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A SIMPLE PROOF FOR THE UNICITY OF THE LIMIT CYCLE IN THE BOGDANOV-TAKENS SYSTEM

BY
CHENGZHI LI, CHRISTIANE ROUSSEAU AND XIAN WANG

ABSTRACT. We show that the Bogdanov-Takens system has at most one limit cycle. Similarly we show that the maximum number of limit cycles in the universal unfolding of the symmetric cusp of order 2 (resp. 3) is one (resp. 2). The proof uses the elementary technique of Liénard's equation, yielding a global result for all values of the parameters.

1. **Introduction.** The findings presented in this paper arise from the study of the bifurcation diagrams for local singularities of vector fields. Our discussion focuses on planar vector fields having a nilpotent linear part:

$$\dot{x} = y \quad \dot{y} = 0. \tag{1.1}$$

Such a singularity is of minimum codimension 2. In the codimension 2 case, called *Bogdanov-Takens bifurcation* ([1], [2], and [15]) (or *cusp of order 2* [6]) we have a non-degenerate quadratic part:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x^2 + \eta xy \quad \eta = \pm 1. \end{aligned} \tag{1.2}$$

To obtain the bifurcation diagram we study the following universal unfolding:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \epsilon_1 + \epsilon_2 y + x^2 + \eta xy. \end{aligned} \tag{1.3}$$

It is shown that for small ϵ's the family (1.3) has at most one limit cycle. This was first proved for (1.3) by Bogdanov [1] and [2], then by Cushman and Sanders [5]. Results of Petrov [12] and Mardesic [9] later proved that, in the universal unfolding of the cusp of order n, there are at most $(n-1)$ limit cycles (the cusp of order n is the singularity $\dot{x} = y, \dot{y} = x^2 + x^{[3(n-1)/2]}y$ [6] and [13]). All proofs involve the use of elliptic integrals, from which Picard-Fuchs equations, and then a Riccati equation are

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deduced. Petrov also uses the complex analytic continuation of the elliptic integrals involved [12].

Here we want to give an elementary proof. For this we transform our equation (1.3) into a Liénard equation, and we use results obtained by Zhang [16–19] and Cherkas [4]. From Perko's results [10] on rotated vector fields, it is easy to get the monotonic growth of the limit cycle, as shown previously by Bogdanov in [1]. Unfortunately we did not prove the hyperbolicity of the limit cycle.

We also discuss the case of the *symmetric cusp* of order 2 (resp. 3) for which we find a maximum number of one (resp. 2) limit cycles [3], [7], [8]. Here we consider a system symmetric under the inversion $(x, y) \longmapsto (-x, -y)$, and with linear part (1.1). The symmetric cusps of order 2 and 3 have universal unfoldings:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \epsilon_1 x + \epsilon_2 y + x^3 - x^2 y, \end{aligned} \tag{1.4}$$

and:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \epsilon_1 x + \epsilon_2 y + \epsilon_3 x^2 y + x^3 - x^4 y. \end{aligned} \tag{1.5}$$

REMARK. Our result concerning the uniqueness of the limit cycle in the Bogdanov's system is used by Perko in [11].

2. **Proof of the results.** All our families are of the form:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -g(x) - f(x)y. \end{aligned} \tag{2.1}$$

A limit cycle can exist if the system has a singular point which is a focus or a node (i.e. a singular point whose eigenvalues have nonzero real parts with the same sign). In (1.4) and (1.5) the origin is such a singular point. System (1.3) has a focus or node for $\epsilon_1 < 0$, and also a saddle point. A limit cycle cannot contain the saddle point. Looking at the vector field on the line $x = \sqrt{-\epsilon_1}$, we see that limit cycles may occur only in the region $x < \sqrt{-\epsilon_1}$. Similarly (1.4) and (1.5) can have limit cycles only for $\epsilon_1 < 0$, and these occur in the region $x^2 < -\epsilon_1$. Translating the focus or node to $x = 0$, all our systems satisfy:

$$xg(x) > 0 \tag{2.2}$$

in the region where they can have limit cycles.

We let:

$$y = Y - F(x), F(x) = \int_0^x f(x)dx, \tag{2.3}$$

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and we get (with $Y \longmapsto y$):

$$\begin{aligned} \dot{x} &= y - F(x) \\ \dot{y} &= -\, g(x). \end{aligned} \tag{2.4}$$

NOTATION for equation (2.4) throughout the paper:

$$G(x) = \int_0^x g(x)dx \quad f(x) = F'(x). \tag{2.5}$$

The maximum of one limit cycle for the Bogdanov-Takens system and for the symmetric cusp of order 2 follows from the following theorems:

THEOREM 2.1. (Zhang [18], [19]). *We consider Liénard's equation* (2.4) *satisfying* (2.2) *in a region* $x \in [x_1, x_2], x_1 < 0 < x_2$, *and also:*

(i) $g(x)$ *is a continuous function satisfying Lipschitz condition in any finite interval;* $G(x_1) = +\infty$ *if* $x_1 = -\infty$, *and* $G(x_2) = +\infty$ *if* $x_2 = +\infty$.

(ii) $f(x)$ *is a continuous function,* $f(x)/g(x)$ *is non-decreasing, as* x *is increasing inside* $(x_1, 0) \cup (0, x_2)$; $f(x)/g(x)$ *is not identically zero for* $|x| < \delta, \delta$ *sufficiently small.*

Then (2.4) *has at most one limit cycle for* $x \in (x_1, x_2)$. *The limit cycle is stable if it exists.*

THEOREM 2.2. (Cherkas [4]). *We consider Liénard's equation* (2.4) *satisfying* (2.2) *in an interval* (x_1, x_2), $x_1 < 0 < x_2$. *If the equations*

$$F(u) = F(v), \quad G(u) = G(v) \tag{2.6}$$

have no solutions for $x_1 < u < 0 < v < x_2$, *then* (2.4) *has no limit cycle for* $x \in (x_1, x_2)$.

DEFINITION 2.3. [10] *The family of vector fields:*

$$\dot{x} = P(x, y, \delta) \quad \dot{y} = Q(x, y, \delta) \tag{2.7}$$

is a semicomplete family $(\mathrm{mod}\, Q = 0)$ *of rotated polynomial vector fields if*

(i) *The singular points remain fixed for any* δ.

(ii) Q *is independent of* δ.

(iii) $Q(\partial P/\partial \delta) < 0$ *except possibly on* $Q(x, y) = 0$.

(iv) $P/Q \longrightarrow \pm\infty$ *as* $\delta \longrightarrow \pm\infty$.

This yields the following three theorems for semicomplete families of rotated vector fields.

THEOREM 2.4. [10] *Stable and unstable limit cycles of a semicomplete polynomial family* $(\mathrm{mod}\, Q = 0)$ *expand or contract monotonically as* δ *varies in a fixed direction. The motion covers an annular neighborhood of the initial position.*

THEOREM 2.5. [10] *A semistable limit cycle of a semicomplete polynomial family* (mod $Q = 0$) *which crosses* $Q = 0$ *a finite number of times splits into a stable limit cycle and an unstable limit cycle if* δ *is varied in a suitable direction. If* δ *is varied in the opposite direction, the semistable limit cycle disappears.*

THEOREM 2.6. *Let* $L(\delta)$ *be a limit cycle of a semicomplete polynomial family* (mod $Q = 0$) *which crosses* $Q = 0$ *a finite number of times and let* R *be the region covered by* $L(\delta)$ *as* δ *varies through* $(-\infty, +\infty)$. *Then the inner (outer) boundary of* R *consists of either a single rest point, a separatrix cycle or a semistable limit cycle.*

THEOREM 2.7. *The universal unfolding* (1.3) *for the Bogdanov-Takens system has at most one limit cycle for all values of* ϵ_1 *and* ϵ_2. *The limit cycle is unstable (resp. stable) in* (1.3) *when* $\eta = 1$, *(resp.* $\eta = -1$*). For fixed* ϵ_1 *the limit cycle expands monotonically from a focus to a separatrix loop as* $\delta = \eta\sqrt{-\epsilon_1} - \epsilon_2$ *decreases.*

PROOF. (i) We only need to consider the case $\epsilon_1 < 0$: for $\epsilon_1 > 0$ (resp. $\epsilon_1 = 0$), the system has no singular point (resp. a singular point of index zero), and therefore no limit cycle. We suppose $\eta = -1$, the case $\eta = 1$ can be treated by a change of coordinates $y \longmapsto -y$, $t \longmapsto -t$. The system (1.3) can be transformed to the form (2.4) by (2.3) where:

(2.8) $$F(x) = bx + x^2/2, g(x) = ax - x^2,$$

and:

(2.9) $$a = 2\sqrt{-\epsilon_1} > 0, b = -\sqrt{-\epsilon_1} - \epsilon_2.$$

Consider equations (2.6) for $-\infty < u < 0 < v < a$. $F(u) = F(v)$ implies

(2.10) $$u + v = -2b,$$

and $G(u) = G(v)$ implies

(2.11) $$a(u+v)/2 = (u^2 + v^2 + uv)/3.$$

Since $uv < 0$, Theorem 2.2 implies that $b(a+b) < 0$ (i.e. $-\sqrt{-\epsilon_1} < \epsilon_2 < \sqrt{-\epsilon_1}$) if (1.3) has a limit cycle. Furthermore, we consider the derivative $\dot{W}$ of $W(x,y) = y^2/2 + G(x)$ along the trajectories of (2.4):

(2.12) $$\dot{W} = -x^2(b + x/2)(a - x) > 0 \quad \forall x \in (-\infty, a),$$

when $b \leqq -a/2$. Therefore (1.3) has a limit cycle only if $-a/2 < b < 0$, i.e. $-\sqrt{-\epsilon_1} < \epsilon_2 < 0$. A simple calculation shows that:

(2.13) $$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{1}{x^2(a-x)^2}\left[(x+b)^2 - b(a+b)\right] > 0,$$

for $x \in (-\infty, 0) \cup (0, a)$ and $b(a+b) < 0$. By using Theorem 2.1 the result follows.

(ii) For fixed ϵ_1 our system is a semicomplete family of rotated vector fields (mod $Q = 0$):

$$\begin{aligned} \dot{x} &= y - [(\eta\sqrt{-\epsilon_1} - \epsilon_2)x - \eta x^2/2] = P(x, y, \delta) \\ \dot{y} &= -(2\sqrt{-\epsilon_1}x - x^2) = Q(x, y, \delta), \end{aligned} \tag{2.14}$$

with $\delta = \eta\sqrt{-\epsilon_1} - \epsilon_2$. Because of Theorem 2.5 we know that the system has no semistable limit cycle. The last part follows from Theorem 2.6. □

THEOREM 2.8. *The universal unfolding* (1.4) *has at most one hyperbolic limit cycle, for all values of* ϵ_1 *and* ϵ_2. *The limit cycle is stable if it exists. For fixed* ϵ_1, *it expands monotonically from a focus to a separatrix cycle through two saddle points as* ϵ_2 *increases.*

PROOF. (i) We note first that (1.4) has no closed orbit for $\epsilon_1 \geqq 0$. In fact, in case $\epsilon_1 > 0$ (resp. $\epsilon_1 = 0$), the unique equilibrium (0, 0) is a saddle (resp. a point of index -1). Thus we suppose $\epsilon_1 < 0$, and we consider only the region $x^2 < -\epsilon_1$, where (1.4) can have limit cycles. By (2.3), equation (1.4) can be transformed to the form (2.4) where

$$F(x) = x^3/3 - \epsilon_2 x, g(x) = -(\epsilon_1 x + x^3). \tag{2.15}$$

We consider equation (2.6) for $-\sqrt{-\epsilon_1} < u < 0 < v < \sqrt{-\epsilon_1}$. Since G is even, $u = -v$. From $F(-v) = F(v)$, we get $v^2 = 3\epsilon_2$. Noting that $0 < v < \sqrt{-\epsilon_1}$ and using Theorem 2.2 we obtain $0 < \epsilon_2 < -\epsilon_1/3$ if (1.4) has a limit cycle. In this case it is easy to show that

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{1}{x^2(\epsilon_1 + x^2)^2}[x^4 - (\epsilon_1 + 3\epsilon_2)x^2 - \epsilon_1\epsilon_2] > 0, \tag{2.16}$$

when $x^2 < -\epsilon_1$. The desired results follow from Theorem 2.1.

(ii) We must now show that the limit cycle is hyperbolic. For this purpose, we consider the return map $P(y)$ along the positive y-axis. Limit cycles correspond to zeroes of $H(y) = (P(y)^2 - y^2)/2$. $H(y)$ is viewed as the variation along the trajectory of $W(x, y) = y^2/2 + G(x)$. We have:

$$H(y) = \int \dot{W}\, dt = \int -F(x)g(x)dt. \tag{2.17}$$

$F(x)$ is negative inside $(0, \sqrt{3\epsilon_2}) \cup (-\infty, -\sqrt{3\epsilon_2})$ and positive elsewhere. We consider y^* such that the trajectory starting at the point $(0, y^*)$ arrives at the point $(\sqrt{3\epsilon_2}, 0)$. Then, for $y \leqq y^*$, (2.13) gives $H(y) > 0$. The limit cycle, if it exists, occurs for $y > y^*$. In the case where it exists, $H(y)$ has a unique zero at $y = \tilde{y}$. We show that $H'(\tilde{y}) < 0$ by calculating $H'(y)$ as

$$H'(y) = \lim_{\Delta y \to 0} [H(y + \Delta y) - H(y)]/\Delta y. \tag{2.18}$$

(2.19) $$H(y+\Delta y)-H(y)=\int_{\Gamma_1}\dot{W}\,dt-\int_{\Gamma_2}\dot{W}\,dt.$$

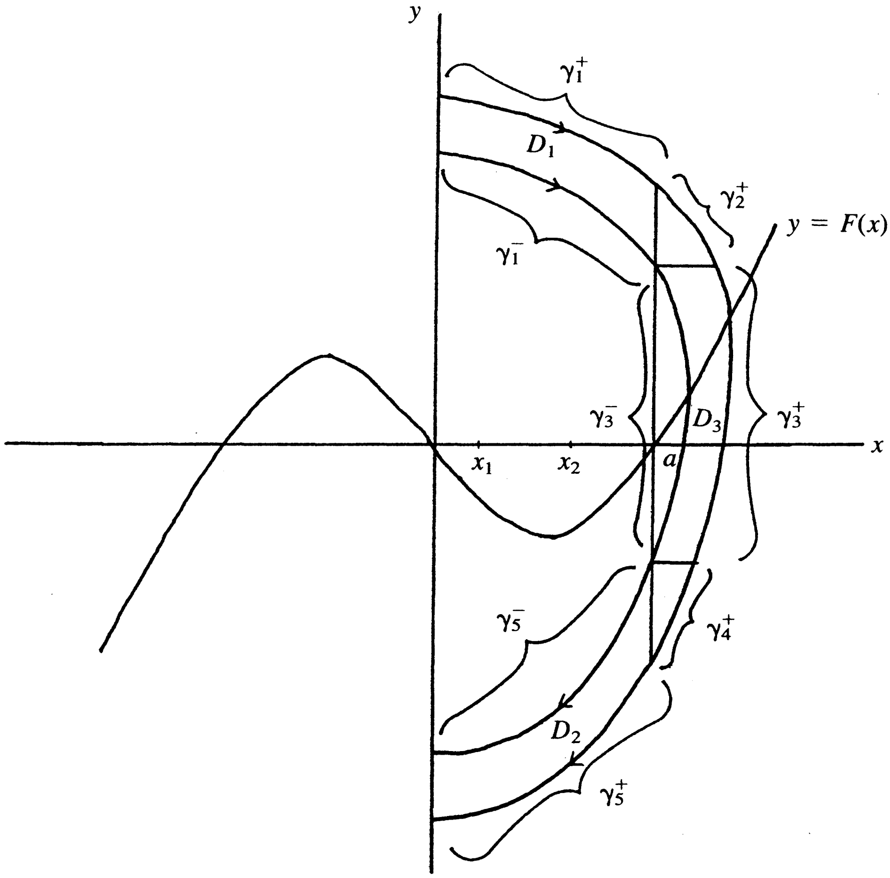

FIGURE 1

These integrals will be divided into several parts (Figure 1)

(2.20) $$\int_{\gamma_{1+}}\dot{W}\,dt-\int_{\gamma_{1-}}\dot{W}\,dt=\int_{\gamma_{1+}}-g(x)F(x)/(y-F(x))dx$$
$$-\int_{\gamma_{1-}}-g(x)F(x)/(y-F(x))dx$$
$$=\iint_{D_1}g(x)F(x)/(y-F(x))^2dxdy=A<0.$$

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Similarly:

$$\int_{\gamma_{5+}} \dot{W}\,dt - \int_{\gamma_{5-}} \dot{W}\,dt < 0, \tag{2.21}$$

$$\int_{\gamma_{2+}} \dot{W}\,dt < 0, \int_{\gamma_{4+}} \dot{W}\,dt < 0, \tag{2.22}$$

$$\begin{aligned} \int_{\gamma_{3+}} \dot{W}\,dt - \int_{\gamma_{3-}} \dot{W}\,dt &= \int_{\gamma_{3+}} F(x)dy - \int_{\gamma_{3-}} F(x)dy \\ &= \iint_{D_3} -F'(x)dxdy < 0, \end{aligned} \tag{2.23}$$

and similarly for the left side. We now look for an estimate for A. Since $|dy/dx| = g(x)/(y - F(x))$ in D_1, we find that vertical distances along trajectories increase with time. Therefore, for $0 < x_1 < x_2 < \sqrt{3\epsilon_2}$:

$$\begin{aligned} |A| &= \iint_{D_1} |g(x)F(x)|/(y - F(x))^2 dxdy \\ &\geqq \Delta y(x_2 - x_1) \min_{\substack{x\in[x_1,x_2] \\ (x,y)\in D_1}} |g(x)F(x)|/(y - F(x))^2 > \lambda\Delta y > 0. \end{aligned} \tag{2.24}$$

(iii) The last part follows as in Theorem 2.7. □

We now study the symmetric cusp of order 3, applying a result obtained by Zhang Zhifen [17], which is in turn an improvement of a result of Rychkov [14]. The theorem we use is the following.

THEOREM 2.9. [17]. *We consider Liénard's equation:*

$$\begin{aligned} \dot{x} &= y - F(x) \\ \dot{y} &= -x \end{aligned} \tag{2.25}$$

satisfying, in a region $x \in (-d, d)$:

(i) *$f(x) = f(-x)$, where $f(x) = F'(x)$ is continuous over $(-d, d)$.*

(ii) *$f(x)$ has two positive zeros $\alpha_1 < \alpha_2 \in (-d, d)$; $F(\alpha_1) > 0$, $F(\alpha_2) < 0$.*

(iii) *$f(x)$ increases monotonically for $x > \alpha_2$ and $x \in (-d, d)$.*

Then the system (2.25) *has at most two limit cycles.*

THEOREM 2.10. *The system* (1.5) *has at most two limit cycles (one stable, one unstable) for all values of ϵ_1 and ϵ_2. For fixed ϵ_1, in the case of two limit cycles, the unstable limit cycle expands (resp. contracts) monotonically to a semistable limit cycle (resp. to a singular point), and the stable limit cycle contracts (resp. expands) monotonically to a semistable limit cycle (resp. to a separatrix cycle) as one of the parameters ϵ_2 or ϵ_3 is varied in a fixed direction. A single stable (unstable) limit cycle, on the other hand, expands monotonically from a singular point to a separarix loop for suitable variation of one of the parameters ϵ_2 or ϵ_3.*

PROOF. The system has the form:

$$\begin{aligned}(2.26)\qquad \dot{x} &= y - (-\epsilon_2 x - \epsilon_3 x^3/3 - x^5/5) = y - F(x)\\ \dot{y} &= -(-\epsilon_1 x + x^3) = -g(x).\end{aligned}$$

We are interested only in the case $\epsilon_1 < 0$. In the case $\epsilon_1 \geqq 0$, the system has a unique singular point of index -1, and therefore possesses no limit cycle. The following classical change of coordinates:

$$\begin{aligned}(2.27)\qquad X &= X(x) = \operatorname{sgn}(x)\sqrt{2G(x)}\\ \tau &= tg(x)\operatorname{sgn}(x)/\sqrt{2G(x)}\end{aligned}$$

changes (2.26) into (2.25). A small calculation shows that the hypothesis of Theorem 2.9 is satisfied in the region $x^2 < -\epsilon_1$, if we change $y \longmapsto -y$, $t \longmapsto -t$. The conclusion follows from Theorem 2.9.

The second part follows from Theorem 2.4, 2.5, 2.6, and from the remark that the unstable limit cycle is inside the stable limit cycle. □

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Department of Mathematics
Peking University, Beijing, PRC

Département de Mathématiques et de Statistique
Université de Montréal
C.P. 6128 Succ. A, Montréal
Qué., H3C 3J7, Canada

Department of Mathematics
Nanjing University, Nanjing, PRC