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TESSELATIONS OF S^2 AND EQUATIONS OVER TORSION-FREE GROUPS

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Let G be a torsion free group, F the free group generated by t. The equation $r(t)=1$ is said to have a solution over G if there is a solution in some group that contains G. In this paper we generalize a result due to Klyachko who established the solution when the exponent sum of t is one.

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0. Labelled patterns

Consider the following definitions.

A pattern P is a directed tree embedded in R^2 with a specified vertex, called the centre, which is adjacent to each edge. If we reverse the orientation of each edge of the pattern P and reflect this directed tree in R^2, we obtain a new pattern denoted $\bar{P}$.

If we label some of the corners at the centre of P with distinct positive labels taken from some alphabet X, we get a labelled pattern denoted P_X. The inverse labelled pattern, $\bar{P}_X$ is obtained from $\bar{P}$ by labelling the corner corresponding to the corner of P_X labelled α with the label $\bar{\alpha}$.

A directed graph Γ embedded in S^2 is said to be a P-graph if each vertex v of Γ has the same degree and looks locally like the centre of P (in which case we call v a positive vertex) or of $\bar{P}$ (whence v is a negative vertex) with respect to the direction of its incident edges.

If, in addition, certain corners of Γ are labelled with elements of $X \cup \bar{X}$ so that the vertices look like the labelled pattern P_X or $\bar{P}_X$, we call Γ a P_X-graph. See for instance Figure 1.

Lemma 1. *Let $m, n \geqq 1$, and let $P_{\{a,b\}}$ and $\bar{P}_{\{a,b\}}$ be the patterns depicted in Figure 2. If Γ is a $P_{\{a,b\}}$-graph, then there exist at least two regions of Γ all of whose corners are labelled with the same letter up to exponent.*

Note we do not assume any diagrams are reduced in the sense of Sieradski [7].

Proof. Stallings in [8] shows that there must be at least two regions of Γ whose boundaries are consistently oriented, i.e. as one traverses the boundary of these regions

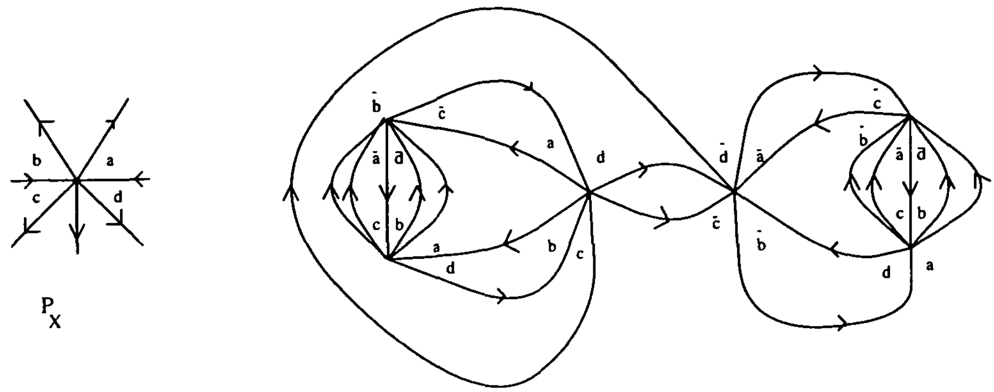

FIGURE 1

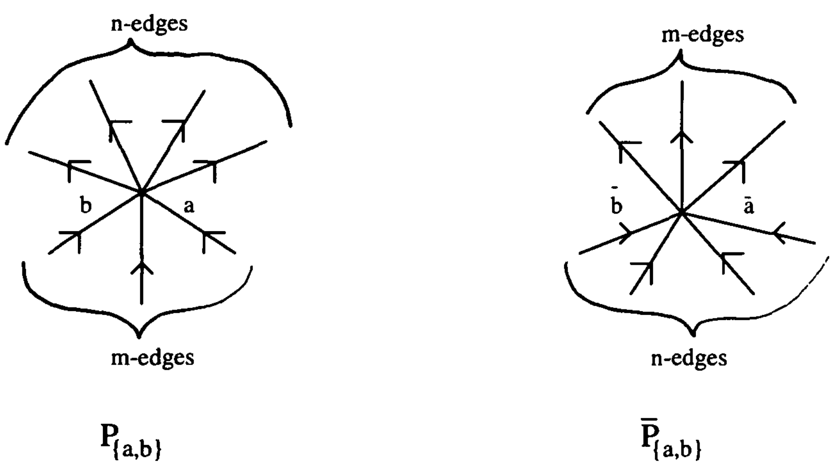

FIGURE 2

in Γ the orientation of the edges always agree or disagree with the motion. The result follows.

Let Γ be a P_X-graph. If D is a region of Γ whose boundary is consistently oriented and all of whose corners are labelled with the same label up to exponent, then D is called a consistent region. If P_X is a labelled pattern so that every P_X-graph Γ has at least two consistent regions, then P_X is said to be of type K. For instance, Figure 1 shows a pattern that is not of type K.

In [5], Klyachko has shown that a labelled pattern of the form depicted in Figure 3 is of type K.

This tesselation result was an important step in settling the Kervaire conjecture for torsion-free groups. In this paper, we prove that a larger class of labelled patterns is of type K. This result enables us, using the techniques of Howie [4], to exhibit a large

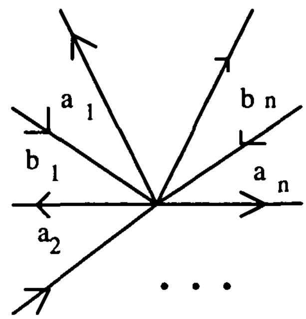

FIGURE 3

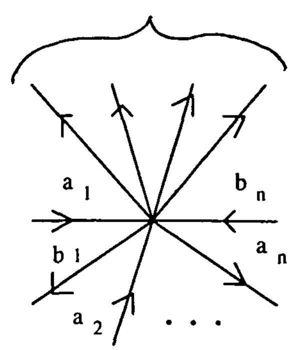

FIGURE 4

class of equations with torsion-free coefficients which are solvable. Combinatorial results such as this were anticipated by Stallings in [**8**, p. 147]. There seem to be two techniques for proving facts concerning graphs embedded on S^2; curvature (weight tests) (see for example [**1**, **2**, **3**, **7**, **8**]), and minimal circle techniques (see [**5**, **6**]). As it is a generalization of the proof of Klyachko, our proof uses the latter.

1. P_X^m

Main Lemma. *Let* $m \geqq 2$, $n \geqq 1$, $X = \{a_1, b_1, \ldots, a_n, b_n\}$ *and let* P_X^m *be the pattern depicted in Figure* 4. *Then* P_X^m *is of type K.*

Proof. Let Γ be a P_X^m-graph. We will add a new set of edges E to Γ on S^2 as follows. Let R be a region of Γ whose boundary is not consistently oriented. We pair the corners of R which are sources and sinks so that an edge runs from each source corner to the sink corner to which it has been paired. We do this in such a way as to keep the added edges from intersection (see Figure 5).

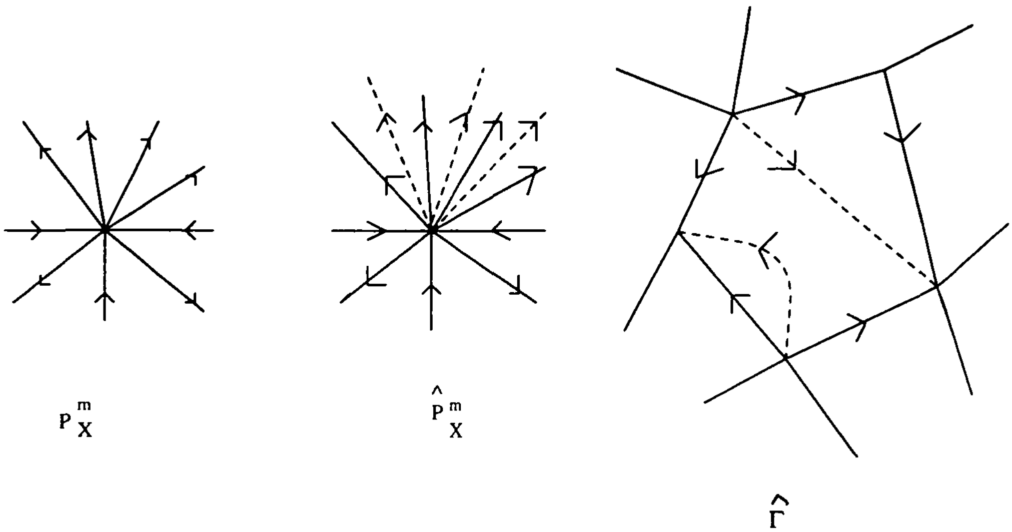

FIGURE 5

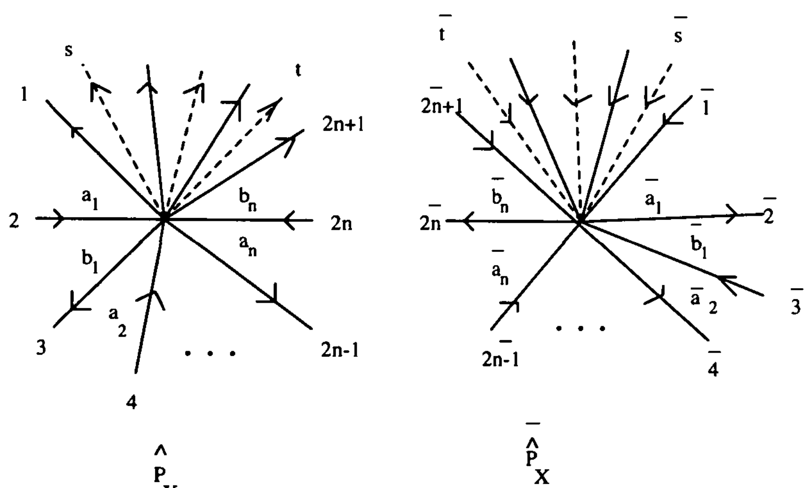

FIGURE 6

We now have a $\hat{P}_X$-graph $\hat{\Gamma}$ where $\hat{P}_X$ is the pattern depicted in Figure 5.

It is clear that each new added edge connects a positive vertex to a negative vertex. We now label the germs of some of the edges of $\hat{\Gamma}$ to correspond to the following labelling of $\hat{P}_X$ as in Figure 6. A germ of an edge is a "small" interval contained in the edge, one of whose endpoints is either the initial or terminal vertex of the edge. Thus each edge has two disjoint germs.

Note if $m=2$ then $s=t$.

Let $u,v\in\{s,t,1,2,\dots,2n+1\}$. Then a $(u,\bar{v})$-path p is a simple closed path in $\hat{\Gamma}$ so that as one travels around the path one leaves positive vertices on the germ labelled u and negative vertices on the germ labelled $\bar{v}$.

Let A be the following set of ordered pairs:

$$A=\{(2,\bar{1}),(1,\bar{2}),(2n,2\bar{n}+1),(2n+1,2\bar{n}),(i,\bar{s}),(t,\bar{j}):\ 2\leqq i,\ j\leqq 2n\}$$

We call p an A-path if p is a $(u,\bar{v})$-path for some $(u,\bar{v})$ in A.

An acceptable path is a pair (p,D) where p is an A-path, and D is a disk on S^2 whose topological boundary is p. If (p,D) and (q,E) are acceptable paths, then we say that (p,D) strictly contains (q,E) if D strictly contains E. We say (p,D) is minimal if it strictly contains no other acceptable paths. If (p,D) is an acceptable path so that p is clockwise (resp. counter-clockwise) with respect to D, then we say that (p,D) is clockwise (resp. counter-clockwise).

Forcing Lemma. *Let Γ be a P-graph and let D be a disk on S^2 whose boundary is a simple closed path p in Γ. Assume that at each positive vertex on p, the germ labelled i lies either in D or on the boundary of D, and that at each negative vertex on p, the germ labelled j lies in D or on the boundary of D. Furthermore, assume there is a vertex v on p so that the appropriate germ at v lies in the interior of D. Then, if $(i,\bar{j})\in A$, there is an acceptable path (q,E) strictly contained in (p,D).*

Proof. We shall construct q as follows. Start at the vertex v. Now, follow the germ labelled i or j as appropriate into the interior of D. At each vertex, leave on the appropriate germ. The assumptions on the germs of the vertices on p assure us that whenever we leave from a vertex on the boundary of D, we do not leave D.

If $i=1$ or $2n+1$ (respectively $j=\bar{1}$ or $2n\mp 1$) then both i and $\bar{j}$ point away from (resp. in toward) the adjacent vertex. If e is an edge of $\hat{\Gamma}$ with a germ labelled s, t, $\bar{s}$, or $\bar{t}$, then e is one of the edges that were added to Γ to make $\hat{\Gamma}$. If e has a germ labelled with a number, then e is an edge of Γ. This assures us that in constructing q, we never leave a vertex on the edge with which we entered that vertex.

Since $\hat{\Gamma}$ is a finite graph, eventually, we shall arrive at a vertex which we have already visited, thus completing a simple closed path, q, which is the A-path for which we were looking. Now q bounds two disks, one of which, E, is strictly contained in D. So, (q,E) is an acceptable path strictly contained in (p,D).

Lemma. *If $\hat{\Gamma}$ is connected, (p,D) is an acceptable path, and D is a region of Γ, then D is a consistent region of Γ.*

Proof. Notice that if p is an $(i,\bar{s})$ path and D is a region of $\hat{\Gamma}$, then p does not contain any negative vertices. Similarly, if p is a $(t,\bar{j})$ path, then p does not contain any positive vertices. The result follows.

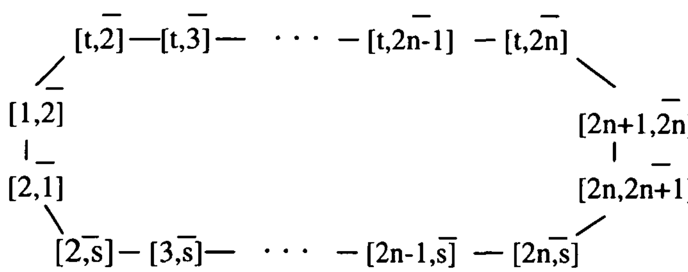

FIGURE 7

Let p be any A-path, thus p bounds two discs D and E. Both (p,D) and (p,E) are acceptable paths and thus each contain minimal acceptable paths, which are distinct, hence the previous lemma reduces the main lemma to the following.

Lemma. *If (p,D) is an acceptable path, and D is not a region, then (p,D) is not minimal.*

Proof. Let (p,D) be an acceptable path with D not a region. Consider the cyclic pattern of elements of A as depicted in Figure 7.

It is our contention that if (p,D) is a clockwise (resp. counter-clockwise) acceptable path with p an $(i,\bar{j})$-path, then (p,D) strictly contains an $(i_1,\bar{j}_1)$-path, where $(i_1,\bar{j}_1)$ is directly clockwise (resp. counter-clockwise) of $(i,\bar{j})$ in the above pattern. (Here, we consider a clockwise $(2,\bar{j})$-path that has only positive vertices to be a $(2,\bar{1})$-path, and a counter-clockwise $(2,\bar{j})$-path that has only positive vertices to be a $(2,\bar{s})$-path. Similar conventions are taken for $(2n,\bar{j})$, $(i,\bar{2})$ and $(i,\overline{2n})$-paths.

For example, let p be a clockwise $(1,\bar{2})$-path. Let v be a positive vertex on p. Now, since every germ labelled either 1 or $\bar{2}$ points away from the corresponding vertex, it is clear that the germ on which p arrives at v is labelled $2k$ for some $1 \leqq k \leqq n$ (i.e. this germ points into v). Since p is clockwise, the germ labelled t at v must lie in the interior of D. Since this is true for all positive vertices on p, the Forcing Lemma implies that (p,D) contains a $(t,\bar{2})$ path q. This path, and the disk it bounds in D make up an acceptable path strictly contained by (p, D).

The similar arguments that complete the proof are left to the reader. This ends the proof of the Main Lemma.

2. The Magnus derivative

If P_X is a labelled pattern, then a corner of P_X which is neither a source nor a sink corner will be referred to as a neutral corner. In what follows we shall only consider labelled patterns that have labels assigned to each of their neutral corners.

If P_X is a pattern with centre vertex v, define $\sigma(P_X) = outdeg(v) - indeg(v)$. Then, the Magnus Derivative of P_X, denoted P'_X, is obtained by adding an edge pointing out of v

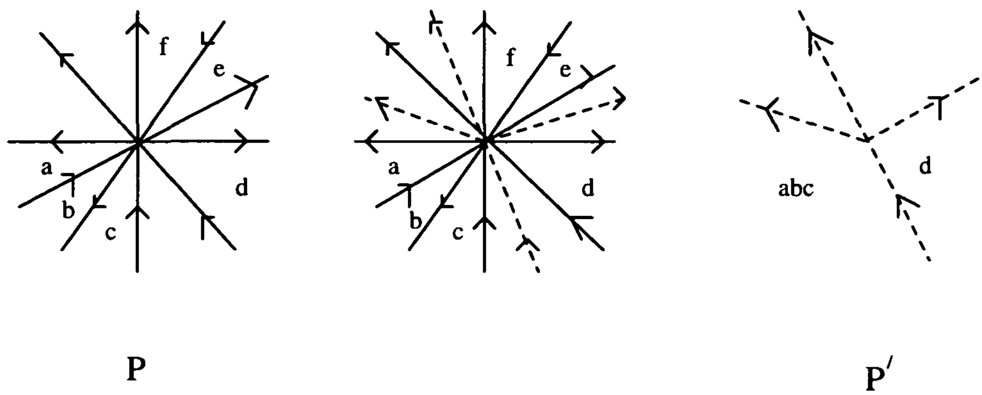

FIGURE 8

in each source corner, and an edges pointing in towards v in each sink corner, and then removing all of the original edge of P_X. Each neutral corner of P'_X contained a non-empty set of neutral corners of P_X, and is labelled as a word in the free semigroup with basis the alphabet X. This word is obtained by taking the product of the labels read counterclockwise. We observe that since all neutral corners in P_X are labelled then all neutral corners in P'_X are also labelled. The labels for the corners of P' come from the free semigroup with basis X not from X. We have chosen to use the same subscript in the interest of brevity. There should not be any confusion in this technical point. It is clear that $\bar{P}'_X=(\bar{P}_X)'$, and that $\sigma(P'_X)=\sigma(P_X)$.

If Γ is a P_X-graph, we can obtain Γ', a P'_X-graph, as follows:

Let R be a region of Γ whose boundary is not consistently oriented. We pair the corners of R which are sources and sinks and add edges interior to R so that an edge runs from each source corner to the sink corner with which it has been paired. We do this in such a way as to keep the added edges from intersecting. Now, by labelling the corners appropriately, the collection of added edges forms Γ' (see Figure 8).

Theorem. *If P'_X is of type K, then P_X is of type K.*

Proof. Let Γ be a P_X-graph. We may assume that Γ has no regions that are loops. Construct a P'_X- graph Γ' as above, superimposed over Γ. Since Γ' is of type K, we need to show that any consistent region of Γ' contains a consistent region of Γ.

Let D be a consistent region of Γ'. Then for some fixed t, each corner of D contains $2t$ Γ-germs, alternately pointing in towards and out from the adjacent vertex, and $2t-1$ Γ-corners. The innermost of these Γ-corners has the same label (up to exponent) at each Γ'-corner of D. Since there are no loops in Γ, it is an easy exercise to show that there must be a consistent Γ-region with this label.

Recursively define $P_X^{(1)}=P'_X$ and $P_X^{(k+1)}=(P_X^{(k)})'$.

Corollary. *If P_X is a pattern so that for some k, $P_X^{(k)}$ is of the form described in the Main Lemma, then P_X is of type K.*

3. Applications to equations over groups

Let $e=a_1t^{m_1}a_2t^{m_2}\dots a_kt^{m_k}$ be an equation in the variable t with coefficients $S=\{a_i: 1\leqq i\leqq k\}$. We will associate to e the following labelled pattern P_e with centre v and labelled with elements of S. We will consider edges leaving v to have a positive direction, and edges entering v to have a negative direction. The labelled pattern P_e has $|m_1|+|m_2|+\cdots+|m_k|$ edges directed so that as one circles v in a counter-clockwise direction starting at the corner labelled a_1, one encounters $|m_1|$ edges directed with the sign of m_1; then the corner labelled a_2; then $|m_2|$ edges directed with the sign of m_2; then the corner labelled a_3; continuing until one reaches the $|m_k|$ edges directed with the sign of m_k, and then returns to the corner labelled a_1. For example, if $e=at^2bt^{-1}ct^3dt^{-1}$, then P_e is shown in Figure 1. If we define $\bar{e}=\bar{a}_1t^{-m_k}\bar{a}_kt^{-m_{k-1}}\dots\bar{a}_2t^{-m_1}$, then $P_{\bar{e}}=\bar{P}_e$.

Similarly, if P_X is a labelled pattern, then we define e_P to be that equation with coefficients from X so that $P_{e_P}=P_X$. (e_P is defined up to cyclic conjugation.) If e is an equation, we define e', the derivative of e to be $e_{P'}$ where $P'=(P_e)'$. Recursively, $e^{(k+1)}=(e^k)'$.

Let G be a group. An assignment of the coefficients S to the group G is a function $\alpha: S\to G$. We use the convention $\alpha(a_i)=g_i$. We call an assignment proper if for all $i(\text{mod}\,k)$, $m_{i-1}m_i<0$ implies $g_i\neq 1$. We may consider e as an equation over G i.e. $\hat{e}=g_1t^{m_1}g_2t^{m_2}\dots g_kt^{m_k}$ is an element of $G*\langle t\rangle$ where $\langle t\rangle$ is the infinite cyclic group generated by t.

We say that $\hat{e}$ is solvable over G if the natural homomorphism $\phi: G\to G*\langle t\rangle/\hat{e}$ is an inclusion, and that e is solvable over G if $\hat{e}$ is solvable over G for any proper assignment of the coefficients of e. We say that e is of type K if for every group G, and every proper assignment $\alpha: S\to G$ such that for all i, g_i has infinite order, $\hat{e}$ is solvable over G. In particular, if e is of type K and G is torsion free, then e is solvable over G.

If $\alpha: S\to G$ is a proper assignment, and Γ is a P_e-graph, then a region R of Γ is singular if the labels of R being read counter-clockwise yield a relation of G (via α). The following lemma is just the dual situation to a well-known result of Howie [4], and shall be stated without proof.

Lemma. *Let $e=a_1t^{m_1}a_2t^{m_2}\dots a_kt^{m_k}$ be an equation in the variable t with coefficients $S=\{a_i: 1\leqq i\leqq k\}$. Let G be a group, and $\alpha: S\to G$ be a proper assignment. Then if $\hat{e}$ is not solvable over G, there exists a P_e-graph Γ and a region R_0 of Γ so that each region $R\neq R_0$ of Γ is singular, but R_0 is not singular.*

Corollary. *If P_e is of type K, then e is of type K.*

Proof. Assume e is not of type K. Then there is a group G and a proper assignment $\alpha: S\to G$ so that each g_i is of infinite order, and so that $\hat{e}$ is not solvable over G.

Then let Γ be the P_e-graph described by the previous lemma. We may assume that Γ is minimal with respect to the number of its vertices. Since P_e is of type K, Γ has at least one region R which is both singular and consistently labelled with some label a_i. If R had both positive and negative occurrences of a_i, then Γ could be reduced using standard methods (c.f. [7]). Since R is singular, it follows that g_i has finite order. This contradiction proves the corollary.

The following theorem is an immediate consequence of this corollary and the Main Lemma.

Theorem. *The equation* $e=(\prod_{i=1}^{n} a_i t^{-1} b_i t)(\prod_{j=1}^{m-1} c_j t)$ *is solvable over the torsion-free group* G *for* $m \geqq 2$ *and* $n \geqq 1$.

Corollary. *If* e *is an equation so that some derivative of* e *has the form described in the above theorem, then* e *is solvable over torsion free groups.*

Assume e has exponent sum one in t. If there is only one occurrence of t in e, then for any group G, and any assignment α, the natural homomorphism $\phi: G \to G * \langle t \rangle / \hat{e}$ is an isomorphism, so e is solvable over G. So we will assume that t occurs more than once in e.

In this case, it is clear that if one takes repeated derivatives of P_e, eventually one will reach a labelled pattern with exactly one edge, i.e. $deg((P_e)^n)=1$ for some n. If n is minimal in this regard, then $(P_e)^{n-1}$ will be of the form shown in Figure 3.

This proves the following corollary:

Corollary. *If* $\sigma(P_X)=1$ *and* $deg(P_X)>1$, *then* P_X *is of type K.*

We are now in a position to recover the result of Klyachko.

Corollary. *If the exponent sum of* t *in* e *is one, and* G *is torsion-free, then* e *is solvable over* G.

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