i = 1, \dots, 2n$; and $U(t,\cdot) \in C([0,t), \mathcal{L}(Z_j))$ and $U(t,\cdot) \in C([0,t), \mathcal{L}(Y_i))$ for all $j \leq p_{2n}$ and every $i \leq 2n$.

(iii) *$f \in C([0,a] \times C([-r,0]; Z_{p_i+1}); Y_i) \cap C([-r,0]; Y_k), Z_{p_k})$ for all $i = 1, \dots, 2n-1$, and $k = 1, \dots, 2n$, $f \in C([0,a] \times C([-r,0]; Z_{j+1}); Z_j)$ for all j, and exits $L_f > 0$ such that*

$$\|f(t,\psi_1) - f(t,\psi_2)\| \leq L_f \|\psi_1 - \psi_2\|_{\mathcal{C}}, \quad t \in [0,a], \psi_i \in \mathcal{C}.$$

If $\varphi \in C([-r,0]; Y_{2n})$, then there exists a unique mild solution of the neutral system (1.1)–(1.2) *on $[-r, nr_1 \wedge a]$.*

Proof To simplify, we assume $nr_1 < a$. The other case can be proved arguing as in the rest of this proof. Let $M > 0$ such that $\|U(t,s)\| \leq M$ for all $t > s$, $t, s \in [0,a]$. Let $\Gamma\colon C([-r,r_1]; X) \to C([-r,r_1]; X)$ be the map given by $(\Gamma u)_0 = \varphi$ and

$$\Gamma u(t) = U(t,0)[\varphi(0) + g(0, \varphi(-r_1))] - g(t, \varphi(t-r_1)) \\ - \int_0^t U(t,s)A(s)g(s, \varphi(s-r_1))\, ds + \int_0^t U(t,s) f(s, u_s)\, ds, \quad t \in [0, r_1]. \tag{2.5}$$

From (i) it follows that the function $s \to A(s)g(s, \varphi(s-r_1))$ belongs to $C([0,r_1]; Y_2)$ which permits us to affirm that $s \to U(t,s)A(s)g(s,\varphi(s-r_1))ds \in L^1([0,t); X)$ for all $t \in [0, r_1]$. Now it is easy to show that $\Gamma u \in C([-r,r_1]; X)$ for all $u \in C([-r,r_1]; X)$.

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On the other hand, from the inequality

$$\sup_{\theta\in[0,t]} \|\Gamma^k u(\theta) - \Gamma^k v(\theta)\| \le \frac{(ML_f)^k}{k!} \int_0^t \sup_{\theta\in[0,s]} \|u(\theta) - v(\theta)\| \, ds, \quad t \in [0, r_1],$$

it follows that $\Gamma^k(\cdot)$ is a contraction for k large enough and there exits a unique fixed point $u^1(\cdot)$ of Γ. Obviously, $u^1(\cdot)$ is the unique mild solution of (1.1)–(1.2) on $[-r, r_1]$.

Next, we show that $u^1 \in C([-r, r_1]; Y_{2n-2})$. From the assumptions it is easy to see that $s \to g(s, \varphi(s - r_1))$ and $s \to A(s)g(s, \varphi(s - r_1))$ are functions in $C([0, r_1]; Y_1)$. From this fact and the inequality

$$\begin{aligned}\|u^1(t)\|_{Z_2} \le \|U(t,0)(\varphi(0) + g(0, \varphi(-r_1))\|_{Z_2} + \|g(t, \varphi(t - r_1))\|_{Z_2} \\ + \int_0^t \|U(t,s)A(s)g(s, \varphi(s - r_1))\|_{Z_2} \, ds \\ + \int_0^t \|U(t,s)\|_{\mathcal{L}(X,Z_2)} \|f(s, u_s^1)\| \, ds,\end{aligned}$$

we infer $u^1 \in C([-r, r_1]; Z_2)$ and $s \to f(s, u_s^1) \in C([0, r_1]; Z_1)$.

If $p_1 = 2$, then

$$\begin{aligned}\|u^1(t)\|_{Y_1} \le \|U(t,0)\varphi(0) + g(0, \varphi(-r_1))\|_{Y_1} + \|g(t, \varphi(t - r_1))\|_{Y_1} \\ + \int_0^t \|U(t,s)A(s)g(s, \varphi(s - r_1))\|_{Y_1} \, ds \\ + \int_0^t \|U(t,s)\|_{\mathcal{L}(Z_1,Y_1)} \|f(s, u_s^1)\|_{Z_1} \, ds,\end{aligned}$$

and hence, $u^1 \in C([-r, r_1]; Y_1)$. On the other hand, if $p_1 > 2$ we get

$$\begin{aligned}\|u^1(t)\|_{Z_3} \le \|U(t,0)(\varphi(0) + g(0, \varphi(-r_1)))\|_{Z_3} + \|g(t, \varphi(t - r_1))\|_{Z_3} \\ + \int_0^t \|U(t,s)A(s)g(s, \varphi(s - r_1))\|_{Z_3} \, ds \\ + \int_0^t \|U(t,s)\|_{\mathcal{L}(Z_3,Z_1)} \|f(s, u_s^1)\|_{Z_1} \, ds,\end{aligned}$$

which implies that $u^1 \in C([-r, r_1]; Z_3)$ and $s \to f(s, u_s^1) \in C([0, r_1]; Z_2)$. Continuing as above, we infer that $u^1 \in C([-r, r_1]; Y_{2n-2})$.

We can now repeat the above process with $(u^1)_{r_1}$ and Y_{2n-2} in place of φ and Y_{2n}, and prove the existence of a unique mild solution $u^2 \in C([r_1 - r, 2r_1]; Y_{2n-4})$ for the neutral system

$$\frac{d}{dt}[x(t) + g(t, x(t - r_1))] = Ax(t) + f(t, x_t), \quad t \in [r_1, 2r_1],$$

$$x_{r_1} = (u^1)_{r_1}.$$

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From the above steps, for each $i = 1, \dots, n$ there exists a unique mild solution $u^i \in C([(i-1)r_1 - r, ir_1]; Y_{2n-2i})$ of the neutral system

$$\frac{d}{dt}[x(t) + g(t, x(t - r_1))] = Ax(t) + f(t, x_t), \quad t \in [(i-1)r_1, ir_1],$$
$$x_{(i-1)r_1} = (u^{i-1})_{(i-1)r_1}.$$

Finally, by defining the function $u\colon [-r, nr_1] \to X$ by $u(t) = \varphi(t)$ for $t \leq 0$ and $u(t) = u^i(t)$ for $t \in [(i-1)r_1, ir_1]$, $i = 1, 2, \dots, n$, we obtain a mild solution of (1.1)–(1.2) on $[-r, nr_1]$. ∎

Assuming that $U(t,s)$ is compact for $t > s$, we can also prove the existence of a mild solutions for (1.1)–(1.2).

Theorem 2.4 *Assume the conditions* (i)–(ii) *in Theorem* 2.3 *are satisfied,* $U(t,s)$ *is compact for every* $t > s$ *and* $U(t, \cdot) \in C([0,t), \mathcal{L}(X))$ *for all* $t \in [0,a]$. *Suppose* $f \in C([0,a] \times C([-r,0]; Z_{p_i+1}); Y_i) \cap C([0,a] \times C([-r,0]; Z_{j+1}); Z_j)$ *for all* $j \in \mathbb{N}$ *and every* $i = 1, \dots, 2n-1$, *and there are* $m \in C([0,a]; [0,\infty))$ *and a non-decreasing function* $W\colon [0,\infty) \to (0,\infty)$ *such that* $\|f(t,\psi)\| \leq m(t)W(\|\psi\|_{\mathcal{C}})$, *for each* $(t,\psi) \in [0,a] \times \mathcal{C}$. *If* $\varphi \in C([-r,0]; Y_4)$ *and*

$$M \int_0^{r_1} m(s)\, ds < \int_{C(\varphi)}^{\infty} \frac{ds}{W(s)},$$

where $M = \sup\{\|U(t,s)\|; t > s,\ t,s \in [0,a]\}$ *and*

$$C(\varphi) = M(\|\varphi(0) + g(0, \varphi(-r_1))\| + \sup_{\theta \in [0,r_1]} \|g(\theta, \varphi(\theta - r_1))\| + \|\varphi\|_{\mathcal{C}} + Mr_1 \sup_{\theta \in [0,r_1]} \|A(\theta) g(\theta, \varphi(\theta - r))\|.$$

Then there exists a mild solution of (1.1)–(1.2) *on* $[-r_1, b]$ *for some* $r_1 < b \leq a$.

Proof Let $\Gamma(\,\cdot\,)$ be the map defined by (2.5). From [21, Lemma 3.1] we infer that Γ is completely continuous. In order to use the Leray–Schauder alternative theorem ([14, Theorem 6.5.4]), we next establish *a priori* estimates for the solutions of $z = \lambda\Gamma z$, $\lambda \in (0,1)$. Let $\lambda \in (0,1)$, z^λ be a solution of $z = \lambda\Gamma z$ and $\alpha^\lambda\colon [0,r_1] \to \mathbb{R}$ be defined by $\alpha^\lambda(t) = \|\varphi\|_{\mathcal{C}} + \sup_{\theta \in [0,t]} \|z^\lambda(\theta)\|$. Then for $t \in [0,r_1]$ we get

$$\|z^\lambda(t)\| \leq M\|\varphi(0) + g(0, \varphi(-r_1))\| + \|g(t, \varphi(t - r_1))\| + M\int_0^t \|A(\theta) g(\theta, \varphi(\theta - r_1))\|\, d\theta + M\int_0^t m(s) W(\|z_s^\lambda\|_{\mathcal{C}})\, ds,$$

which implies that

$$\alpha^\lambda(t) \leq C(\varphi) + M\int_0^t m(s) W(\alpha^\lambda(s))\, ds. \tag{2.6}$$

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If $\beta_\lambda(t)$ represents the right-hand side of (2.6), then $\beta_\lambda'(t) \le Mm(t)W(\beta_\lambda(t))$ and

$$\int_{C(\varphi)}^{\beta_\lambda(t)} \frac{ds}{W(s)} \le M \int_0^{r_1} m(s)\,ds < \int_{C(\varphi)}^{\infty} \frac{ds}{W(s)}\,ds,$$

which shows that $\{\beta_\lambda(\cdot) : \lambda \in (0,1)\}$ is bounded in $C([0,r_1])$, and as a consequence, that $\{z^\lambda(\cdot) : \lambda \in (0,1)\}$ is bounded in $C([-r,r_1];X)$.

From [14, Theorem 6.5.4] there exists a fixed point $u^1(\cdot)$ of $\Gamma(\cdot)$. Moreover, from the proof of Theorem 2.3 it follows that $u^1 \in C([0,r_1];Y_2)$, $s \to g(s,u^1(s-r_1)) \in C([r_1,2r_1];Y_1)$, and $s \to A(s)g(s,u^1(s-r_1)) \in C([r_1,2r_1];X)$.

Let $r_1 < b \le 2r_1$ be such that

$$M\int_{r_1}^{b} m(s)\,ds < \int_{C(u^1)}^{\infty} \frac{ds}{W(s)},$$

where

$$C(u^1) = M\|u^1(r_1) + g(r_1,u^1(0))\| + \sup_{\theta\in[r_1,2r_1]} \|g(\theta,u^1(\theta-r_1))\| \\ + \|(u^1)_{r_1}\|_{\mathcal{C}} + Mb \sup_{\theta\in[r_1,2r_1]} \|A(\theta)g(\theta,u^1(\theta-r))\|.$$

Arguing as in the first part of this proof, we infer the existence of a mild solution $u^2 \in C([r_1-r,b];X)$ of

$$\frac{d}{dt}[x(t)+g(t,x(t-r_1))] = Ax(t) + f(t,x_t), \quad t \in [r_1,b],$$
$$x_{r_1} = (u^1)_{r_1}.$$

To finish, we note that the function obtained by pasting the functions $u^1(\cdot)$ and $u^2(\cdot)$ is a mild solution (1.1)–(1.2) on $[-r,b]$. ■

Remark 2.5 It is relevant to observe that the assumptions in the above results are not restrictive. Assume, for instance, $A(t) = A$ for all $t \in [0,a]$. In this case, A is the generator of a analytic semigroup $(T(t))_{t\ge 0}$ on X, $U(t,s) = T(t-s)$ for $t \ge s$ and we can consider, among several alternatives, the spaces $Y_i = [D(A^i)]$, $i \in \mathbb{N}$, and $Z_j = [D((-A)^{j+\beta_j})]$, $\beta_j \in (0,1)$, $j \in \mathbb{N}$, where $(-A)^\beta$ denotes a fractional power of A. We note that $Y_i \hookrightarrow Z_j$ if $i > j$, $U(t,\cdot) \in L^1([0,t];\mathcal{L}(Z_{j_1},Z_{j_2}))$ when $|j_1+\beta_1-j_2-\beta_2| < 1$, and $U(\cdot)$ is a strongly continuous operator family on each one of these spaces. The assumptions on f and g, are verified, for instance, by functions which are continuously invariant on these spaces; consider, for example, $g \in C([0,a]\times C([-r,0];Y_i);Y_i)$ and $f \in C([0,a]\times C([-r,0];Z_j);Z_j)$ for all i, j.

In the non-autonomous case, we can think in the interpolation spaces $(X,\mathcal{D})_{\alpha,p}$, $\alpha \in (0,1)$ and $p > 1$, defined as in Lunardi [27]. We remark that $(X,\mathcal{D})_{\beta,p} \hookrightarrow (X,\mathcal{D})_{\theta,p}$ for $\beta > \theta$,

$$U(t,\cdot) \in L^1([0,t];\mathcal{L}((X,\mathcal{D})_{\theta,p},\mathcal{D})) \text{ and } U(t,\cdot) \in L^1([0,t];\mathcal{L}(X,(X,\mathcal{D})_{\theta,p}))$$

for $\theta \in (0,1)$, and $U(t,\cdot) \in L^1([0,t];\mathcal{L}((X,\mathcal{D})_{\theta,p},(X,\mathcal{D})_{\beta,p}))$ when $0 < \theta < \beta < 1$. The assumptions on g, f, are satisfied by functions continuously invariant on these spaces.

2.1 Existence of Classical Solutions.

We complete this section by studying the existence of classical solutions in $C^\alpha([0,b];\mathcal{D})$ for the neutral system

$$\frac{d}{dt}[x(t)+g(t,x(t-r_1))] = A(t)x(t)+f(t)x_t, \quad t\in[0,a], \tag{2.7}$$

$$x_0=\varphi\in\mathcal{C}_{\mathcal{D}}=C([-r,0];\mathcal{D}), \tag{2.8}$$

where $f\in C^\alpha([0,a];\mathcal{L}(\mathcal{C}_{\mathcal{D}},X))$.

For simplicity, in the rest of this paper we assume $A=A(0)$, $\mathcal{D}$ is the space $[D(A)]$ with the norm $\|x\|_{\mathcal{D}} = \|Ax\|$, and $\mathcal{C}_{\mathcal{D}}$ is the space $C([-r,0];\mathcal{D})$ with the norm $\|\psi\|_{\mathcal{C}_{\mathcal{D}}} = \sup_{\theta\in[-r,0]}\|A\psi(\theta)\|$. In addition, $(T(t))_{t\geq 0}$ represents the semigroup generated by A and C_i, $i\in\mathbb{N}$, are positive constants such that $\|A^iT(t)\|\leq C_it^{-i}$ for every $t>0$. To obtain our results, we need consider some interpolation spaces. The notation $(X,\mathcal{D})_{\eta,\infty}$, $\eta\in(0,1)$ stands for the space

$$(X,\mathcal{D})_{\eta,\infty}=\{x\in X:[x]_{\eta,\infty}=\sup_{t\in(0,1)}\|t^{1-\eta}AT(t)x\|<\infty\},$$

endowed with the norm $\|x\|_{\eta,\infty}=[x]_{\eta,\infty}+\|x\|$. Next, we assume that the numbers $C_k=\sup_{s\in(0,a]}\|s^kA^kT(s)\|$ and $C^k_{\eta,\infty}=\sup_{s\in(0,a]}s^{1-\eta}\|A^kT(s)\|_{\mathcal{L}((X,\mathcal{D})_{\eta,\infty},X)}$ are finite for all $k\in\mathbb{N}\cup\{0\}$; see [27] for additional details.

We introduce now the following concept of a classical solution.

Definition 2.6 A function $u\in C([-r,b];X)$, $0<b\leq a$, is called a classical solution of (2.7)–(2.8) in $C^\alpha([0,b];\mathcal{D})$ if $u|_{[0,b]}\in C^\alpha([0,b];\mathcal{D})$ and u verifies (2.7)–(2.8) on $[0,b]$.

To prove the main result of this section, we need some preliminary lemmas. The proof of Lemma 2.7 follows from the steps in the proof of [27, Theorem 4.3.1]. We omit the proof.

Lemma 2.7 *Let $\xi\in C^\alpha([0,b];X)$, $x\in\mathcal{D}$ and $u\colon[0,b]\to X$ be the function given by*

$$u(t)=T(t)x+\int_0^t T(t-s)\xi(s)\,ds.$$

If $Ax+\xi(0)\in(X,\mathcal{D})_{\alpha,\infty}$, then $u\in C^\alpha([0,b];\mathcal{D})\cap C^{1+\alpha}([0,b];X)$, $u'(t)\in(X,\mathcal{D})_{\alpha,\infty}$ for all $t\in[0,b]$, $u'(t)=Au(t)+\xi(t)$ for every $t\in[0,b]$ and

$$[|u|]_{C^\alpha([0,b];\mathcal{D})}\leq\Lambda_1[|\xi|]_{C^\alpha([0,b];X)}+\frac{C^1_{\alpha,\infty}}{\alpha}\|Ax+\xi(0)\|_{\alpha,\infty},$$

$$\|u\|_{C([0,b];\mathcal{D})}\leq C_0\|Ax\|+\frac{C_1b^\alpha}{\alpha}[|\xi|]_{C^\alpha([0,b];X)}+2C_0\|\xi\|_{C([0,b];X)},$$

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where $\Lambda_1 = \frac{2C_1}{\alpha} + 3C_0 + 1 + \frac{C_2}{\alpha(1-\alpha)}$.

Lemma 2.8 *Let the assumptions in Lemma 2.7 hold and assume* $\xi(0) \in (X, \mathcal{D})_{\alpha,\infty}$. *Then*

$$\|u\|_{C([0,b];\mathcal{D})} \leq C_0\|Ax\| + \Lambda_2[|\xi|]_{C^\alpha([0,b];X)} b^\alpha + \frac{C^1_{\alpha,\infty}}{\alpha}\|\xi(0)\|_{\alpha,\infty} b^\alpha,$$

where $\Lambda_2 = (C_0 + \frac{C_1}{\alpha} + 1)$.

Proof By re-writing the function u in the form

$$u(t) = T(t)x + \int_0^t T(t-s)(\xi(s) - \xi(t))\, ds + \int_0^t T(t-s)\xi(t)\, ds,$$

we obtain

$$Au(t) = AT(t)x + \int_0^t AT(t-s)(\xi(s) - \xi(t))\, ds + T(t)\xi(t) - \xi(t).$$

Consequently,

$$\begin{aligned}
\|Au(t)\| &\leq C_0\|Ax\| + \int_0^t \frac{C_1[|\xi|]}{t-s}(t-s)^\alpha ds + \|\xi(0) - \xi(t)\| \\
&\quad + \Big\| \int_0^t AT(s)\xi(0) d s \Big\| + \|T(t)(\xi(t) - \xi(0))\| \\
&\leq C_0\|Ax\| + \frac{C_1}{\alpha}[|\xi|]_{C^\alpha([0,b];X)} b^\alpha + [|\xi|]_{C^\alpha([0,b];X)} b^\alpha \\
&\quad + \int_0^t \frac{C^1_{\alpha,\infty}\|\xi(0)\|_{\alpha,\infty}}{(t-s)^{1-\alpha}}\, ds + C_0[|\xi|]_{C^\alpha([0,b];X)} b^\alpha,
\end{aligned}$$

which permits us to finish the proof. ■

Remark 2.9 In what follows, for $u \in C([-r,b];\mathcal{D})$ we denote by F_u and P_u the functions $F_u\colon [0,b] \to X$ and $P_u\colon [0,b] \to \mathcal{C}_\mathcal{D}$ defined by $F_u(t) = (A(t) - A)u(t) + f(t)u_t$ and $P_u(t) = u_t$. The notation $\mathcal{C}^\alpha_\mathcal{D}(b)$, $0 < b \leq a$, is used for the space

$$\mathcal{C}^\alpha_\mathcal{D}(b) = \{u \in C^\alpha([-r,b];\mathcal{D}) : P_u \in C^\alpha([0,b];\mathcal{C}_\mathcal{D})\}$$

with the norm $\|u\|_{\mathcal{C}^\alpha_\mathcal{D}(b)} = \|P_u\|_{C^\alpha([0,b];\mathcal{C}_\mathcal{D})}$. In addition, $y\colon [-r,a] \to X$ is the function defined by $y_0 = \varphi$ and $y(t) = T(t)\varphi(0)$ for $t \in [0,a]$, Λ_1, Λ_2 are the constants introduced in Lemmas 2.7 and 2.8, respectively, and $g_1\colon [0,r_1] \to X$ is the function defined by $g_1(t) = g(t, \varphi(t - r_1))$.

Lemma 2.10 *Let* $u, v \in \mathcal{C}^\alpha_\mathcal{D}(b)$ *with* $u_0 = v_0$. *Then* $F_u \in C^\alpha([0,b];X)$ *and*

$$\|F_u\|_{C^\alpha(X)} \leq ([|A|]_{C^\alpha(\mathcal{L}(\mathcal{D},X))}(2b^\alpha + 1) + \|f\|_{C^\alpha(\mathcal{L}(\mathcal{C}_\mathcal{D},X))})\|u\|_{\mathcal{C}^\alpha_\mathcal{D}(b)},$$

$$\|F_{(u-v)}\|_{C^\alpha(X)} \leq (2[|A|]_{C^\alpha(\mathcal{L}(\mathcal{D},X))} b^\alpha + \|f\|_{C^\alpha(\mathcal{L}(\mathcal{C}_\mathcal{D},X))}(2b^\alpha + 1))\|u - v\|_{\mathcal{C}^\alpha_\mathcal{D}(b)},$$

where $\|\cdot\|_{C^\alpha(Z)} = \|\cdot\|_{C^\alpha([0,b];Z)}$ *for* $Z = X, \mathcal{L}(\mathcal{D},X)$ *and* $Z = \mathcal{L}(\mathcal{C}_\mathcal{D},X)$.

Proof We only prove the second inequality. Let $w = u - v$. Using that $w = (u - v)|_{[0,b]} \in C^{\alpha}([0,b];D)$ and $w_0 = 0$, for $t, s \in [0,b]$ we find that

$$\begin{aligned}
&\|F_w(t) - F_w(s)\| \\
&\quad \leq \|(A(t) - A(s))\|_{\mathcal{L}(\mathcal{D},X)}\|w(t)\|_{\mathcal{D}} + \|A(s) - A)\|_{\mathcal{L}(\mathcal{D},X)}\|w(t) - w(s)\|_{\mathcal{D}} \\
&\qquad + \|f(t) - f(s)\|_{\mathcal{L}(\mathcal{C}_{\mathcal{D}},X)}\|w_t\|_{\mathcal{C}_{\mathcal{D}}} + \|f(s)\|_{\mathcal{L}(\mathcal{C}_{\mathcal{D}},X)}\|w_t - w_s\|_{\mathcal{C}_{\mathcal{D}}} \\
&\quad \leq \big([|A|]_{C^{\alpha}(\mathcal{L}(\mathcal{D},X))}[|w|]_{C^{\alpha}(\mathcal{D})}t^{\alpha} + [|A|]_{C^{\alpha}(\mathcal{L}(\mathcal{D},X))}s^{\alpha}[|w|]_{C^{\alpha}(\mathcal{D})}\big)(t-s)^{\alpha} \\
&\qquad + \big([|f|]_{C^{\alpha}(\mathcal{L}(\mathcal{C}_{\mathcal{D}},X))}[|P_w|]_{C^{\alpha}(\mathcal{C}_{\mathcal{D}})}b^{\alpha} + \|f\|_{C(\mathcal{L}(\mathcal{C}_{\mathcal{D}},X))}[|P_w|]_{C^{\alpha}(\mathcal{C}_{\mathcal{D}})}\big)(t-s)^{\alpha},
\end{aligned}$$

so that

$$[|F_w|]_{C^{\alpha}(X)} \leq (2[|A|]_{C^{\alpha}(\mathcal{L}(\mathcal{D},X))}b^{\alpha} + \|f\|_{C^{\alpha}(\mathcal{L}(\mathcal{C}_{\mathcal{D}},X))}(b^{\alpha}+1))[|P_w|]_{C^{\alpha}(\mathcal{C}_{\mathcal{D}})}.$$

A similar procedure permits us to prove that

$$\|F_w\|_{C(X)} \leq [|A|]_{C^{\alpha}(\mathcal{L}(\mathcal{D},X))}b^{\alpha}\|P_w\|_{\mathcal{C}_{\mathcal{D}}} + \|f\|_{C(\mathcal{L}(\mathcal{C}_{\mathcal{D}},X))}b^{\alpha}[|P_w|]_{C^{\alpha}(\mathcal{C}_{\mathcal{D}})}.$$

From these estimates we obtain

$$\|F_w\|_{C^{\alpha}(X)} \leq \big(2[|A|]_{C^{\alpha}(\mathcal{L}(\mathcal{D},X))}b^{\alpha} + \|f\|_{C^{\alpha}(\mathcal{L}(\mathcal{C}_{\mathcal{D}},X))}(2b^{\alpha}+1)\big)\|w\|_{\mathcal{C}^{\alpha}_{\mathcal{D}}(b)}. \qquad \blacksquare$$

In the next result, $D_1 g(t,\psi)$ denotes the derivative of $g(t,\psi)$ with respect to t and $D_2 g(t,\psi)$ denotes the Fréchet derivative of $g(t,\psi)$ with respect to ψ. We can now prove the main result of this section.

Theorem 2.11 *Assume $y \in C^{\alpha}_{\mathcal{D}}(a)$, $A\varphi(0) \in (X,\mathcal{D})_{\alpha,\infty}$ and the following conditions hold.*

(i) *$g \in C^1([0,a] \times X, X)$, g is differentiable from $[0,a] \times (X,\mathcal{D})_{\alpha,\infty}$ into $(X,\mathcal{D})_{\alpha,\infty}$ and there are positive constants L^i_g, $i = 1,2$, such that*

$$\|D_i g(t,x) - D_i g(s,y)\|_{\mathcal{L}(X)} \leq L^i_g(|t-s|^{\alpha} + \|x-y\|), \quad s,t \in [0,a], x,y \in X.$$

(ii) *$f \in C([0,a] \times \mathcal{C}_{\mathcal{D}};(X,\mathcal{D})_{\alpha,\infty})$, $g_1 \in C^{1+\alpha}([0,r_1];X) \cap C^1([0,r_1];(X,\mathcal{D})_{\alpha,\infty})$, $f(0)\varphi - \frac{d}{dt}g_1(t)|_{t=0} \in (X,\mathcal{D})_{\alpha,\infty}$ and there is $r_1 < \delta \leq a$ such that*

$$\Lambda_3(\delta)[\Lambda_1 + \Lambda_2]\max\{r_1^{\alpha},1\} < 1,$$

where $\Lambda_3(\delta) = \big(2[|A|]_{C^{\alpha}([0,\delta];\mathcal{L}(\mathcal{D},X))}\delta^{\alpha} + \|f\|_{C^{\alpha}([0,\delta];\mathcal{L}(\mathcal{C}_{\mathcal{D}},X))}(2\delta^{\alpha}+1)\big)$.

Then there exists a classical solution of (2.7)–2.8 *in $C^{\alpha}([0,b];\mathcal{D})$ for some $r_1 < b \leq a$.*

Proof On the space $\mathcal{C}^{\alpha}_{\mathcal{D}}(\varphi,r_1) = \{u \in \mathcal{C}^{\alpha}_{\mathcal{D}}(r_1) : u_0 = \varphi\}$ endowed with the metric $d(u,v) = \|u - v\|_{\mathcal{C}^{\alpha}_{\mathcal{D}}(r_1)}$ we define the map $\Gamma\colon \mathcal{C}^{\alpha}_{\mathcal{D}}(\varphi,r_1) \to \mathcal{C}^{\alpha}_{\mathcal{D}}(\varphi,r_1)$ by $(\Gamma u)_0 = \varphi$ and

$$\Gamma u(t) = T(t)\varphi(0) + \int_0^t T(t-s)\Big[-\frac{d}{ds}g_1(s) + F_u(s)\Big]\,ds, \quad t \in [0,r_1].$$

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From Lemmas 2.7 and 2.10 we have that $\Gamma u \in C^{\alpha}([0,r_1];\mathcal{D})$. In order to prove that Γ is a contraction in $\mathcal{C}^{\alpha}_{\mathcal{D}}(\varphi,r_1)$, we introduce the decomposition $\Gamma u = \Gamma^1 u + y$. For $t \in [0,r_1)$ and $h > 0$ such that $t+h \in [0,r_1]$, we get

$$\begin{aligned}
\|P_{\Gamma u}(t+h) - P_{\Gamma u}(t)\|_{\mathcal{C}_{\mathcal{D}}} &= \|(\Gamma u)_{(t+h)} - (\Gamma u)_t\|_{\mathcal{C}_{\mathcal{D}}} \\
&\le \|(\Gamma u)_h - \varphi\|_{\mathcal{C}_{\mathcal{D}}} + \sup_{s\in[0,r_1]} \|\Gamma u(s+h) - \Gamma u(s)\|_{\mathcal{D}} \\
&\le \|y_h - \varphi\|_{\mathcal{C}_{\mathcal{D}}} + \|(\Gamma^1 u)_h\|_{\mathcal{C}_{\mathcal{D}}} + [|\Gamma u|]_{C^{\alpha}([0,r_1];\mathcal{D})} h^{\alpha} \\
&\le [|P_y|]_{C^{\alpha}([0,r_1];\mathcal{C}_{\mathcal{D}})} h^{\alpha} + \|(\Gamma^1 u)_h\|_{\mathcal{C}_{\mathcal{D}}} + [|\Gamma u|]_{C^{\alpha}([0,r_1];\mathcal{D})} h^{\alpha}.
\end{aligned}$$

To estimate $\|(\Gamma^1 u)_h\|_{\mathcal{C}_{\mathcal{D}}}$ we use Lemma 2.8 with $x = 0$. For $\theta \in [-h,0]$ we see that

$$\begin{aligned}
\|\Gamma^1 u(\theta+h)\|_{\mathcal{D}} \le \Lambda_2 \Big(\Big\| \frac{d}{dt} g_1 \Big\|_{C^{\alpha}([0,r_1];X)} + \|F_u\|_{C^{\alpha}([0,r_1];X)} \Big) h^{\alpha} \\
+ \frac{C^1_{\alpha,\infty}}{\alpha} \|f(0)\varphi - \frac{d}{dt} g_1(t)|_{t=0}\|_{\alpha,\infty} h^{\alpha},
\end{aligned}$$

from which we infer that

$$\begin{aligned}
[|P_{\Gamma u}|]_{C^{\alpha}([0,r_1];\,\mathcal{C}_{\mathcal{D}})} \le [|P_y|]_{C^{\alpha}([0,r_1];\mathcal{C}_{\mathcal{D}})} \\
+ \Lambda_2 \Big(\Big\| \frac{d}{dt} g_1 \Big\|_{C^{\alpha}([0,r_1];X)} + \|F_u\|_{C^{\alpha}([0,r_1];X)} \Big) \\
+ \frac{C^1_{\alpha,\infty}}{\alpha} \Big\| f(0)\varphi + \frac{d}{dt} g_1(t)|_{t=0} \Big\|_{\alpha,\infty} + [|\Gamma u|]_{C^{\alpha}([0,r_1];\mathcal{D})},
\end{aligned}$$

and $\Gamma u \in \mathcal{C}^{\alpha}_{\mathcal{D}}(\varphi,r_1)$. Moreover, from this estimate and Lemmas 2.7 and 2.10, for $u,v \in \mathcal{C}^{\alpha}_{\mathcal{D}}(\varphi,r_1)$ we find that

$$\begin{aligned}
[|P_{\Gamma u} - P_{\Gamma v}|]_{\mathcal{C}^{\alpha}([0,r_1],\,\mathcal{C}_{\mathcal{D}})} &\le \Lambda_2 \|F_{u-v}\|_{C^{\alpha}([0,r_1];X)} + [|\Gamma(u-v)|]_{C^{\alpha}([0,r_1];\mathcal{D})} \\
&\le \Lambda_2 \Lambda_3(\delta) \|u-v\|_{\mathcal{C}^{\alpha}_{\mathcal{D}}(r_1)} + \Lambda_1 [|F_{u-v}|]_{C^{\alpha}([0,r_1];X)} \\
&\le \Lambda_2 \Lambda_3(\delta) \|u-v\|_{\mathcal{C}^{\alpha}_{\mathcal{D}}(r_1)} + \Lambda_1 \Lambda_3(\delta) \|u-v\|_{\mathcal{C}^{\alpha}_{\mathcal{D}}(r_1)},
\end{aligned}$$

and hence,

$$\begin{aligned}
[|P_{\Gamma u} - P_{\Gamma v}|]_{C^{\alpha}([0,r_1];\mathcal{C}_{\mathcal{D}})} &\le \Lambda_3(\delta)\,[\Lambda_2 + \Lambda_1]\,\|u-v\|_{\mathcal{C}^{\alpha}_{\mathcal{D}}(r_1)}, \\
\|P_{\Gamma u} - P_{\Gamma v}\|_{C([0,r_1];\mathcal{C}_{\mathcal{D}})} &\le \Lambda_3(\delta)\,[\Lambda_2 + \Lambda_1]\, r_1^{\alpha} [|P_u - P_v|]_{C^{\alpha}([0,r_1];\mathcal{C}_{\mathcal{D}})},
\end{aligned}$$

since $P_{\Gamma u}(0) = P_{\Gamma v}(0)$. From these estimates it follows that

$$d(\Gamma u, \Gamma v) \le \Lambda_3(\delta)\,[\Lambda_1 + \Lambda_2] \max\{r_1^{\alpha}, 1\} d(u,v),$$

and Γ has a unique fixed point $u^1 \in \mathcal{C}^{\alpha}_{\mathcal{D}}(\varphi,r_1)$.

To continue, we remark on some properties of $u^1(\cdot)$. From Lemma 2.7, $u^1(\cdot)$ is a classical solution of (2.7)–(2.8) in $C^\alpha([0,r_1];\mathcal{D})$, $u^1 \in C^{1+\alpha}([0,r_1];X)$ and $\frac{d}{dt}u^1(t) = A(t)u^1(t) - \frac{d}{dt}g_1(t) + f(t)u_t \in (X,\mathcal{D})_{\alpha,\infty}$ for all $t \in [0,r_1]$. This implies $A(t)u^1(t) \in (X,\mathcal{D})_{\alpha,\infty}$ for all $t \in [0,r_1]$, since $-\frac{d}{dt}g_1(t) + f(t)u_t \in (X,\mathcal{D})_{\alpha,\infty}$ for each $t \in [0,r_1]$. Moreover, if $g_2\colon [r_1,2r_1] \to X$ is the function given by $g_2(t) = g(t,u^1(t-r_1))$, then $f(r_1)(u^1)_{r_1} - \frac{d}{dt}g_2(t)|_{t=r_1} \in (X,\mathcal{D})_{\alpha,\infty}$ and $\frac{d}{dt}g_2 \in C^\alpha([r_1,2r_1];X)$. In addition, if $y^1\colon [r_1-r,a] \to X$ is the function defined by $(y^1)_{r_1} = (u^1)_{r_1}$ and $y^1(t) = T(t-r_1)u^1(r_1)$ for $t \in [r_1,a]$, a straightforward estimation permits us to show that $t \to (y^1)_t \in C^\alpha([r_1,a];\mathcal{C}_{\mathcal{D}})$.

Let $r_1 < b \le \delta$ be such that $\widetilde{\Lambda_3}\,[\Lambda_1 + \Lambda_2]\max\{(b-r_1)^\alpha, 1\} < 1$, where

$$\widetilde{\Lambda_3} = 2[|A|]_{C^\alpha([r_1,b];\mathcal{L}(\mathcal{D},X))}(b-r_1)^\alpha + \|f\|_{C^\alpha([r_1,b];\mathcal{L}(\mathcal{C}_{\mathcal{D}},X))}(2(b-r_1)^\alpha + 1).$$

Considering the above remarks and proceeding as in the first part of this proof, we see that there exists a unique solution $u^2 \in C^\alpha([r_1-r,b];\mathcal{D})$ of the delayed integral equation

$$x(t) = T(t-r_1)u^1(0) + \int_{r_1}^t T(t-s)\Big[-\frac{d}{ds}g_2(s) + F_x(s)\Big]\,ds, t \in [r_1,b],$$
$$x_{r_1} = u^1_{r_1},$$

which from Lemma 2.7, is a classical solution in $C^\alpha([r_1,b];\mathcal{D})$ of the neutral system

$$\frac{d}{dt}[x(t) + g(t,x(t-r_1))] = A(t)x(t) + f(t)x_t, \quad t \in [r_1,b],$$
$$x_{r_1} = u^1_{r_1}.$$

Finally, by defining $u\colon [-r,b] \to X$ by $u(t) = u^1(t)$ for $t \in [-r,r_1]$ and $u(t) = u^2(t)$ for $t \in [r_1,b]$, we obtain a classical solution of (2.7)–(2.8) in $C^\alpha([0,b];\mathcal{D})$. ■

3 Applications

In this section we study the existence of solutions for a concrete partial differential equation. Consider the neutral differential system

$$(3.1)\quad \frac{\partial}{\partial t}\,(\,u(t,\xi) + \beta_1(t)u(t-r_1,\xi)\,) = \gamma(t)\frac{\partial^2}{\partial\xi^2}u(t,\xi) + \sum_{i=2}^m \beta_i(t)u(t-r_i,\xi) + \alpha(t)\int_{t-r}^t \eta(s-t)u(s,\xi)\,ds,$$

$$(3.2)\qquad u(t,0) = u(t,\pi) = 0, \quad t \in [0,a],$$

$$(3.3)\qquad u(s,\xi) = \varphi(s,\xi), \qquad s \in [-r,0],\ \xi \in [0,\pi],$$

for $(t,\xi) \in [0,a]\times[0,\pi]$. For simplicity, we assume $\alpha,\beta_i \in C([0,a];\mathbb{R})$, $\eta \in L^2([0,a];\mathbb{R})$, $\gamma \in C^\alpha([0,a];(0,\infty))$, and $0 < r_i \le r \le a$ for all $i = 1,\dots,m$.

To treat this system we consider the space $X = L^2([0,\pi])$ and the linear operators $A, A(s)\colon \mathcal{D} \subset X \to X$ defined by $Ax = x''$ and $A(s)x = \gamma(s)Ax$ on the domain $\mathcal{D} = \{x \in X : x'' \in X, x(0) = x(\pi) = 0\}$. The operator A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on X, and A has a discrete spectrum with eigenvalues of the form $-n^2, n \in \mathbb{N}$ with corresponding normalized eigenfunctions given by $z_n(\xi) := \left(\frac{2}{\pi}\right)^{1/2}\sin(n\xi)$. The set of functions $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis for X, $T(t)x = \sum_{n=1}^{\infty} e^{-n^2 t}\langle x, z_n\rangle z_n$ for $x \in X$ and $Ax = -\sum_{n=1}^{\infty} n^2\langle x, z_n\rangle z_n$ for $x \in D(A)$. Moreover, it us possible to define fractional powers of A; see [27, Chapter 2]. In particular, for $x \in X$ and $\alpha > 0$, $(-A)^{-\alpha}x = \sum_{n=1}^{\infty} n^{-2\alpha}\langle x, z_n\rangle z_n$ and $(-A)^{\alpha}\colon D((-A)^{\alpha}) \subseteq X \to X$ is given by $(-A)^{\alpha}x = \sum_{n=1}^{\infty} n^{2\alpha}\langle x, z_n\rangle z_n$, for

$$x \in D((-A)^{\alpha}) = \Big\{x \in X : \sum_{n=1}^{\infty} n^{2\alpha}\langle x, z_n\rangle z_n \in X\Big\}.$$

To represent the system (3.1)–(3.3) in the abstract form (2.7)–(2.8), we introduce the functions $g\colon [0,a] \times X \to X$ and $f\colon [0,a] \to \mathcal{L}(\mathcal{C}, X)$ defined by $g(t,x)(\xi) = \beta_1(t)x(\xi)$ and

$$[f(t)\psi](\xi) = \sum_{i=2}^{m} \beta_i(t)\psi(-r_i,\xi) + \alpha(t)\int_{-r}^{0} \eta(s)\psi(s,\xi)\,ds.$$

It is easy to see that $f \in C([0,a];\mathcal{L}(\mathcal{C}_{\mathcal{D}}, X))$, $g \in C([0,b] \times X, X)$, $g(t,\cdot)$ is a bounded linear operator for all $t \in [0,a]$, $\|g\|_{C([0,b]\times X,X)} \leq \|\beta_1\|_{C([0,a]),\mathbb{R})}$, and

$$\|f\|_{C([0,b];\mathcal{L}(\mathcal{C}_{\mathcal{D}},X))} \leq \sum_{i=2}^{m} \|\beta_i\|_{C([0,b];\mathbb{R})} + \|\alpha\|_{C([0,b];\mathbb{R})}^{1/2} r^{1/2}\|\eta\|_{L^2([0,b];\mathbb{R})},$$

for all $0 < b \leq a$. Moreover, $A(\cdot) \in C^{\alpha}([0,a];\mathcal{L}(\mathcal{D}, X))$ and $[|A|]_{C^{\alpha}([0,b];\mathcal{L}(\mathcal{D},X))} \leq \|\gamma\|_{C^{\alpha}([0,b];\mathbb{R})}$ for all $0 < b \leq a$.

In what follows, we say that a function $u \in C([-r,b];X)$, $b > 0$, is a mild solution (resp. a classical solution in $C^{\alpha}([0,b];X)$) of (3.1)–(3.3) if u is a mild solution (resp. a classical solution in $C^{\alpha}([0,b];X)$) of the associated abstract neutral system (1.1)–(1.2).

Proposition 3.1 *Assume $\varphi \in C([-r,0];[D(A^{2n})])$ for some $n \in \mathbb{N}$. Then there exists a mild solution of* (3.1)–(3.3) *in $C([-r, nr_1 \wedge a];X)$.*

Proof The assertion follows directly from Theorem 2.3 and Remark 2.5 by considering the spaces $Y_i = [D(A^i)]$ and $Z_j = [D(-A)^{j+\gamma_j}]$ for $i = 1,\dots,2n$ and $j \in \mathbb{N}$ and $\gamma_j \in (0,1)$. We omit the additional details.

To finish our paper we consider the problem of the existence of classical solutions. In the next result, y, $g_1(\cdot)$, Λ_1, and Λ_2 are as in Remark 2.9. In addition to the above assumptions, we will assume that $\beta_1 \in C^{1+\alpha}([0,a];\mathbb{R})$, $\alpha, \beta_i \in C^{\alpha}([0,a];\mathbb{R})$, $i = 2,\dots,m$. If $[D(A)] \hookrightarrow Z \hookrightarrow X$ and the above conditions are verified, it is easy to see that $f \in C^{\alpha}([0,a];\mathcal{L}(\mathcal{C}_{\mathcal{D}}, Z))$, $g \in C^1([0,a] \times Z, Z)$,

$$\|Dg(t,x) - Dg(s,y)\|_{\mathcal{L}(X)} \leq \|\beta_1'\|_{C^{\alpha}([0,b];\mathbb{R})}(|t-s|^{\alpha} + \|x-y\|_{\mathcal{D}}),$$

for all $s, t \in [0, b]$, $0 < b \leq a$, and every $x, y \in \mathcal{D}$, and

$$\begin{aligned}&\|f\|_{C^\alpha([0,b];\mathcal{L}(\mathcal{C}_{\mathcal{D},Z}))}\\ &\leq \Theta(b) = \sum_{i=1}^{m} \|\beta_i\|_{C^\alpha([0,b];\mathbb{R})} + \left[\|\alpha\|_{C([0,b];\mathbb{R})}^{1/2} + \|\alpha\|_{C^\alpha([0,b];\mathbb{R})}^{\frac{1}{2}}\right] r^{1/2} \|\eta\|_{L^2([0,b];\mathbb{R})}\end{aligned}$$

for all $0 < b \leq a$. ∎

The next result follows directly from Theorem 2.11.

Proposition 3.2 *Assume $\varphi \in C^{1+\alpha}([-r, 0]; (X, \mathcal{D})_{\alpha,\infty})$, $A\varphi(0) \in (X, \mathcal{D})_{\alpha,\infty}$, $\beta_1 \in C^{1+\alpha}([0, a]; \mathbb{R})$, $\alpha, \beta_i \in C^\alpha([0, a]; \mathbb{R})$, $i = 2, \ldots, m$, $y \in \mathcal{C}^\alpha_{\mathcal{D}}(a)$ and there exists $\delta > r_1$ such that*

$$[2\|\gamma\|_{C^\alpha([0,\delta])}\delta^\alpha + \Theta(\delta)(2\delta^\alpha + 1)][\Lambda_1 + \Lambda_2]\{r_1^\alpha, 1\} < 1.$$

Then there exists a classical solution u of (3.1)–(3.3) in $C^\alpha([0, b]; \mathcal{D})$ for some $r_1 < b \leq a$. Equivalently, there exits $u \in C^\alpha([0, b]; W^2([0, \pi]) \cap W_0^1([0, \pi])) \cap C([-r, b]; X)$, such that u verifies (3.1)–(3.2) a.e. for $(t, \xi) \in [0, b] \times [0, \pi]$ and u satisfies (3.3) a.e. for $(s, \xi) \in [-r, 0] \times [0, \pi]$.

References

[1] M. Adimy and K. Ezzinbi, *A class of linear partial neutral functional-differential equations with nondense domain.* J. Differential Equations **147**(1998), no. 2, 285–332. http://dx.doi.org/10.1006/jdeq.1998.3446

[2] K. Balachandran, G. Shija, and J. H. Kim, *Existence of solutions of nonlinear abstract neutral integrodifferential equations.* Comput. Math. Appl. **48**(2004), no. 10-11, 1403–1414. http://dx.doi.org/10.1016/j.camwa.2004.08.002

[3] K. Balachandran and R. Sakthivel, *Existence of solutions of neutral functional integrodifferential equation in Banach spaces.* Proc. Indian Acad. Sci. Math. Sci. **109**(1999), no. 3, 325–332. http://dx.doi.org/10.1007/BF02843536

[4] M. Benchohra and S. K. Ntouyas, *Nonlocal Cauchy problems for neutral functional differential and integrodifferential inclusions in Banach spaces.* J. Math. Anal. Appl. **258**(2001), no. 2, 573–590. http://dx.doi.org/10.1006/jmaa.2000.7394

[5] M. Benchohra, J. Henderson, and S. K. Ntouyas, *Existence results for impulsive multivalued semilinear neutral functional differential inclusions in Banach spaces.* J. Math. Anal. Appl. **263**(2001), no. 2, 763–780. http://dx.doi.org/10.1006/jmaa.2001.7663

[6] P. Cannarsa and D. Sforza, *Global solutions of abstract semilinear parabolic equations with memory terms.* NoDEA Nonlinear Differential Equations. Appl. **10**(2003), no. 4, 399–430. http://dx.doi.org/10.1007/s00030-003-1004-2

[7] P. Clément and J. A. Nohel, *Asymptotic behavior of solutions of nonlinear Volterra equations with completely positive kernels.* SIAM J. Math. Anal. **12**(1981), no. 4, 514–535. http://dx.doi.org/10.1137/0512045

[8] P. Clément and J. Prüss, *Global existence for a semilinear parabolic Volterra equation.* Math. Z. **209**(1992), no. 1, 17–26. http://dx.doi.org/10.1007/BF02570816

[9] R. Datko, *Linear autonomous neutral differential equations in a Banach space.* J. Diff. Equations **25**(1977), no. 2, 258–274. http://dx.doi.org/10.1016/0022-0396(77)90204-2

[10] J. P. Dauer and K. Balachandran, *Existence of solutions of nonlinear neutral integrodifferential equations in Banach spaces.* J. Math. Anal. Appl. **251**(2000), no. 1, 93–105. http://dx.doi.org/10.1006/jmaa.2000.7022

[11] H. Fang and J. Li, *On the existence of periodic solutions of a neutral delay model of single-species population growth.* J. Math. Anal. Appl. **259**(2001), no. 1, 8–17. http://dx.doi.org/10.1006/jmaa.2000.7340

[12] H. I. Freedman and Y. Kuang, *Some global qualitative analyses of a single species neutral delay differential population model.* Rocky Mountain J. Math. **25**(1995), no. 1, 201–215. http://dx.doi.org/10.1216/rmjm/1181072278

[13] X. Fu and X. Liu, *Existence of periodic solutions for abstract neutral non-autonomous equations with infinite delay.* J. Math. Anal. Appl. **325**(2007), no. 1, 249–267. http://dx.doi.org/10.1016/j.jmaa.2006.01.048

[14] A. Granas. and J. Dugundji, *Fixed Point Theory.* Springer-Verlag, New York, 2003.

[15] M. E. Gurtin and A. C. Pipkin, *A general theory of heat conduction with finite wave speed.* Arch. Rational Mech. Anal. **31**(1968), no. 2, 113–126. http://dx.doi.org/10.1007/BF00281373

[16] J. K. Hale *Partial neutral functional-differential equations.* Rev. Roumaine Math. Pures Appl. **39**(1994), no. 4, 339–344.

[17] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional-Differential Equations.* Applied Mathematical Sciences 99. Springer-Verlag, New York, 1993.

[18] H. R. Henríquez, *Periodic solutions of abstract neutral functional differential equations with infinite delay.* Acta Math. Hungar. **121**(2008), no. 3, 203–227. http://dx.doi.org/10.1007/s10474-008-7009-x

[19] E. Hernández, *Existence results for partial neutral integrodifferential equations with unbounded delay.* J. Math. Anal. Appl. **292**(2004), no. 1, 194–210. http://dx.doi.org/10.1016/j.jmaa.2003.11.052

[20] E. Hernández and D. O'Regan, *Existence results for abstract partial neutral differential equations.* To appear in Proc. Amer. Math. Soc.

[21] E. Hernández and M. McKibben, *Some comments on: "Existence of solutions of abstract nonlinear second-order neutral functional integrodifferential equations."* [Comput. Math. Appl. 46 (2003), no. 8-9] Comput. Math. Appl. **50**(2005), no. 5-6, 655–669. http://dx.doi.org/10.1016/j.camwa.2005.08.001

[22] E. Hernández and H. Henríquez, *Existence results for partial neutral functional-differential equation with unbounded delay.* J. Math. Anal. Appl. **221**(1998), no. 2, 452–475. http://dx.doi.org/10.1006/jmaa.1997.5875

[23] ———, *Existence of periodic solution of partial neutral functional-differential equation with unbounded delay.* J. Math. Anal. Appl. **221**(1998) no. 2, 499–522. http://dx.doi.org/10.1006/jmaa.1997.5899

[24] Y. Kuang, *Qualitative analysis of one- or two-species neutral delay population models.* SIAM J. Math. Anal. **23**(1992), no. 1, 181–200. http://dx.doi.org/10.1137/0523009

[25] Q. Li, J. Cao, and S. Wan, *Positive periodic solution for a neutral delay model in population.* J. Biomath. **13**(1998), no. 4, 435–438.

[26] A. Lunardi, *On the linear heat equation with fading memory.* SIAM J. Math. Anal. **21**(1990), no. 5, 1213–1224. http://dx.doi.org/10.1137/0521066

[27] ———, *Analytic semigroups and optimal regularity in parabolic problems.* Progress in Nonlinear Differential Equations and their Applications 16. Birkhäauser Verlag, Basel, 1995.

[28] J. W. Nunziato, *On heat conduction in materials with memory.* Quart. Appl. Math. **29**(1971), 187–204.

[29] J. Wu and H. Xia, *Self-sustained oscillations in a ring array of coupled lossless transmission lines.* J. Differential Equations **124** (1996), no. 1, 247–278. http://dx.doi.org/10.1006/jdeq.1996.0009

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