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INTEGRAL MEANS OF FUNCTIONS WITH POSITIVE REAL PART

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1. We denote by $\mathscr{P}$ the class of functions of the form

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$$

that are regular in $\Delta = \{z : |z| < 1\}$ and satisfy $\operatorname{Re} h(z) > 0$ there. For $0 \leqq r < 1$, we write

$$I_p(r) = I_p(r, h) = \frac{1}{2\pi} \int_0^{2\pi} |h(re^{i\theta})|^p \, d\theta, \quad (p > 0),$$

$$I(r) = I_1(r),$$

$$A(r) = A(r, h) = \sup\{\operatorname{Re} h(z) : |z| = r\},$$

$$M(r) = M(r, h) = \sup\{|h(z)| : |z| = r\}.$$

We note that, for $h \in \mathscr{P}$, the inequality

$$M(r) \leqq \frac{1+r}{1-r}$$

is classical.

Let now $h \in \mathscr{P}$ and write $h(z) = u(r, \theta) + iv(r, \theta)$ for $z = re^{i\theta} \in \Delta$. Then

$$I(r) \leqq \frac{1}{2\pi} \int_0^{2\pi} u(r, \theta) d\theta + \frac{1}{2\pi} \int_0^{2\pi} |v(r, \theta)| d\theta$$

$$= 1 + \frac{1}{2\pi} \int_0^{2\pi} |v(r, \theta)| d\theta$$

by the normalization $h(0) = 1$. Furthermore, by Zygmund's theorem [**1**, p. 58],

$$\int_0^{2\pi} |v(r, \theta)| d\theta \leqq \int_0^{2\pi} u(r, \theta) \log^+ u(r, \theta) d\theta + 6\pi e$$

$$\leqq 2\pi \log A(r) + 6\pi e,$$

since $\log^+ u(r, \theta) = \max\{\log u(r, \theta), 0\} \leqq \log A(r)$, as $A(r) \geqq 1$. We have thus proved the first part of our opening theorem.

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THEOREM 1. *Let* $h \in \mathscr{P}$ *and* $0 \leqq r < 1$. *Then*

$$I(r) \leqq \log A(r) + B,$$

and, for $p > 1$,

$$(1.1) \quad I_p(r) \leqq B_p A(r)^{p-1}.$$

(Throughout this paper B, C, K denote positive absolute constants and B_p, C_p denote positive constants which depend only on p, but the constant denoted by each symbol may differ at different occurrences.)

To prove the second part of Theorem 1 we need only note that, by M. Riesz's theorem [**1**, p. 54], for $p > 1$,

$$\int_0^{2\pi} |h(re^{i\theta})|^p d\theta \leqq B_p \int_0^{2\pi} u(r, \theta)^p d\theta \leqq 2\pi B_p A(r)^{p-1}.$$

For $h \in \mathscr{P}$, therefore,

$$(1.2) \quad \begin{aligned} I(r) &= O(\log A(r)), \\ I_p(r) &= O(A(r)^{p-1}),\ p > 1, \end{aligned}$$

as $r \to 1$. For $0 < p < 1$, of course, it is well known that

$$I_p(r) = O(1),\ r \to 1,$$

for such h. The question now arises whether, in some sense, the relations in (1.2) are best possible. One might ask, for instance, whether there is a positive function ϕ on $(0, 1)$ such that if $h \in \mathscr{P}$ and $A(r) = O(\phi(r))$, then

$$I(r) = o(\log A(r))$$

as $r \to 1$. Using examples constructed by Salem, we prove a general theorem which implies that the answer to this question is in the negative.

THEOREM 2. *Let* ϕ *be any positive function continuous and increasing to infinity on* $[0, 1)$ *such that* $(1 - r)\,\phi(r)$ *decreases on* $[0, 1)$. *Then there is a function* $f \in \mathscr{P}$ *with* $A(r) = 0(\phi(r))$, $r \to 1$, *for which*

$$(1.3) \quad \liminf_{r \to 1} \frac{I(r)}{\log \phi(r)} > 0,$$

and

$$(1.4) \quad \liminf_{r \to 1} \frac{I_p(r)}{\phi(r)^{p-1}} > 0$$

for each $p > 1$.

Remark. The hypothesis that $(1 - r)\,\phi(r)$ is decreasing is not an unnatural one here since it can be shown (cf. Lemma 5 below) that, for $h \in \mathscr{P}$, $(1 - r)(1 + r)^{-1}A(r)$ is a decreasing function of r on $[0, 1)$.

The proof of Theorem 2 is given in Sections 2 and 3 but we conclude this section by noting that the first part of the theorem extends a recent result due to Lewis [**6**]. This author has shown that, given any number ϵ in $(0, 1)$, there exists a function $h \in \mathscr{P}$ satisfying $M(r) = O((1-r)^{-\epsilon})$ for which

$$\text{(1.5)} \quad \liminf_{r\to 1} \frac{I(r)}{\log 1/(1-r)} > 0.$$

Hayman [**4**] had earlier established a similar result but with, in (1.5), lim sup in place of lim inf. If we take $\phi(r) = (1-r)^{-\epsilon}$ above and note that $A(r) = O((1-r)^{-\epsilon})$ implies $M(r) = O((1-r)^{-\epsilon})$ for $h \in \mathscr{P}$ (this follows easily from (5.7) below, for example), then we see that (1.5) is a special case of (1.3).

2. Proof of theorem 2. In this section we state and prove two lemmas that we need.

LEMMA 1. *Let $h \in \mathscr{P}$ and write $u(r,\theta) = \operatorname{Re} h(re^{i\theta})$. Then, for $p > 1$,*

$$\text{(2.1)} \quad \int_0^{2\pi} u(r,\theta)^p d\theta \log\left\{\frac{1}{2\pi}\int_0^{2\pi} u(r,\theta)^p d\theta\right\} \leqq (p-1)\int_0^{2\pi} u(r,\theta)^p \log u(r,\theta) d\theta$$

for $0 \leqq r < 1$.

Proof. Fix $r \in [0, 1)$ and, for $h \in \mathscr{P}$, set

$$\mu_r(\theta) = \frac{1}{2\pi}\int_0^\theta u(r,t)dt$$

for $\theta \in [0, 2\pi]$. Then μ_r is an increasing function of θ and $\mu_r(2\pi) - \mu_r(0) = 1$. Hence, if

$$J_p(r) = \frac{1}{2\pi}\int_0^{2\pi} u(r,\theta)^p d\theta = \int_0^{2\pi} u(r,\theta)^{p-1} d\mu_r(\theta),$$

then [**9**, p. 73] $[J_p(r)]^{1/(p-1)}$ is an increasing function of p in $(1, \infty)$. Consequently,

$$\frac{d}{dp}\left\{\frac{1}{p-1}\log J_p(r)\right\} \geqq 0$$

for $p > 1$, from which it easily follows that

$$\frac{1}{p-1} J_p(r) \log J_p(r) \leqq \frac{d}{dp} J_p(r) = \frac{1}{2\pi}\int_0^{2\pi} u(r,\theta)^p \log u(r,\theta) d\theta,$$

and we have proved Lemma 1.

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Some preliminaries are necessary before we can state our second lemma. If $h \in \mathscr{P}$, then, by the Herglotz representation theorem, there is a function μ increasing on $(-\infty, \infty)$ satisfying

$$\mu(t + 2\pi) - \mu(t) = 1$$

for $t \in (-\infty, \infty)$, such that

$$(2.2) \quad h(z) = \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t)$$

for $z \in \Delta$. Then, for $0 \leqq r < 1$,

$$(2.3) \quad \operatorname{Re} h(re^{i\theta}) = \int_0^{2\pi} P(r, \theta - t) d\mu(t)$$

where

$$P(r, \psi) = \frac{1 - r^2}{1 - 2r\cos\psi + r^2} = \frac{1 - r^2}{(1 - r)^2 + 4r\sin^2\frac{1}{2}\psi}$$

is the Poisson kernel. We note that, for $|\theta - t| < \pi$ and $\frac{1}{2} \leqq r < 1$,

$$(2.4) \quad P(r, \theta - t) \leqq \frac{1 - r^2}{(1 - r)^2 + 4r\pi^{-2}(\theta - t)^2} \leqq \frac{\pi^2(1 - r)}{(1 - r)^2 + (\theta - t)^2}.$$

Finally, for $\delta > 0$, we write

$$\omega(\delta, \mu) = \sup\{\mu(\theta + \delta) - \mu(\theta) : 0 \leqq \theta < 2\pi\}$$

so that $\omega(\delta, \mu)$ is the "modulus of continuity" of μ.

LEMMA 2. *Let $h \in \mathscr{P}$ and μ be related as in* (2.2). *Then*

$$A(r, h) \leqq K\omega(1 - r, \mu)/(1 - r)$$

for $\frac{1}{2} \leqq r < 1$.

Proof. For $m = 0, 1, 2, \ldots$, write

$$F_m = F_m(r, \theta) = \{t : m(1 - r) \leqq |\theta - t| < (m + 1)(1 - r)\}.$$

Then, for $r \in [\frac{1}{2}, 1)$ and $\theta \in [0, 2\pi]$,

$$\begin{aligned} \operatorname{Re} h(re^{i\theta}) &\leqq \int_{\theta-\pi}^{\theta+\pi} \frac{\pi^2(1 - r)}{(1 - r)^2 + (\theta - t)^2} d\mu(t) \\ &\leqq \pi^2(1 - r) \sum_{m=0}^{\infty} \int_{F_m} \frac{d\mu(t)}{(1 - r)^2 + (\theta - t)^2} \\ &\leqq \frac{2\pi^2\omega(1 - r, \mu)}{1 - r} \sum_{m=0}^{\infty} \frac{1}{1 + m^2} = K\frac{\omega(1 - r, \mu)}{1 - r} \end{aligned}$$

by (2.3) and (2.4). This proves Lemma 2.

3. As we have already said in Section 1, our proof of Theorem 2 is based on a result of Salem [**10**] which we now state as

LEMMA 3. *Let the function ψ be defined and increasing on $(0, \infty)$ and satisfy $\psi(\delta)/\delta \to \infty$ as $\delta \to 0$. Suppose also that, for every integer $n > 1$ and $\delta \in (0, \infty)$, $\psi(n\delta) \leqq n\psi(\delta)$. Then there exists a function F defined and increasing in $(-\infty, \infty)$ with $F(t + 2\pi) - F(t) = 1$ for $t \in (-\infty, \infty)$ whose modulus of continuity $\omega(\delta, F)$ satisfies*

$$(3.1) \quad \omega(\delta, F) \leqq \psi(\delta), 0 < \delta \leqq 2\pi,$$

and such that if

$$c_n = \int_0^{2\pi} e^{-int} dF(t), \quad n \geqq 1,$$

then

$$(3.2) \quad \sum_{k=1}^{n} |c_k|^2 \geqq Cn\psi\left(\frac{1}{n}\right), \quad n \geqq 1.$$

Suppose now that ϕ is the function defined in Theorem 2 and set

$$\psi(\delta) = \begin{cases} \delta\phi(1 - \delta), & 0 < \delta < 1, \\ \delta\phi(0), & \delta \geqq 1. \end{cases}$$

Then it is easily verified that ψ satisfies the conditions of Lemma 3. Let F be the corresponding function obtained in the lemma and write

$$f(z) = \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} dF(t) = 1 + 2 \sum_{n=1}^{\infty} c_n z^n, \quad z \in \Delta.$$

Then $f \in \mathscr{P}$ and, by Lemma 2 and (3.1),

$$(3.3) \quad A(r, f) \leqq K \frac{\omega(1 - r, F)}{1 - r} \leqq K \frac{\psi(1 - r)}{1 - r} = K\phi(r)$$

for $\frac{1}{2} \leqq r < 1$. We show next that

$$(3.4) \quad I_2(r, f) \geqq B\phi(r), \tfrac{1}{2} \leqq r < 1.$$

Let $r \in [\frac{1}{2}, 1)$ and let n be the integer that $1 - 1/n \leqq r < 1 - 1/(n + 1)$. Then, using Parseval's theorem and (3.2),

$$I_2(r, f) = 1 + 4 \sum_{m=1}^{\infty} |c_m|^2 r^{2m} \geqq 4 \sum_{m=1}^{n+1} |c_m|^2 \left(1 - \frac{1}{n}\right)^{2m} \geqq K \sum_{m=1}^{n+1} |c_m|^2$$
$$\geqq B(n + 1)\psi\left(\frac{1}{n + 1}\right) = B\phi\left(1 - \frac{1}{n + 1}\right) \geqq B\phi(r),$$

since ϕ is increasing. This proves (3.4), and (1.4) follows for the case $p = 2$.

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We next use (3.4) to establish (1.4) for arbitrary $p > 1$. By Holder's inequality, with $1/q = 1 - 1/p$ and $p > 1$, we have

$$I_2(r) \leqq I_p(r)^{1/p} I_q(r)^{1/q}$$

and so, by (3.4), (1.1) and (3.3),

$$B\phi(r) \leqq I_p(r)^{1/p} C_p \phi(r)^{1/p}$$

i.e.,

$$I_p(r) \geqq B_p \phi(r)^{p-1}$$

for $\frac{1}{2} \leqq r < 1$. This clearly gives (1.4).

It remains only to show that (1.3) holds. Write $u(r, \theta) = \operatorname{Re} f(z)$ for $z = re^{i\theta} \in \Delta$. Then by (2.1), with $p = 2$, (3.4) and (3.3),

$$B\phi(r) \log B\phi(r) \leqq \int_0^{2\pi} u(r, \theta)^2 \log u(r, \theta) d\theta$$
$$\leqq \int_0^{2\pi} u(r, \theta)^2 \log^+ u(r, \theta) d\theta \leqq K\phi(r) \int_0^{2\pi} u(r, \theta) \log^+ u(r, \theta) d\theta,$$

that is,

$$\int_0^{2\pi} u(r, \theta) \log^+ u(r, \theta) d\theta \geqq C \log B\phi(r), \tag{3.5}$$

for $\frac{1}{2} \leqq r < 1$. But, by a converse [**1**, p. 60] to the theorem of Zygmund used in Section 1, it follows that, since $f \in \mathscr{P}$,

$$\int_0^{2\pi} u(r, \theta) \log^+ u(r, \theta) d\theta \leqq \frac{\pi}{2} \int_0^{2\pi} |\operatorname{Im} f(re^{i\theta})| d\theta + K$$
$$\leqq \frac{\pi}{2} \int_0^{2\pi} |f(re^{i\theta})| d\theta + K$$

for $0 \leqq r < 1$, and this, together with (3.5), clearly implies (1.3). The proof of Theorem 2 is now complete.

4. Some refinements of theorems 1 and 2. The function F constructed by Salem in [**10**] to establish Lemma 3 is a singular function, so it is natural to ask whether Theorem 2 can be proved with a function $f \in \mathscr{P}$ which is 'generated', according to (2.2), by an increasing function μ which is absolutely continuous. The subclass of $\mathscr{P}$ of such functions will be denoted by $\mathscr{P}_{\text{ac}}$. That the answer to the question is in the negative, at least in a special case, has already been proved by Keogh [**5**] who has shown, essentially, that if $h \in \mathscr{P}_{\text{ac}}$, then

$$I(r) = o\left(\log \frac{1}{1-r}\right)$$

as $r \to 1$. We prove here the following more complete result.

THEOREM 3. *Let $h \in \mathscr{P}_{ac}$ and suppose that $A(r, h) \to \infty$ as $r \to 1$. Then, as $r \to 1$,*

(4.1) $\quad I(r, h) = o(\log A(r, h))$

and, for each $p > 1$,

(4.2) $\quad I_p(r, h) = o(A(r, h)^{p-1})$.

Proof. To prove (4.1) it is enough, by Zygmund's theorem again, to show that

$$\int_0^{2\pi} u(r, \theta) \log^+ u(r, \theta) d\theta = o(\log A(r, h)),$$

where $u(r, \theta) = \operatorname{Re} h(re^{i\theta})$. Let μ be the absolutely continuous increasing function related to h by (2.2). Then (as in Section 2)

$$u(r, \theta) = \int_0^{2\pi} P(r, \theta - t) d\mu(t) = \int_0^{2\pi} P(r, \theta - t) g(t) dt,$$

for some $g \in L[0, 2\pi]$, and it is familiar from harmonic function theory that we then have

(4.3) $\quad \lim_{r \to 1} u(r, \theta) = g(\theta)$

a.e. in $[0, 2\pi]$. We write next

$$\int_0^{2\pi} u(r, \theta) \log^+ u(r, \theta) d\theta = \int_0^{2\pi} [\log^+ u(r, \theta)] g(\theta) d\theta + \int_0^{2\pi} [\log^+ u(r, \theta)] \{u(r, \theta) - g(\theta)\} d\theta$$
$$= J_1 + J_2, \text{ say.}$$

Now $\log^+ u(r, \theta) \{\log A(r)\}^{-1}$ is uniformly bounded in Δ and, because of (4.3), tends to 0 a.e. in $[0, 2\pi]$ as $r \to 1$. Hence, by Lebesgue's dominated convergence theorem,

$$J_1 \{\log A(r)\}^{-1} \to 0 \text{ as } r \to 1.$$

Also

(4.4) $\quad J_2 \{\log A(r)\}^{-1} \leqq B \int_0^{2\pi} |u(r, \theta) - g(\theta)| d\theta$

and, since, trivially,

$$\int_0^{2\pi} u(r, \theta) d\theta \to \int_0^{2\pi} g(\theta) d\theta \quad \text{as } r \to 1,$$

it follows from (4.3) (see, for example, [**1**, p. 21]) that the integral on the right of (4.4) tends to 0 as $r \to 1$. This completes the proof of (4.1).

The relation (4.2) can be obtained by a similar argument or it can be deduced from (4.1) by means of Lemma 1; in either case the details are easy and are left to the reader.

Both (4.1) and (4.2) are best possible but before proving this we mention another result of Keogh [loc. cit.]. The result is stated by the author in terms of starlike functions but it can be given an equivalent formulation for $\mathscr{P}_{\mathrm{ac}}$ as follows: given any positive function η, defined in $[0, 1)$ with $\eta(r) \to 0$ $(r \to 1)$, there exists $h \in \mathscr{P}_{\mathrm{ac}}$ such that

$$\sup \left\{ \left| \int_0^r \frac{\operatorname{Re} h(\rho e^{i\theta}) - 1}{\rho} \, d\rho \right| : z = re^{i\theta} \in \Delta \right\} < \infty$$

and

$$\limsup_{r \to 1} \frac{I(r)}{\eta(r) \log 1/(1-r)} > 0.$$

We strengthen and extend this result by proving

THEOREM 4. *Let ϕ be as in Theorem 2 and let η be any positive function defined in $[0, 1)$ with $\eta(r) \to 0 (r \to 1)$. Then for $p \geqq 1$, there are functions $g_p \in \mathscr{P}_{\mathrm{ac}}$ satisfying $A(r, g_p) = O(\phi(r))$, $r \to 1$, such that*

$$\liminf_{r \to 1} \frac{I(r, g_1)}{\eta(r) \log \phi(r)} > 0 \tag{4.5}$$

and

$$\liminf_{r \to 1} \frac{I_p(r, g_p)}{\eta(r) \phi(r)^{p-1}} > 0 \tag{4.6}$$

for each $p > 1$.

This theorem will be shown to be a consequence of Lemma 3 and the following lemma.

LEMMA 4. *Let $f(z) = 1 + 2\sum_{n=1}^{\infty} c_n z^n \in P$ and let $(\lambda_n)_0^{\infty}$, where $\lambda_0 = 1$, be a convex sequence of positive numbers which converges to 0. Let*

$$g(z) = 1 + 2 \sum_{n=1}^{\infty} \lambda_n c_n z^n, \quad z \in \Delta.$$

Then $g \in \mathscr{P}_{\mathrm{ac}}$ and $A(r, g) \leqq A(r, f)$ for $0 \leqq r < 1$.

Proof. Since, for $0 \leqq r < 1$, $(\lambda_n r^n)_0^{\infty}$ is a convex sequence which converges to 0, it follows [**12**, p. 183] that

$$\operatorname{Re} \left\{ \tfrac{1}{2} + \sum_{n=1}^{\infty} \lambda_n z^n \right\} \geqq 0$$

for $z = re^{i\theta} \in \Delta$. Hence, by (2.2),

$$\lambda_n = \int_0^{2\pi} e^{-int} d\mu(t)$$

for some function μ increasing on $[0, 2\pi]$ with $\mu(2\pi) - \mu(0) = 1$. Thus, for $z \in \Delta$,

$$g(z) = \int_0^{2\pi} \left\{1 + 2\sum_{n=1}^{\infty} c_n r^n e^{in(\theta - t)}\right\} d\mu(t) = \int_0^{2\pi} f(re^{i(\theta-t)}) d\mu(t)$$

and so $g \in \mathscr{P}$ and $A(r, g) \leqq A(r, f)$. We next use the fact [**12**, p. 179] that if $\frac{1}{2}a_0 + \sum_1^\infty (a_n \cos nx + b_n \sin nx)$ is a Fourier–Stieltjes series and $(\lambda_n)_0^\infty$ is a convex sequence tending to 0, then $\frac{1}{2}a_0\lambda_0 + \sum_1^\infty (a_n \cos nx + b_n \sin nx)\lambda_n$ is a Fourier series. This result implies here that there is a function $G \in L[0, 2\pi]$ such that

$$\lambda_n c_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-int} G(t) dt \quad (n \geqq 0)$$

(where $c_0 = 1$), from which we obtain

$$g(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} G(t) dt.$$

Since

$$G(t) = \lim_{r \to 1} \operatorname{Re} g(re^{it}) \geqq 0$$

a.e. on $[0, 2\pi]$, it follows that $g \in \mathscr{P}_{\mathrm{ac}}$. This completes the proof of the lemma.

Proof of Theorem 4. For $n \geqq 1$, let

$$\epsilon_n = \sup\{\eta(r)^{1/2}: 1 - 1/n \leqq r \leqq 1 - 1/(n+1)\}.$$

Then $\epsilon_n \to 0$ as $n \to \infty$. Let $(\lambda_n)_0^\infty$ be a convex decreasing sequence such that $\lambda_{n+1} \geqq \epsilon_n$ $(n \geqq 1)$ and $\lambda_n \to 0$. (Such a sequence is easily constructed.) Let $f(z) = 1 + 2\sum c_n z^n$ be the function defined in the proof of Theorem 2 and let

$$g(z) = 1 + 2\sum_1^\infty \lambda_n c_n z^n, \quad z \in \Delta.$$

Then, by Lemma 4, $A(r, g) \leqq A(r, f) = O(\phi(r))$ and $g \in \mathscr{P}_{\mathrm{ac}}$. Now fix $r \in (0, 1)$ and choose n such that $1 - 1/n < r \leqq 1 - 1/(n + 1)$. By the argument used to prove (3.4),

$$I_2(r, g) \geqq B\lambda_{n+1}^2 \phi(r) \geqq B\eta(r)\phi(r)$$

and it is clear that (4.6) follows in the case $p = 2$ on taking $g_2 = g$.

The method used in Section 3 to deduce (1.4) from (3.4) now gives

$$I_p(r, g_2) \geqq B_p \eta(r)^p \phi(r)^{p-1} \quad (\tfrac{1}{2} \leqq r < 1)$$

for every $p > 1$, and it is clear from this that a $g_p \in \mathscr{P}_{\mathrm{ac}}$ exists for which (4.6) holds for all such p.

Finally, by the argument used to prove (1.3),

$$I(r, g_2) \geqq C\eta(r) \log [B\eta(r)\phi(r)]$$

and assuming, as we may, that $\eta(r) \geqq K\phi(r)^{-1/2}$, say, for all r sufficiently near 1, we immediately deduce (4.5) with $g_1 = g_2$. This completes the proof of Theorem 4.

5. The maximum modulus. As a consequence of Theorem 1 we have, for $h \in \mathscr{P}$,

$$I(r) \leqq \log M(r) + A$$

and

$$I_p(r) \leqq B_p M(r)^{p-1} \ (p > 1)$$

for $0 \leqq r < 1$. We now turn our attention, in this final section, to the problem of obtaining lower estimates for the integral means in terms of the maximum modulus. We begin by deriving a simple inequality of this type for functions that are merely regular in Δ.

Suppose, initially, that $f(z) = \sum_0^\infty a_n z^n$ is regular in Δ and continuous in the closure $\bar{\Delta}$. For $0 \leqq r < 1$, we have

$$M(r,f) \leqq \sum_0^\infty |a_n|r^n \leqq \left(\sum_0^\infty |a_n|^2\right)^{1/2}\left(\sum_0^\infty r^{2n}\right)^{1/2}$$

and so, by Parseval's theorem,

$$(1 - r^2)M(r,f)^2 \leqq \frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^2 d\theta.$$

Fix now $r \in (0, 1)$ and let g be a function regular in Δ. Let the zeros of g in $\{z: |z| \leqq r\}$ be, with due account of multiplicity, $z_1, z_2, \ldots z_n$ and write

$$B_n(z) = \prod_{k=1}^n \frac{z - z_k}{1 - \bar{z}_k z}, \quad z \in \bar{\Delta},$$

Then $|B_n(z)| < 1$ for $z \in \Delta$ and $|B_n(z)| = 1$ when $|z| = 1$. Now write

$$F(z) = g(rz)B_n(z)^{-1}, \ z \in \Delta.$$

Then F is regular and non-zero in Δ and continuous in Δ and so, given any $p > 0$, we can define a regular branch of $F^{p/2}$ in Δ which is also continuous on $\bar{\Delta}$. Hence, using (5.1) with $f = F^{p/2}$,

$$(1 - r^2)M(r^2, g)^p \leqq (1 - r^2)M(r, F^{p/2})^2 \leqq \frac{1}{2\pi}\int_0^{2\pi} |F(e^{i\theta})|^p d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} |g(re^{i\theta})|^p d\theta,$$

that is,

(5.2) $(1-r^2)M(r^2, g)^p \leqq I_p(r, g)$

for $p > 0$.

This inequality is sharp for each $p > 0$, as the example $g(z) = (1-z)^{-2/p}$ shows, and is essentially a known result. The case $p = 1$, for instance, is proved in [**7**] and the general result, in however a less precise form, is obtained in [**2**]. Although (5.2) is sharp, it can be improved in one direction to yield the following more delicate result: if g is regular in Δ, then

(5.3) $$\int_0^r M(t, g)^p dt \leqq r\pi I_p(r, g)$$

for $p > 0$. A proof of this result can be found in [**8**]; but see also [**3**]. We note also that if, for some $p > 0$, g belongs to the Hardy class H^p, i.e., if

$$\sup_{0 \leqq r < 1} I_p(r, g) < \infty,$$

then it is known [**3**], and is, in fact, an easy consequence of (5.3), that in this case (5.2) can be improved to

(5.4) $M(r, g) = o((1-r)^{-1/p}),\ r \to 1.$

For the class $\mathscr{P}$, of course, inequalities (5.2) and (5.4) are of interest only when $p \geqq 1$, since (as already noted in Section 1) $h \in \mathscr{P}$ implies $M(r, h) = O((1-r)^{-1}))$, $r \to 1$. In the case $p = 1$ both (5.2) and (5.4) can be improved for the class $\mathscr{P}$, as we now show.

THEOREM 5. *Let $h \in \mathscr{P}$. Then, for $O < r < 1$,*

(5.5) $$M(r, h) \leqq \frac{2r\pi I(r, h)}{(1-r)\log 1/(1-r)}.$$

If, further, $h \in H^1$, then

(5.6) $$M(r, h) = o\left(\left[(1-r)\log\frac{1}{1-r}\right]^{-1}\right), \quad r \to 1.$$

We show that this theorem is a consequence of inequality (5.3) and the following lemma.

LEMMA 5. *Let $h \in \mathscr{P}$. Then*

$$\frac{1-r}{1+r} M(r, h)$$

is a decreasing function of r on $[0, 1)$.

Proof. For $h \in \mathscr{P}$,

(5.7) $$|h'(z)| \leqq \frac{2}{1-r^2} \operatorname{Re} h(z), \quad z \in \Delta.$$

(This classical inequality follows easily from (2.2)). Fix $\theta \in [0, 2\pi]$. Then, for $0 < r < 1$,

$$\frac{\partial}{\partial r}\frac{1-r}{1+r}|h(re^{i\theta})| = \frac{1-r}{r(1+r)}|h(z)|\left\{\operatorname{Re}\frac{zh'(z)}{h(z)} - \frac{2r}{1-r^2}\right\}(z = re^{i\theta})$$

$\leqq 0$ by (5.7). Hence $[(1-r)/(1+r)]|h(re^{i\theta})|$ decreases on $[0, 1)$ for each fixed $\theta \in [0, 2\pi]$. Let now $0 < r_1 < r_2 < 1$ and choose θ_0 such that

$$|h(r_2{}^{i\theta_0})| = M(r_2, h).$$

Then

$$\frac{1-r_2}{1+r_2}M(r_2) = \frac{1-r_2}{1+r_2}|h(r_2e^{i\theta_0})| \leqq \frac{1-r_1}{1+r_1}|h(r_1e^{i\theta_0})|$$
$$\leqq \frac{1-r_1}{1+r_1}M(r_1),$$

and we have established Lemma 5.

We now prove Theorem 5. By (5.3), with $p = 1$,

$$r\pi I(r, h) \geqq \int_0^r M(t, h)dt \geqq \frac{1-r}{1+r}M(r, h)\int_0^r \frac{1+t}{1-t}dt$$
$$\geqq \tfrac{1}{2}(1-r)M(r, h)\log\frac{1}{1-r},$$

where we have used Lemma 5. This proves (5.5).

If $h \in H^1$ then, by (5.3) again,

$$\int_0^1 M(r, h)dr < \infty$$

and an obvious refinement of the above argument gives (5.6). We omit the details but remark that they can be found in [**11**] where similar arguments have been used.

Our last theorem shows that (5.6) is, in a certain sense, best possible.

THEOREM 6. *Let $\epsilon(r)$ be any positive function defined on $[0, 1)$ such that $\epsilon(r) \to 0$ $(r \to 1)$. Then there exists a function $h \in \mathscr{P}$ such that $h \in H^1$ and*

$$(5.8)\qquad \limsup_{r\to 1}\frac{(1-r)M(r, h)\log 1/(1-r)}{\epsilon(r)} > 0.$$

(A lim inf result is clearly not possible in general here because of (5.3).)

Proof. Let (r_n) be a sequence of positive real numbers increasing to 1 such that

$$\sum_{n=1}^{\infty}\epsilon(r_n) < \infty.$$

Let

$$A^{-1} = \sum_{n=1}^{\infty} \epsilon(r_n)\left(\log \frac{1}{1-r_n}\right)^{-1}$$

and set

$$\lambda_n = A\epsilon(r_n)\left(\log \frac{1}{1-r_n}\right)^{-1}, \quad n \geqq 1,$$

so that $\sum \lambda_n = 1$. For $z \in \Delta$, we now define

$$h(z) = \sum_{n=1}^{\infty} \lambda_n \frac{1+r_n z}{1-r_n z}.$$

Then $h \in \mathscr{P}$ and, for $0 \leqq r < 1$,

$$I(r, h) \leqq \sum_{n=1}^{\infty} \lambda_n \int_0^{2\pi} \left|\frac{1+r_n z}{1-r_n z}\right| d\theta \leqq B \sum_{n=1}^{\infty} \epsilon(r_n),$$

so that $h \in H^1$. Finally, for $n \geqq 1$,

$$M(r_n, h) \geqq \operatorname{Re} h(r_n) \geqq \lambda_n \frac{1+r_n^2}{1-r_n^2} \geqq \frac{A\epsilon(r_n)}{2(1-r_n)\log 1/(1-r_n)},$$

and (5.8) follows. This proves Theorem 6.

Inequality (5.4) is also best possible in the same sense for the class $\mathscr{P}$ for each $p > 1$. This can be proved using examples similar to those constructed in the proof of Theorem 6 above. The details are left to the reader.

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