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ON IWASAWA THEORY OF RUBIN–STARK UNITS AND NARROW CLASS GROUPS

YOUNESS MAZIGH

Faculté des sciences de Meknès, Département de mathématiques, Université Moulay Ismail, B.P. 11201 Zitoune, Meknès 50000, Maroc
e-mail: y.mazigh@edu.umi.ac.ma

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Abstract. Let K be a totally real number field of degree r. Let K_∞ denote the cyclotomic $\mathbb{Z}_2$-extension of K, and let L_∞ be a finite extension of K_∞, abelian over K. The goal of this paper is to compare the characteristic ideal of the χ-quotient of the projective limit of the narrow class groups to the χ-quotient of the projective limit of the rth exterior power of totally positive units modulo a subgroup of Rubin–Stark units, for some $\overline{\mathbb{Q}_2}$-irreducible characters χ of $\mathrm{Gal}(L_\infty/K_\infty)$.

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1. Introduction. Let K be a number field, and let Cl_K^+ denote the narrow class group of K, that is, the quotient group of the group of fractional ideals of K modulo the subgroup of principal fractional ideals generated by a totally positive element α of K, i.e., α is an element of K^* such that $\sigma(\alpha)$ is positive for every embedding $\sigma : K \longrightarrow \mathbb{R}$. The natural homomorphism of Cl_K^+ onto the ideal class group of K induces, for every odd prime p, an isomorphism of the p-primary component of Cl_K^+ onto the p-class group of K. But, the 2-primary components are not necessarily isomorphic. Before we explain our results in details, we set some notation.

Let K be a totally real number field of degree $r = [K : \mathbb{Q}]$. Let K_∞ denote the cyclotomic $\mathbb{Z}_2$-extension of K and L_∞ a finite extension of K_∞, abelian over K. Fix a decomposition of

$$\mathrm{Gal}(L_\infty/K) = \mathrm{Gal}(L_\infty/K_\infty) \times \Gamma, \ \Gamma \simeq \mathbb{Z}_2.$$

Then, the fields $L := L_\infty^\Gamma$ and K_∞ are linearly disjoint over K.

If F/K is a finite abelian extension of K, we write $A^+(F)$ for the 2-part of the narrow class group of F and $\mathcal{E}^+(F)$ for the group of the totally positive units of F. For a $\mathbb{Z}$-module M, let $\widehat{M} = \varprojlim M/2^n M$ denote the 2-adic completion of M. Let

$$A_\infty^+ := \varprojlim A^+(F) \qquad \text{and} \qquad \widehat{\mathcal{E}_\infty^+} := \varprojlim \widehat{\mathcal{E}^+(F)},$$

where the projective limit is taken over all finite sub-extensions of L_∞, with respect to the norm maps. Let

$$\chi : G_K \longrightarrow \overline{\mathbb{Q}}_2^\times,$$

be a non-trivial totally even character of the absolute Galois G_K of K (i.e., it is trivial on all complex conjugations inside G_K) that factors through L. Denote the ring generated by the values of χ over $\mathbb{Z}_2$ by $\mathcal{O}$, and let Δ be the Galois group $\mathrm{Gal}(L/K)$. Let $\mathcal{O}(\chi)$ denote the ring $\mathcal{O}$ on which Δ acts via χ. For any $\mathbb{Z}_2[\Delta]$-module M, we define the χ-quotient M_χ of M by

$$M_\chi := M \otimes_{\mathbb{Z}_2[\Delta]} \mathcal{O}(\chi).$$

For any profinite group $\mathcal{G}$, we define the Iwasawa algebra

$$\mathcal{O}[[\mathcal{G}]] := \varprojlim \mathcal{O}[\mathcal{G}/\mathcal{H}],$$

where the projective limit is over all finite quotients $\mathcal{G}/\mathcal{H}$ of $\mathcal{G}$. In case $\mathcal{G} = \Gamma$, we shall write

$$\Lambda := \mathcal{O}[[\Gamma]].$$

Let L_χ denote the fixed field of $\ker(\chi)$, and let $K(1)$ be the maximal 2-extension inside the Hilbert class field of K. In the sequel, we will assume (for simplicity) that

$$L = L_\chi \quad \text{and} \quad K = L \cap K(1).$$

In particular, L is totally real.

For a 2-adic prime $\mathfrak{p}$ of K, let $\mathrm{Frob}_\mathfrak{p}$ denote a Frobenius element at $\mathfrak{p}$ inside the absolute Galois group of K. Assume that

($\mathcal{H}_1$) *the extension $L/\mathbb{Q}$ is unramified at* 2,
($\mathcal{H}_2$) *for any 2-adic prime $\mathfrak{p}$ of K, we have* $\chi(\mathrm{Frob}_\mathfrak{p}) \neq 1$,
($\mathcal{H}_3$) *the Leopoldt conjecture holds for every finite extension F of L in L_∞ for the prime* 2.

We will denote by $\widehat{\mathrm{St}_\infty^+}$ the projective limit $\varprojlim_n \widehat{St_n^+}$, where St_n^+ is the module constructed by the Rubin–Stark elements (see Definition 3.1). In particular, $\widehat{\mathrm{St}_\infty^+}$ is a submodule of $\bigwedge^r \widehat{\mathcal{E}_\infty^+}$.

In [**9**], for a fixed odd rational prime p, we used the theory of Euler systems to bound the size of the χ-quotient of the p-class groups by the characteristic ideal of the χ-quotient of the rth exterior power of units modulo Rubin–Stark units, in the non-semi-simple case, thus extending the results of [**4**].

In this paper, we consider the case $p = 2$. More precisely, we use the Euler system formed by Rubin–Stark elements to compare the characteristic ideal of the χ-quotient of the projective limit of the 2-part of the narrow class groups to the χ-quotient of the projective limit of the rth exterior power of totally positive units modulo $\widehat{\mathrm{St}_\infty^+}$. We draw the attention of the reader to the fact that, because of many complications, the case $p = 2$ is not often treated in the literature, unlike [**6**, **14**]. The following theorem summarizes our results.

THEOREM 1.1. *Assume that Hypotheses $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ hold. Then,*

$$\mathrm{char}((A_\infty^+)_\chi) \quad \textit{divides} \quad \lambda \cdot \mathrm{char}\left(\left(\left(\bigwedge^r \widehat{\mathcal{E}_\infty^+}\right)/\widehat{\mathrm{St}_\infty^+}\right)_\chi\right),$$

where λ is a power of 2 explicitly given in formula (18).

Treating the case $p = 2$ leads to several complications. The first comes from the non-triviality of the cohomology groups of the absolute Galois group of $\mathbb{R}$. More precisely, for a number field F and a real place w of F, the cohomology group $H^i(F_w, M)$ is not necessarily trivial, where M is a $\mathbb{Z}_p[G_{F_w}]$-module. Hence, the result of [**1**, Proposition 3.8] does not apply, since the cohomological dimension of $G_{K,\Sigma}$ is infinite. The second complication is the need to modify the canonical Selmer structure $\mathcal{F}_{can}$, and to study the Λ-structure of the projective limit of these Selmer groups. This problem is treated in Section 2.3. For this, we use a relation between the universal norms in $\mathbb{Z}_p$-extension and the Λ-structure of certain modules. This is already known, thanks to Vauclair who applied some homological proprieties in [**19**, **20**] to determine this relation.

To control the contributions from infinite places, we use a slight variant of Galois cohomology, the so-called totally positive Galois cohomology $H^*_+(G_{K,\Sigma}, .)$, see Section 2.2, introduced by Kahn in [**8**], based on ideas of Milne [**12**]. Totally positive Galois cohomology has been used by several authors, such as Chinburg et al. [**5**] and Assim and Movahhedi [**2**].

2. Iwasawa theory of Selmer groups.

2.1. Selmer structures. In this subsection, we recall some definitions concerning the notion of Selmer structure introduced by Mazur and Rubin in [**10**, **11**]. For any field k and a fixed separable algebraic closure $\overline{k}$ of k, we write $G_k := \mathrm{Gal}(\overline{k}/k)$ for the Galois group of $\overline{k}/k$. Let $\mathcal{O}$ be the ring of integers of a finite extension Φ of $\mathbb{Q}_2$, and let D denote the divisible module $\Phi/\mathcal{O}$. For a 2-adic representation T with coefficients in $\mathcal{O}$, we define

$$D(1) = D \otimes \mathbb{Z}_2(1), \qquad T^* = \mathrm{Hom}_{\mathcal{O}}(T, D(1)),$$

where $\mathbb{Z}_2(1) := \varprojlim \mu_{2^n}$ is the Tate module.

For a number field F, let F_w denote the completion of F at a given place w of F. Let us recall the local duality theorem (cf. [**12**, Corollary I.2.3]): For $i = 0, 1, 2$, there is a perfect pairing

$$\begin{aligned} H^{2-i}(F_w, T) \times H^i(F_w, T^*) &\xrightarrow{\langle\,,\,\rangle_w} H^2(F_w, D(1)) \cong D, \text{ if } w \text{ is finite},\\ \widehat{H}^{2-i}(F_w, T) \times \widehat{H}^i(F_w, T^*) &\xrightarrow{\langle\,,\,\rangle_w} \widehat{H}^2(F_w, D(1)), \text{ if } w \text{ is infinite}, \end{aligned} \tag{1}$$

where $\widehat{H}^*(F_w, .)$ denotes the Tate cohomology group.

DEFINITION 2.1. Let T be a 2-adic representation of G_F with coefficients in $\mathcal{O}$, and let w be a non-2-adic prime of F. A local condition $\mathcal{F}$ at the prime w on T is a choice of an $\mathcal{O}$-submodule $H^1_{\mathcal{F}}(F_w, T)$ of $H^1(F_w, T)$. For the 2-adic primes, a local condition at 2 will be a choice of an $\mathcal{O}$-submodule $H^1_{\mathcal{F}}(F_2, T)$ of the semi-local cohomology group:

$$H^1(F_2, T) := \oplus_{w|2} H^1(F_w, T).$$

Let I_w denote the inertia subgroup of G_{F_w}. We say that T is unramified at w if the inertia subgroup I_w of w acts trivially on T. We assume in the sequel that T is unramified outside a finite set of places of F.

Definition 2.2. A Selmer structure $\mathcal{F}$ on T is a collection of the following data:

- a finite set $\Sigma(\mathcal{F})$ of places of F, including all infinite places, all 2-adic places and all places where T is ramified,
- a local condition on T, for every $w \in \Sigma(\mathcal{F})$.

If $w \notin \Sigma(\mathcal{F})$, we will write $H^1_{\mathcal{F}}(F_w, T) = H^1_{ur}(F_w, T)$, where $H^1_{ur}(F_w, T)$ is the subgroup of unramified cohomology classes:

$$H^1_{ur}(F_w, T) = \ker(\ H^1(F_w, T) \longrightarrow H^1(I_w, T)\ ).$$

If $\mathcal{F}$ is a Selmer structure on T, we define the Selmer group $H^1_{\mathcal{F}}(F, T) \subset H^1(F, T)$ to be the kernel of the localization map

$$H^1(G_{F,\Sigma(\mathcal{F})}, T) \longrightarrow \bigoplus_{w \in \Sigma(\mathcal{F})} (H^1(F_w, T)/H^1_{\mathcal{F}}(F_w, T)),$$

where $G_{F,\Sigma(\mathcal{F})} := \mathrm{Gal}(F_{\Sigma(\mathcal{F})}/F)$ is the Galois group of the maximal algebraic extension of F unramified outside $\Sigma(\mathcal{F})$.

A Selmer structure $\mathcal{F}$ on T determines a Selmer structure $\mathcal{F}^*$ on T^*. Namely,

$$\Sigma(\mathcal{F}) = \Sigma(\mathcal{F}^*), \quad H^1_{\mathcal{F}^*}(F_w, T^*) := H^1_{\mathcal{F}}(F_w, T)^{\perp}, \ \text{ if } w \in \Sigma(\mathcal{F}^*) - \Sigma_2,$$

under the local Tate pairing $\langle\ ,\ \rangle_w$ and

$$H^1_{\mathcal{F}^*}(F_2, T^*) := H^1_{\mathcal{F}}(F_2, T)^{\perp},$$

under the pairing $\oplus_{w|2}\langle\ ,\ \rangle_w$. Here, Σ_2 denotes the set of 2-adic places of F.

There is a natural partial ordering on the set of Selmer structures on T. Namely, we will say that $\mathcal{F} \leq \mathcal{F}'$ if and only if

$$H^1_{\mathcal{F}}(F_w, T) \subset H^1_{\mathcal{F}'}(F_w, T) \ \text{ for all places } w.$$

If $\mathcal{F} \leq \mathcal{F}'$, we have an exact sequence [**10**, Theorem 2.3.4]

$$\begin{aligned} H^1_{\mathcal{F}}(F, T) &\hookrightarrow H^1_{\mathcal{F}'}(F, T) \longrightarrow \bigoplus_w H^1_{\mathcal{F}'}(F_w, T)/H^1_{\mathcal{F}}(F_w, T) \longrightarrow H^1_{\mathcal{F}^*}(F, T^*)^{\vee} \\ &\twoheadrightarrow H^1_{\mathcal{F}'^{,*}}(F, T^*)^{\vee}, \end{aligned} \tag{2}$$

where $()^{\vee}$ denotes the Pontryagin dual.

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EXAMPLE 2.3. Let w be a place of F, and let F_w^{ur} denote the maximal unramified extension of F_w. Define the subgroup of universal norms

$$H^1(F_w, T)^u = \bigcap_{F_w \subset k \subset F_w^{ur}} \mathrm{cor}_{k,F_w} H^1(k, T),$$

where the intersection is over all finite unramified extensions k of F_w. Let $H^1(F_w, T)^{u,sat}$ denote the $\mathcal{O}$-saturation of $H^1(F_w, T)^u$ in $H^1(F_w, T)$, i.e., $H^1(F_w, T)/H^1_{\mathcal{F}_{ur}}(F_w, T)$ is a free $\mathcal{O}$-module and $H^1_{\mathcal{F}_{ur}}(F_w, T)/H^1(F_w, T)^u$ has a finite length. For a submodule N of a finitely generated $\mathcal{O}$-module M, the $\mathcal{O}$-saturation N^{sat} of N in M is the pre-image under the canonical map $M \longrightarrow M \otimes_{\mathcal{O}} \Phi$ of $N \otimes_{\mathcal{O}} \Phi$. Following [**11**, Definition 5.1], we define the unramified Selmer structure $\mathcal{F}_{ur}$ on T by

- $\Sigma(\mathcal{F}_{ur}) := \{\mathfrak{q} : T \text{ is ramified at } \mathfrak{q}\} \cup \{\mathfrak{p} : \mathfrak{p} \mid 2\} \cup \{w : w \mid \infty\}$,
- $H^1_{\mathcal{F}_{ur}}(F_w, T) = \begin{cases} H^1(F_w, T)^{u,sat}, & \text{if } w \nmid 2\infty; \\ H^1(F_w, T), & \text{if } w \mid \infty. \end{cases}$, and

  $H^1_{\mathcal{F}_{ur}}(F_2, T) = \bigoplus_{\mathfrak{p}\mid 2} H^1(F_{\mathfrak{p}}, T)^{u,sat}$.

For future use, we record here the following well-known properties of unramified Selmer structure:

(i)

$$H^1_{\mathcal{F}^*_{ur}}(F_w, T^*) = H^1_{ur}(F_w, T^*)_{div}, \quad H^1_{\mathcal{F}^*_{ur}}(F_2, T^*) = \bigoplus_{\mathfrak{p}\mid 2} H^1_{ur}(F_{\mathfrak{p}}, T^*)_{div}. \tag{3}$$

(ii) If $w \nmid 2$ and T is unramified at w, then

$$H^1_{\mathcal{F}_{ur}}(F_w, T) = H^1_{ur}(F_w, T) \quad \text{and} \quad H^1_{\mathcal{F}^*_{ur}}(F_w, T^*) = H^1_{ur}(F_w, T^*).$$

(iii) Let Cl_F denote the ideal class group of F. Then,

$$H^1_{\mathcal{F}^*_{ur}}(F, \mathbb{Q}_2/\mathbb{Z}_2)^\vee \cong Cl_F \otimes \mathbb{Z}_2,$$

where for an abelian group A, A_{div} denotes the maximal divisible subgroup of A.

Assertion (i) follows from [**15**, Section 2.1.1, Lemme] and Assertions (ii) and (iii) follow immediately from [**17**, Lemma 1.3.5] and [**10**, Section 6.1], respectively.

2.2. Totally positive Galois cohomology. Let Σ be a finite set of places of F containing infinite places and all 2-adic places. If F' is an extension of F, we denote also by Σ the set of places of F' lying above places in Σ. Let $G_{F,\Sigma}$ be the Galois group of the maximal algebraic extension F_Σ of F, which is unramified outside Σ. If w is a place of F, we denote the decomposition group of w in $\overline{F}/F$ by G_w.

For a finite $\mathcal{O}[G_{F,\Sigma}]$-module M, we write M_+ for the cokernel of the injective map

$$M \longrightarrow \oplus_{w\mid\infty} \mathrm{Ind}_{G_w}^{G_F} M\ ; \quad 0 \longrightarrow M \longrightarrow \oplus_{w\mid\infty} \mathrm{Ind}_{G_w}^{G_F} M \longrightarrow M_+ \longrightarrow 0,$$

where $\mathrm{Ind}_{G_w}^{G_F} M$ denotes the induced module. Following [**8**], we define the ith totally positive Galois cohomology $H^i_+(G_{F,\Sigma}, M)$ of M by

$$H^i_+(G_{F,\Sigma}, M) := H^{i-1}(G_{F,\Sigma}, M_+).$$

We first list the following facts that hold for an arbitrary number field F.

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PROPOSITION 2.4. *We have the following properties:*

(*i*) *There is a long exact sequence*

$$\cdots \longrightarrow H^i_+(G_{F,\Sigma}, M) \longrightarrow H^i(G_{F,\Sigma}, M) \longrightarrow \oplus_{w|\infty} H^i(F_w, M)$$

$$\longrightarrow H^{i+1}_+(G_{F,\Sigma}, M) \longrightarrow \cdots .$$

(*ii*) *For $i \notin \{1, 2\}$, we have $H^i_+(G_{F,\Sigma}, M) = 0$.*

(*iii*) *If F'/F is an extension unramified outside Σ with Galois group G, then there is a cohomological spectral sequence*

$$H^p(G, H^q_+(G_{F',\Sigma}, M)) \Longrightarrow H^{p+q}_+(G_{F,\Sigma}, M).$$

Proof. See [**8**, Section 5]. □

The following corollary is a direct consequence of (*ii*) in Proposition 2.4.

COROLLARY 2.5. *Let F'/F be a finite Σ-ramified extension with Galois group G. Then, the corestriction map induces an isomorphism*

$$H^2_+(G_{F',\Sigma}, M)_G \xrightarrow{\sim} H^2_+(G_{F,\Sigma}, M) . \qquad \square$$

To go further, we need the following remark. If M_Σ denotes the cokernel of the canonical map $M \longrightarrow \oplus_{w\in\Sigma} \mathrm{Ind}^{G_F}_{G_w} M$, then for all $i \geq 0$, we have

$$H^i(G_{F,\Sigma}, M_\Sigma) = H^{i+1}_c(G_{F,\Sigma}, M), \tag{4}$$

where $H^{i+1}_c(G_{F,\Sigma}, .)$ is the continuous cohomology with compact support (for the definition, see [**13**, Section 5.7.2]). Note that

$$H^i_c(G_{F,\Sigma}, M) \cong H^{3-i}(G_{F,\Sigma}, M^*)^\vee, \tag{5}$$

for all $i \geq 1$, cf. [**13**, Proposition 5.7.4], where $M^* = \mathrm{Hom}_{\mathbb{Z}_2}(M, \mu_{2^\infty})$.

PROPOSITION 2.6. *Let Σ_f denote the set of finite places in Σ. Then, there is a long exact sequence*

$$\oplus_{w\in\Sigma_f} H^1(F_w, M) \longrightarrow H^1(G_{F,\Sigma}, M^*)^\vee \longrightarrow H^2_+(G_{F,\Sigma}, M) \longrightarrow \oplus_{w\in\Sigma_f} H^2(F_w, M)$$
$$\twoheadrightarrow H^0(G_{F,\Sigma}, M^*)^\vee . \tag{6}$$

Proof. Consider the commutative exact diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & M & \longrightarrow & \oplus_{w\in\Sigma} \mathrm{Ind}^{G_F}_{G_w} M & \longrightarrow & M_\Sigma & \longrightarrow 0 \\
 & \| & & \downarrow\!\!\!\downarrow & & \downarrow & \\
0 \longrightarrow & M & \longrightarrow & \oplus_{w|\infty} \mathrm{Ind}^{G_F}_{G_w} M & \longrightarrow & M_+ & \longrightarrow 0.
\end{array}$$

Using the snake lemma, we obtain the exact sequence

$$0 \longrightarrow \oplus_{w\in\Sigma_f}\mathrm{Ind}_{G_w}^{G_F} M \longrightarrow M_\Sigma \longrightarrow M_+ \longrightarrow 0\,. \tag{7}$$

Taking the $G_{F,\Sigma}$-cohomology of the exact sequence (7) and since $H^i_+(G_{F,\Sigma}, M) = 0$ for $i \notin \{1, 2\}$ (see Proposition 2.4), we get the exact sequence

$$\oplus_{w\in\Sigma_f} H^1(F_w, M) \longrightarrow H^1(G_{F,\Sigma}, M_\Sigma) \longrightarrow H^2_+(G_{F,\Sigma}, M) \longrightarrow \oplus_{w\in\Sigma_f} H^2(F_w, M)$$
$$\twoheadrightarrow H^2(G_{F,\Sigma}, M_\Sigma).$$

To obtain the desired result, it suffices to observe that

$$H^1(G_{F,\Sigma}, M_\Sigma) = H^1(G_{F,\Sigma}, M^*)^\vee \quad \text{and} \quad H^2(G_{F,\Sigma}, M_\Sigma) = H^0(G_{F,\Sigma}, M^*)^\vee;$$

this is a consequence of properties (4) and (5). □

2.3. Iwasawa theory. Throughout this subsection, we fix a totally real number field K. Let $r = [K : \mathbb{Q}]$ and $K_\infty = \bigcup_{n\geq 0} K_n$ denote the cyclotomic $\mathbb{Z}_2$-extension of K. Assume that all algebraic extensions of K are contained in a fixed algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$. If F is a finite extension of K and w is a place of F, fix a place $\overline{w}$ of $\overline{\mathbb{Q}}$ lying above w. The decomposition (resp. inertia) group of $\overline{w}$ in $\overline{\mathbb{Q}}/F$ is denoted by G_w (resp. I_w). If v is a place of K and F is a Galois extension of K, we denote the decomposition group of v in F/K by $D_v(F/K)$. Recall that

$$\chi : G_K \longrightarrow \mathcal{O}^\times$$

is a non-trivial totally even character, factoring through a finite abelian extension L of K. Assume that L and K_∞ are linearly disjoint over K. Let $L_n = LK_n$ and let $L_\infty = LK_\infty$ be the cyclotomic $\mathbb{Z}_2$-extension of L. In the sequel, we will denote by T the 2-adic representation

$$T = \mathbb{Z}_2(1) \otimes \mathcal{O}(\chi^{-1}).$$

Let Σ be a finite set of places of K containing all infinite places, all 2-adic places and all places where T is ramified. If F is an extension of K, we denote also by Σ the set of places of F lying above places in Σ.

Let us recall the definition of the canonical Selmer structure $\mathcal{F}_{can}$ on T:

- $\Sigma(\mathcal{F}_{can}) = \Sigma$,
- $H^1_{\mathcal{F}_{can}}(F_w, T) = \begin{cases} H^1_{\mathcal{F}_{ur}}(F_w, T), & \text{if } w \nmid 2\infty \\ H^1(F_w, T), & \text{if } w \mid \infty \end{cases}$ and

  $H^1_{\mathcal{F}_{can}}(F_2, T) = \oplus_{w|2} H^1(F_w, T)$,

where $\mathcal{F}_{ur}$ is the unramified local condition, see Example 2.3. Let

$$H^1_{\mathcal{F}_{can}}(FK_\infty, T) := \varprojlim_n H^1_{\mathcal{F}_{can}}(FK_n, T), \quad H^1_{\mathcal{F}^*_{can}}(FK_\infty, T^*) := \varinjlim_n H^1_{\mathcal{F}^*_{can}}(FK_n, T^*),$$

where the projective (resp. injective) limit is taken with respect to the corestriction (resp. restriction) maps. For an $\mathcal{O}$-module M, we denote by $M^\vee := \mathrm{Hom}_{\mathbb{Z}_2}(M, \mathbb{Q}_2/\mathbb{Z}_2)$ its Pontryagin dual.

Note that the Kolyvagin system (see [**17**, **10**]) machinery permits to obtain bounds on the associated Selmer groups. More precisely, the Kolyvagin–Rubin approach shows (see [**17**, Theorem 2.3.3]) that if a non-trivial Euler system exists, then the index of the Euler system in $H^1_{\mathcal{F}_{can}}(K_\infty, T)$ gives a bound for $H^1_{\mathcal{F}^*_{can}}(K_\infty, T^*)^\vee$. It is well know that the Rubin–Stark elements give rise to Euler systems for the 2-adic representation $T = \mathbb{Z}_2(1) \otimes \mathcal{O}(\chi^{-1})$ [**16**]. To find a bound for the narrow class group, we need to modify the canonical Selmer structure, cf. Proposition 2.12.

Definition 2.7. Let F be a finite extension of K, and let $\mathcal{F}$ be a Selmer structure on T. We define the positive Selmer structure $\mathcal{F}^+$ by

- $\Sigma(\mathcal{F}^+) = \Sigma(\mathcal{F})$,
- $H^1_{\mathcal{F}^+}(F_w, T) = \begin{cases} H^1_{\mathcal{F}}(F_w, T), & \text{if } w \nmid \infty, \\ 0, & \text{if } w \mid \infty. \end{cases}$

The following lemma is a first step towards our purpose.

Lemma 2.8. *Let F be a finite Galois extension of L, and let Cl_F^+ be the narrow class group of F. Then,*

$$H^1_{\mathcal{F}^{+,*}_{ur}}(F, T^*) \cong \mathrm{Hom}(Cl_F^+, T^*).$$

Proof. Let w be a finite place of F. Since χ is a character factoring through L, the decomposition group G_w acts trivially on T^*. Then,

$$H^1_{ur}(F_w, T^*) \cong \mathrm{Hom}(G_w/I_w, T^*).$$

Moreover, G_w/I_w is torsion-free and T^* is divisible, then $\mathrm{Hom}(G_w/I_w, T^*)$ is divisible; therefore,

$$H^1_{\mathcal{F}^*_{ur}}(F_w, T^*) = H^1_{ur}(F_w, T^*),$$

by (3). In particular, $H^1(F_w, T^*)/H^1_{\mathcal{F}^*_{ur}}(F_w, T^*)$ injects into $\mathrm{Hom}(I_w, T^*)$. Hence,

$$\begin{aligned} H^1_{\mathcal{F}^{+,*}_{ur}}(F, T^*) &= \ker\big(\, H^1(G_{F,\Sigma}, T^*) \longrightarrow \textstyle\bigoplus_{w\in\Sigma} H^1(F_w, T^*)/H^1_{\mathcal{F}^{+,*}_{ur}}(F_w, T^*)\,\big) \\ &= \ker\big(\, \mathrm{Hom}(G_{F,\Sigma}, T^*) \longrightarrow \textstyle\bigoplus_{w\in\Sigma_f} \mathrm{Hom}(I_w, T^*)\,\big). \end{aligned}$$

Using class field theory, we obtain the result. □

Let

$$H^1_{\mathcal{F}^+_{can}}(FK_\infty, T) := \varprojlim_n H^1_{\mathcal{F}^+_{can}}(FK_n, T), \quad H^1_{\mathcal{F}^{+,*}_{can}}(FK_\infty, T^*) := \varinjlim_n H^1_{\mathcal{F}^{+,*}_{can}}(FK_n, T^*),$$

where the projective (resp. injective) limit is taken with respect to the corestriction (resp. restriction) maps.

We now want to study the relation between $H^1_{\mathcal{F}^{+,*}_{can}}(K_\infty, T^*)$ and $H^1_{\mathcal{F}^{+,*}_{can}}(L_\infty, T^*)$.

Definition 2.9. We define the Selmer structures $\mathcal{F}_\Sigma$ on T by

- $\Sigma(\mathcal{F}_\Sigma) = \Sigma$,
- $H^1_{\mathcal{F}_\Sigma}(F_w, T) = H^1(F_w, T)$ if $w \in \Sigma$.

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Let

$$H^1_{\mathcal{F}^+_\Sigma}(FK_\infty, T) := \varprojlim_n H^1_{\mathcal{F}^+_\Sigma}(FK_n, T), \quad H^1_{\mathcal{F}^{+,*}_\Sigma}(FK_\infty, T^*) := \varinjlim_n H^1_{\mathcal{F}^{+,*}_\Sigma}(FK_n, T^*),$$

where the projective (resp. injective) limit is taken with respect to the corestriction (resp. restriction) maps.

LEMMA 2.10. *Let $\mathcal{G}$ denote the Galois group* $\mathrm{Gal}(L_\infty/K)$. *Then, the* $\mathcal{O}[[\mathcal{G}]]$*-modules* $H^1_{\mathcal{F}^{+,*}_{can}}(L_\infty, T^*)$ *and* $H^1_{\mathcal{F}^{+,*}_\Sigma}(L_\infty, T^*)$ *are isomorphic.*

Proof. Let Σ_2 denote the set of 2-adic places. Observe that $\mathcal{F}^{+,*}_\Sigma \leq \mathcal{F}^{+,*}_{can}$, then by (2) we have an exact sequence

$$0 \longrightarrow H^1_{\mathcal{F}^{+,*}_\Sigma}(L_n, T^*) \longrightarrow H^1_{\mathcal{F}^{+,*}_{can}}(L_n, T^*) \longrightarrow \oplus_{w\in\Sigma_f-\Sigma_2} H^1_{\mathcal{F}^*_{ur}}(L_{n,w}, T^*)\ .$$

Passing to direct limit over n, the result follows from the proof of [**1**, Proposition 3.5]. □

The following proposition is crucial for our purpose.

PROPOSITION 2.11. *The* Λ*-modules* $H^1_{\mathcal{F}^{+,*}_{can}}(K_\infty, T^*)^\vee$ *and* $(H^1_{\mathcal{F}^{+,*}_{can}}(L_\infty, T^*)^\vee)_{\mathrm{Gal}(L_\infty/K_\infty)}$ *are pseudo-isomorphic.*

Before we prove this result, we need a preliminary result: For every finite Galois extension F of K, we have the exact sequence

$$0 \longrightarrow H^1_{\mathcal{F}^{+,*}_\Sigma}(F, T^*)^\vee \longrightarrow H^2_+(G_{F,\Sigma}, T) \longrightarrow \widetilde{\oplus}_{w\in\Sigma_f} H^2(F_w, T) \longrightarrow 0, \qquad (8)$$

where $\widetilde{\oplus}_{w\in\Sigma_f} H^2(F_w, T)$ denotes the kernel of the map $\oplus_{w\in\Sigma_f} H^2(F_w, T) \longrightarrow H^0(F, T^*)^\vee$.

Indeed, by dualizing the exact sequence defining the module $H^1_{\mathcal{F}^{+,*}_\Sigma}(F, T^*)$

$$0 \longrightarrow H^1_{\mathcal{F}^{+,*}_\Sigma}(F, T^*) \longrightarrow H^1(G_{F,\Sigma}, T^*) \longrightarrow \oplus_{w\in\Sigma_f} H^1(F_w, T^*),$$

we obtain the exact sequence

$$\oplus_{w\in\Sigma_f} H^1(F_w, T) \longrightarrow H^1(G_{F,\Sigma}, T^*)^\vee \longrightarrow H^1_{\mathcal{F}^{+,*}_\Sigma}(F, T^*)^\vee \longrightarrow 0.$$

Hence, the exact sequence (8) follows from Proposition 2.6.

Now, we prove the Proposition 2.11.

Proof. Let n be a nonnegative integer, and let Δ_n denote the Galois group $\mathrm{Gal}(L_n/K_n)$. Then, the exact sequence (8) induces the commutative diagram

$$\begin{array}{ccccccc}
(H^1_{\mathcal{F}^{+,*}_\Sigma}(L_n, T^*)^\vee)_{\Delta_n} & \longrightarrow & H^2_+(G_{L_n,\Sigma}, T)_{\Delta_n} & \longrightarrow & (\widetilde{\oplus}_{w\in\Sigma_f} H^2(L_{n,w}, T))_{\Delta_n} & \longrightarrow & 0 \\
\downarrow N'_n & & \wr\downarrow N_n & & \downarrow N''_n & & \\
0 \longrightarrow H^1_{\mathcal{F}^{+,*}_\Sigma}(K_n, T^*)^\vee & \longrightarrow & H^2_+(G_{K_n,\Sigma}, T) & \longrightarrow & \widetilde{\oplus}_{w\in\Sigma_f} H^2(K_{n,w}, T) & \longrightarrow & 0,
\end{array}$$

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where all vertical maps are induced by the corestriction. The one of the middle is an isomorphism by Corollary 2.5. By the snake lemma, we obtain

$$\mathrm{coker}(N_n') \cong \ker(N_n'') \quad \text{and} \quad \ker(N_n') \cong \mathrm{coker}(\alpha_n),$$

where

$$\alpha_n : H_1(\Delta_n, H^2_+(G_{L_n,\Sigma}, T)) \longrightarrow H_1(\Delta_n, \widetilde{\bigoplus}_{w\in\Sigma_f} H^2(L_{n,w}, T)).$$

The orders of the groups

$$H_0(\Delta_n, \widetilde{\bigoplus}_{w\in\Sigma_f} H^2(L_{n,w}, T)) \quad \text{and} \quad H_1(\Delta_n, \widetilde{\bigoplus}_{w\in\Sigma_f} H^2(L_{n,w}, T)),$$

are bounded independently of n (cf. [**1**, Lemma 3.7]). Therefore, the Λ-modules $H^1_{\mathcal{F}^{+,*}_{can}}(K_\infty, T^*)^\vee$ and $(H^1_{\mathcal{F}^{+,*}_{can}}(L_\infty, T^*)^\vee)_{\mathrm{Gal}(L_\infty/K_\infty)}$ are pseudo-isomorphic. This finishes the proof. □

For a nonnegative integer n, let A_n^+ denote the 2-part of the narrow class group of L_n, and let

$$A_\infty^+ := \varprojlim_n A_n^+,$$

where the injective limit is taken with respect to the norm maps.

Proposition 2.12. *If one of the hypotheses $\mathcal{H}_2$ or $\mathcal{H}_3$ holds, then*

$$\mathrm{char}((A_\infty^+)_\chi) \quad \textit{divides} \quad \mathrm{char}(H^1_{\mathcal{F}^{+,*}_{can}}(K_\infty, T^*)^\vee).$$

Proof. Consider the exact sequence

$$H^1_{\mathcal{F}^*_{ur}}(L_{n,2}, T^*)^\vee \longrightarrow H^1_{\mathcal{F}^{+,*}_{ur}}(L_n, T^*)^\vee \longrightarrow H^1_{\mathcal{F}^{+,*}_{can}}(L_n, T^*)^\vee \longrightarrow 0.$$

Since

$$H^1_{\mathcal{F}^*_{ur}}(L_{n,2}, T^*) \cong \oplus_{w|2}\mathrm{Hom}(G_w/I_w, T^*),$$

we obtain

$$H^1_{\mathcal{F}^*_{ur}}(L_{n,2}, T^*)^\vee \cong \oplus_{v|2}\mathcal{O}(\chi^{-1})[\mathrm{Gal}(L_n/K)/D_v(L_n/K)].$$

Passing to the projective limit and taking the Δ-co-invariants, we get

$$(\mathcal{O}(\chi^{-1})[\mathcal{G}/D_v(L_\infty/K)])_\Delta \simeq \begin{cases} \text{finite}, & \text{if } \chi(D_v(L/K)) \neq 1; \\ \mathcal{O}[\mathrm{Gal}(K_\infty/K)/D_v(K_\infty/K)], & \text{if } \chi(D_v(L/K)) = 1, \end{cases}$$

where $\Delta = \mathrm{Gal}(L_\infty/K_\infty)$. Using Proposition 2.11 and Lemma 2.8, we obtain

$$\mathrm{char}((A_\infty^+)_\chi) \quad \text{divides} \quad \mathcal{J}^s \cdot \mathrm{char}(H^1_{\mathcal{F}^{+,*}_{can}}(K_\infty, T^*)^\vee),$$

where $\mathcal{J}$ is the augmentation ideal of Λ and $s = \#\{v \mid 2\,;\ \chi(\mathrm{Frob}_v) = 1\}$. Since L is totally real, the characteristic ideal $\mathrm{char}((A_\infty)_\chi)$ is prime to $\mathcal{J}$, by Leopoldt conjecture.

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The exact sequence

$$\oplus_{v|\infty} H^1_{Iw}(K_v, T) \longrightarrow H^1_{\mathcal{F}^{+,*}_{ur}}(K_\infty, T^*)^\vee \longrightarrow H^1_{\mathcal{F}^*_{ur}}(K_\infty, T^*)^\vee \longrightarrow 0$$

and Lemma 2.8 show that

$$\mathrm{char}((A^+_\infty)_\chi) \mid \mathrm{char}((A_\infty)_\chi) \cdot \mathrm{char}(\oplus_{v|\infty} H^1_{Iw}(K_v, T)),$$

where $H^1_{Iw}(K_v, T) := \varprojlim_n (\oplus_{w|v} H^1(K_{n,w}, T))$. Since v in an infinite prime, $2H^1_{Iw}(K_v, T) = 0$ and then the characteristic ideal $\mathrm{char}(\oplus_{v|\infty} H^1_{Iw}(K_v, T))$ is prime to $\mathcal{J}$. Hence, $\mathrm{char}((A^+_\infty)_\chi)$ is prime to $\mathcal{J}$. This permits to conclude. □

To obtain some information about the Λ-structure of $H^1_{\mathcal{F}^+_{can}}(K_\infty, T)$, we need some facts from universal norms in $\mathbb{Z}_2$-extension [**19**, **20**]. Let

$$H^1_{Iw}(K, .) := \varprojlim_n H^1(G_{K_n,\Sigma}, .) \quad \text{and} \quad H^1_{Iw,+}(K, .) := \varprojlim_n H^1_+(G_{K_n,\Sigma}, .).$$

The next proposition is a first step towards Theorem 2.14, which claims that the Λ-modules $H^1_{\mathcal{F}^+_{can}}(K_\infty, T)$ and $H^1_{\mathcal{F}_{can}}(K_\infty, T)$ are Λ-free.

PROPOSITION 2.13. *There are canonical isomorphisms*
(1) $H^1_{\mathcal{F}_{can}}(K_\infty, T) \cong H^1_{Iw}(K, T)$.
(2) $H^1_{\mathcal{F}^+_{can}}(K_\infty, T) \cong H^1_{Iw,+}(K, T)$.

Proof. Let n be a nonnegative integer. By definition, we have the exact sequence

$$0 \longrightarrow H^1_{\mathcal{F}_{can}}(K_n, T) \longrightarrow H^1(G_{K_n,\Sigma}, T) \longrightarrow \bigoplus_{w \in \Sigma_f - \Sigma_2} H^1(K_{n,w}, T)/H^1_{\mathcal{F}_{ur}}(K_{n,w}, T).$$

Passing to projective limit over n, Assertion (1) follows from the proof of [**1**, Proposition 3.5]. In order to obtain (2), we have on the one hand the exact sequence

$$0 \longrightarrow H^1_{\mathcal{F}^+_{can}}(K_n, T) \longrightarrow H^1_{\mathcal{F}_{can}}(K_n, T) \longrightarrow \oplus_{w|\infty} H^1(K_{n,w}, T).$$

On the other hand, by Proposition 2.4 we have an exact sequence

$$\oplus_{w|\infty} H^0(K_{n,w}, T) \longrightarrow H^1_+(G_{K_n,\Sigma}, T) \longrightarrow H^1(G_{K_n,\Sigma}, T) \longrightarrow \oplus_{w|\infty} H^1(K_{n,w}, T). \quad (9)$$

Since $T = \mathbb{Z}_2(1) \otimes \mathcal{O}(\chi^{-1})$ and w is a real place, we have $H^0(K_{n,w}, T) = 0$. Hence,

$$H^1_{\mathcal{F}^+_{can}}(K_\infty, T) \cong H^1_{Iw,+}(K, T).$$

□

We will need the following isomorphism: For any Galois extension F/F' of number fields, $K \subset F' \subset F$, the restriction map

$$\mathrm{res} : H^1(F', T) \xrightarrow{\sim} H^1(F, T)^{\mathrm{Gal}(F/F')} \quad (10)$$

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induces an isomorphism. Indeed, since χ has finite order, we can assume that $\chi(G_F) = 1$. Then,

$$\begin{aligned} T^{G_F} &= (\mathbb{Z}_2(1) \otimes \mathcal{O}(\chi^{-1}))^{G_F} \\ &= \mathbb{Z}_2(1)^{G_F} \otimes \mathcal{O}(\chi^{-1}) \end{aligned}$$

is trivial. Hence, the inflation–restriction exact sequence

$$0 \longrightarrow H^1(F/F', T^{G_F}) \longrightarrow H^1(F', T) \longrightarrow H^1(F, T)^{\mathrm{Gal}(F'/F)} \longrightarrow H^1(F/F', T^{G_F})$$

gives isomorphism (10).

For an $\mathcal{O}$-module M, let

$\mathrm{Tor}_{\mathcal{O}}(M)$ is the torsion submodule and

$\mathrm{Fr}_{\mathcal{O}}(M) = M/\mathrm{Tor}_{\mathcal{O}}(M)$ is the maximal torsion-free quotient of M.

THEOREM 2.14. *The Λ-modules $H^1_{\mathcal{F}^+_{can}}(K_\infty, T)$ and $H^1_{\mathcal{F}_{can}}(K_\infty, T)$ are Λ-free.*

Proof. By Proposition 2.13, it suffices to prove that the Λ-modules $H^1_{Iw}(K, T)$ and $H^1_{Iw,+}(K, T)$ are Λ-free. For this, we claim that

(i) the groups $H^1(\Gamma_n, \mathcal{H}^1_+(K_\infty, T))$ and $H^1(\Gamma_n, \mathcal{H}^1(K_\infty, T))$ are finite, and $\mathrm{Tor}_{\mathcal{O}}(\mathcal{H}^1_+(K_\infty, T)) = 0$ and $\mathrm{Tor}_{\mathcal{O}}(\mathcal{H}^1(K_\infty, T)) = 0$,

(ii) the groups $H^1(\Gamma_n, \mathrm{Fr}_{\mathcal{O}}(\mathcal{H}^1_+(K_\infty, T)))$ and $H^1(\Gamma_n, \mathrm{Fr}_{\mathcal{O}}(\mathcal{H}^1(K_\infty, T)))$ are cofinitely generated $\mathcal{O}$-modules,

where Γ_n denotes the Galois group $\mathrm{Gal}(K_\infty/K_n)$, and

$$\mathcal{H}^1(K_\infty, .) := \varinjlim_n H^1(G_{K_n,\Sigma}, .) \quad \text{and} \quad \mathcal{H}^1_+(K_\infty, .) := \varinjlim_n H^1_+(G_{K_n,\Sigma}, .).$$

Using this claim, Theorem 1.9 of [**19**] shows that the Λ-module $H^1_{Iw}(K, T)$ and $H^1_{Iw,+}(K, T)$ are free.

Proof of the claim: On the one hand, the Hochschild–Serre spectral sequence (see Proposition 2.4)

$$H^p(\Gamma_n, \mathcal{H}^q_+(K_\infty, T)) \Longrightarrow H^{p+q}_+(G_{K_n,\Sigma}, T)$$

induces the exact sequence

$$H^1(\Gamma_n, \mathcal{H}^1_+(K_\infty, T)) \hookrightarrow H^2_+(G_{K_n,\Sigma}, T) \longrightarrow \mathcal{H}^2_+(K_\infty, T)^{\Gamma_n} \longrightarrow H^2(\Gamma_n, \mathcal{H}^1_+(K_\infty, T)).$$

Since $H^2_+(G_{K_n,\Sigma}, T)$ is a finitely generated $\mathcal{O}$-module and $H^1(\Gamma_n, \mathcal{H}^1_+(K_\infty, T))$ is $\mathcal{O}$-torsion, the module $H^1(\Gamma_n, \mathcal{H}^1_+(K_\infty, T))$ is finite. On the other hand, the Hochschild–Serre spectral sequence

$$H^p(\Gamma_n, \mathcal{H}^q(K_\infty, T)) \Longrightarrow H^{p+q}(G_{K_n,\Sigma}, T)$$

shows that

$$H^1(\Gamma_n, \mathcal{H}^1(K_\infty, T)) \hookrightarrow H^2(G_{K_n,\Sigma}, T).$$

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Hence, $H^1(\Gamma_n, \mathcal{H}^1(K_\infty, T))$ is finite. Since L_n is totally real, by isomorphism (10), we get $\mathrm{Tor}_{\mathcal{O}}(H^1(G_{K_n,\Sigma}, T)) = 0$. Therefore, the exact sequence (9) shows that $\mathrm{Tor}_{\mathcal{O}}(H^1_+(G_{K_n,\Sigma}, T)) = 0$; hence,

$$\mathrm{Tor}_{\mathcal{O}}(\mathcal{H}^1_+(K_\infty, T)) = \mathrm{Tor}_{\mathcal{O}}(\mathcal{H}^1(K_\infty, T)) = 0.$$

This proves Assertion (i). Assertion (ii) is a direct consequence of (i). □

COROLLARY 2.15. *The* Λ*-modules* $H^1_{\mathcal{F}_{can}}(K_\infty, T)$ *and* $H^1_{\mathcal{F}^+_{can}}(K_\infty, T)$ *have* Λ*-rank* $[K:\mathbb{Q}]$*;*

$$\mathrm{rank}_\Lambda(H^1_{\mathcal{F}^+_{can}}(K_\infty, T)) = \mathrm{rank}_\Lambda(H^1_{\mathcal{F}_{can}}(K_\infty, T)) = [K:\mathbb{Q}].$$

Proof. Since $\mathcal{F}^+_{can} \leq \mathcal{F}_{can}$, by (2) we have an exact sequence

$$0 \longrightarrow H^1_{\mathcal{F}^+_{can}}(K_\infty, T) \longrightarrow H^1_{\mathcal{F}_{can}}(K_\infty, T) \longrightarrow \varprojlim_n(\oplus_{w|\infty} H^1(K_{n,w}, T)).$$

Then,

$$\mathrm{rank}_\Lambda(H^1_{\mathcal{F}^+_{can}}(K_\infty, T)) = \mathrm{rank}_\Lambda(H^1_{\mathcal{F}_{can}}(K_\infty, T)).$$

Using the fact that $H^1(G_{L_n,\Sigma}, \mathbb{Z}_2(1)) \cong U_\Sigma(L_n) \otimes \mathbb{Z}_2$, where $U_\Sigma(L_n)$ denotes the Σ-units of L_n, Dirichlet's unit theorem and isomorphism (10) show that

$$\mathrm{rank}_{\mathcal{O}}(H^1(G_{K_n,\Sigma}, T)) = r2^n + t,$$

where $r = [K:\mathbb{Q}]$ and t is an integer independent of n. Then, $\mathrm{rank}_\Lambda(H^1_{\mathcal{F}_{can}}(K_\infty, T)) = r$. This proves the corollary. □

3. Proof of Theorem 1.1. We will take the notations and the conventions of [**9**]. In particular, the construction of the group of Rubin–Stark units [**9**, Definition 4.5] goes on the same lines.

For a nonnegative integer n, the product of all distinct non-2-adic prime ideals dividing the finite part of the conductor of L_n/K is denoted by $\widetilde{\mathfrak{h}}$, which does not depend on n. For any ideal $\mathfrak{g} \mid \widetilde{\mathfrak{h}}$, the maximal subextension of L_n whose conductor is prime to $\widetilde{\mathfrak{h}}\mathfrak{g}^{-1}$ is denoted by $L_{n,\mathfrak{g}}$. Let us fix a finite set S of places containing all infinite places, and at least one finite place, but does not contain any 2-adic prime of K, and a second finite, nonempty set $\mathcal{T}$ of places of K, disjoint from S and does not contain any 2-adic prime of K. Let $S_{L_{n,\mathfrak{g}}} = S \cup \mathrm{Ram}(L_{n,\mathfrak{g}}/K)$, where $\mathrm{Ram}(L_{n,\mathfrak{g}}/K)$ denotes the set of ramified primes in $L_{n,\mathfrak{g}}/K$. Since $L_{n,\mathfrak{g}}$ is a totally real field, Hypotheses 2.1.1–2.1.5 in [**16**, Hypotheses 2.1] on $S_{L_{n,\mathfrak{g}}}$, $\mathcal{T}$ and r are satisfied.

Let $\mathcal{E}_n$ (resp. $\mathcal{E}^+_n$) denote the group of units (resp. totally positive units) of L_n. Following [**9**], we gave the following definition.

DEFINITION 3.1. Let n be a nonnegative integer. We denote by St^+_n the $\mathbb{Z}[\mathrm{Gal}(L_n/K)]$-module generated by the inverse images of $\varepsilon_{n,\mathfrak{g},\mathcal{T}}$ under the map

$\bigwedge^r \mathcal{E}_n^+ \longrightarrow \mathbb{Q} \otimes \bigwedge^r \mathcal{E}_n$ for all $\mathfrak{g} \mid \widetilde{\mathfrak{h}}$, where $\varepsilon_{n,\mathfrak{g},\mathcal{T}}$ is the Rubin–Stark element of the Rubin–Stark conjecture $\mathbf{RS}(L_{n,\mathfrak{g}}/K, S_{L_{n,\mathfrak{g}}}, \mathcal{T}, r)$ [**16**, Conjecture B′].

Recall that for any number field F, Kummer theory gives a canonical isomorphism

$$H^1(F, \mathbb{Z}_2(1)) \cong F^{\times,\wedge} := \varprojlim F^\times/(F^\times)^{2^n}.$$

Since $\chi(G_{L_n}) = 1$ for every $n \geq 0$,

$$H^1(L_n, \mathbb{Z}_2(1)) \otimes \mathcal{O}(\chi^{-1}) \cong H^1(L_n, \mathbb{Z}_2(1) \otimes \mathcal{O}(\chi^{-1})).$$

Therefore,

$$L_n^{\times,\wedge} \otimes \mathcal{O}(\chi^{-1}) \cong H^1(L_n, \mathbb{Z}_2(1) \otimes \mathcal{O}(\chi^{-1})). \tag{11}$$

For simplicity of notation, we let ε_n stand for the Rubin–Stark element $\varepsilon_{n,\widetilde{\mathfrak{h}},\mathcal{T}}$ for $\mathbf{RS}(L_n/K, S_{L_n}, \mathcal{T}, r)$. Remark 4.2 in [**9**] shows that ε_n can be written as $\varepsilon_1 \wedge \cdots \wedge \varepsilon_r$, with $\varepsilon_i \in \mathbb{Q} \otimes L_n^\times$ (this expression is not unique over $\mathrm{Gal}(L_n/K)$, even though ε_n is). Let

$$\varepsilon_{n,\chi} := \widehat{\varepsilon_1} \otimes 1_{\chi^{-1}} \wedge \cdots \wedge \widehat{\varepsilon_r} \otimes 1_{\chi^{-1}}, \tag{12}$$

where $\widehat{\varepsilon_i}$ is the image of ε_i by the natural map $\mathbb{Q} \otimes L_n^\times \longrightarrow \mathbb{Q}_2 \otimes_{\mathbb{Z}_2} L_n^{\times,\wedge}$. Then, under isomorphism (11), we can view each

$$\varepsilon_{n,\chi} \text{ as an element of } \mathbb{Q}_2 \otimes \bigwedge^r H^1(L_n, \mathbb{Z}_2(1) \otimes \mathcal{O}(\chi^{-1})).$$

For every $n \geq 0$, we define

$$c_n = \mathrm{cor}^{(r)}_{L_n,K_n}(\varepsilon_{n,\chi}), \tag{13}$$

where $\mathrm{cor}^{(r)}_{L_n,K_n}$ is the map

$$\mathbb{Q}_2 \otimes \bigwedge^r H^1(L_n, T) \longrightarrow \mathbb{Q}_2 \otimes \bigwedge^r H^1(K_n, T)$$

induced by the corestriction map

$$\mathrm{cor}_{L_n,K_n} : H^1(L_n, T) \longrightarrow H^1(K_n, T).$$

Let ι denote the composite of the natural maps:

$$\bigwedge^r \varprojlim_n H^1_+(K_n, T) \longrightarrow \varprojlim_n \bigwedge^r H^1_+(K_n, T) \longrightarrow \varprojlim_n (\mathbb{Q}_2 \otimes_{\mathbb{Z}_2} \bigwedge^r H^1(K_n, T)), \tag{14}$$

and let

$$\tau : \bigwedge^r \varprojlim_n H^1_+(K_n, T) \longrightarrow \bigwedge^r \varprojlim_n H^1(K_n, T).$$

By Corollary 2.15, it is clear that τ is injective.

Let $c_\infty := \{c_n\}_{n\geq 0} \in \varprojlim_n (\mathbb{Q}_2 \otimes_{\mathbb{Z}_2} \bigwedge^r H^1(K_n, T))$, where c_n is defined in (13). The collection $\{c_n\}_{n\geq 0}$ gives rise to an Euler system for the 2-adic representation $T = \mathbb{Z}_2(1) \otimes \mathcal{O}(\chi^{-1})$, in the sense of [**17**, Definition 2.1.1] (see, e.g., [**9**, Proposition 4.7]). Recall that we can associate a Kolyvagin derivative class to any Euler system for any 2-adic representation [**17**, Section 4.4]. In the sense of [**10**, Definition 3.1.3], this turn out to construct a Kolyvagin system of the canonical Selmer structure $\mathcal{F}_{can}$ [**10**, Theorem 3.2.4].

LEMMA 3.2. *Let η denote the* lcm *of the* 2*-adic numbers* $1 - \chi(\mathrm{Frob}_{\mathfrak{p}})$*, where* $\mathfrak{p}$ *run through the set of* 2*-adic place of* K*. Let* $\mathbf{c}$ *be an element in* $\iota^{-1}(\eta \cdot c_\infty)$*. Under Hypothesis* $\mathcal{H}_3$*,*

$$\mathrm{char}(H^1_{\mathcal{F}^*_{can}}(K_\infty, T^*)^\vee) \quad \textit{divides} \quad \mathrm{char}((\bigwedge^r H^1_{\mathcal{F}_{can}}(K_\infty, T))/\Lambda\tau(\mathbf{c})).$$

Proof. The proof is identical to the proof of Theorem 6.3 of [**9**] line by line. To obtain Theorem [**9**, Theorem 6.3], we proved a variant of Rubin's theorem [**17**, Theorem 2.3.3], loc. cit. [**9**, Theorem 6.1] by constructing an ad-hoc Selmer structure [**9**, Definition 5.6] and an associated Kolyvagin system [**9**, Lemma 5.13]. The construction uses the structure of $H^1_{Iw}(K(\mathfrak{r})_v, T)$ [**9**, Theorem 5.1], deduced from a result of Greither [**7**, Theorem 2.2]. Since Greither's result is also available for $p = 2$ [**6**, Proposition 2.10], the strategy used to obtain [**9**, Theorem 6.3] is also applicable for Lemma 3.2. □

For each place v of K, let

$$H^1_{Iw}(K_v, T) = \varprojlim_n (\oplus_{w|v} H^1(K_{n,w}, T)).$$

By a standard argument (see [**10**, Lemma 5.3.1]), we have

$$H^1_{Iw}(K_v, T) \cong H^1(K_v, T \otimes \Lambda).$$

Hence, Proposition 4.2.3 of [**13**] shows that $H^1_{Iw}(K_v, T)$ is a finitely generated Λ-module.

The following proposition is the key to the proof of our main theorem.

PROPOSITION 3.3. *With the assumptions of Lemma 3.2, we have*

$$\mathrm{char}(H^1_{\mathcal{F}^{+,*}_{can}}(K_\infty, T^*)^\vee) \;\; \textit{divides} \;\; \mathrm{char}((\bigwedge^r H^1_{\mathcal{F}^+_{can}}(K_\infty, T))/\Lambda\mathbf{c}) \cdot \mathrm{char}(\oplus_{v|\infty} H^1_{Iw}(K_v, T)).$$

Proof. Since $\mathcal{F}^+_{can} \leq \mathcal{F}_{can}$, we have an exact sequence

$$\begin{aligned} H^1_{\mathcal{F}^+_{can}}(K_\infty, T) &\hookrightarrow H^1_{\mathcal{F}_{can}}(K_\infty, T) \longrightarrow \oplus_{v|\infty} H^1_{Iw}(K_v, T) \longrightarrow H^1_{\mathcal{F}^{+,*}_{can}}(K_\infty, T^*)^\vee \\ &\twoheadrightarrow H^1_{\mathcal{F}^*_{can}}(K_\infty, T^*)^\vee . \end{aligned} \tag{15}$$

Corollary 2.15 shows that the Λ-modules $H^1_{\mathcal{F}^+_{can}}(K_\infty, T)$ and $H^1_{\mathcal{F}_{can}}(K_\infty, T)$ are Λ-free of rank $r = [K : \mathbb{Q}]$, and therefore, the injection $H^1_{\mathcal{F}^+_{can}}(K_\infty, T) \overset{\beta}{\hookrightarrow} H^1_{\mathcal{F}^+_{can}}(K_\infty, T)$

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induces an exact sequence:

$$0 \longrightarrow \left(\bigwedge^r H^1_{\mathcal{F}^+_{can}}(K_\infty, T)\right)/\Lambda\mathbf{c} \longrightarrow \left(\bigwedge^r H^1_{\mathcal{F}_{can}}(K_\infty, T)\right)/\Lambda\mathbf{c} \longrightarrow\!\!\!\!\!\rightarrow \operatorname{coker}(\beta^{(r)}),$$

where $\beta^{(r)}$ denotes the map induced on the rth exterior power. Using the fact that

$$\operatorname{char}(\operatorname{coker}(\beta)) = \operatorname{char}(\operatorname{coker}(\beta^{(r)}))$$

(cf. [**3**, p. 258]), we get

$$\begin{aligned} &\operatorname{char}((\bigwedge^r H^1_{\mathcal{F}^+_{can}}(K_\infty, T))/\Lambda\mathbf{c}) \cdot \operatorname{char}(\oplus_{v|\infty} H^1_{Iw}(K_v, T)) \cdot \operatorname{char}(H^1_{\mathcal{F}^*_{can}}(K_\infty, T^*)^\vee) \\ = {} &\operatorname{char}((\bigwedge^r H^1_{\mathcal{F}_{can}}(K_\infty, T))/\Lambda\mathbf{c}) \cdot \operatorname{char}(H^1_{\mathcal{F}^{+,*}_{can}}(K_\infty, T^*)^\vee). \end{aligned}$$

Lemma 3.2 permits to conclude. □

Let n be a nonnegative integer, we write A^+_n for the 2-part of the narrow class group of L_n, $\mathcal{E}'_n$ for the 2-units of L_n and $\mathcal{E}'^{,+}_n$ for the totally positive 2-units of L_n. Let

$$A^+_\infty := \varprojlim_n A^+_n, \qquad \widehat{\mathcal{E}'_\infty} := \varprojlim_n \widehat{\mathcal{E}'_n}, \qquad \widehat{\mathcal{E}'^{,+}_\infty} := \varprojlim_n \widehat{\mathcal{E}'^{,+}_n},$$

where all inverse limits are taken with respect to norm maps. It is well known that

$$\varprojlim_n H^1(G_{L_n,\Sigma}, \mathbb{Z}_2(1)) \cong \widehat{\mathcal{E}'_\infty}.$$

Since L_n is a totally real field, Proposition 2.4 leads an exact sequence

$$0 \longrightarrow H^1_+(G_{L_n,\Sigma}, \mathbb{Z}_2(1)) \longrightarrow H^1(G_{L_n,\Sigma}, \mathbb{Z}_2(1)) \longrightarrow \oplus_{w|\infty} H^1(L_{n,w}, \mathbb{Z}_2(1)).$$

Hence,

$$\varprojlim_n H^1_+(G_{L_n,\Sigma}, \mathbb{Z}_2(1)) \cong \widehat{\mathcal{E}'^{,+}_\infty}. \tag{16}$$

Recall that St^+_n denotes the $\mathbb{Z}[\mathrm{Gal}(L_n/K)]$-module constructed by the Rubin–Stark elements (see Definition 3.1). Recall also that

$$c_n = \operatorname{cor}^{(r)}_{L_n,K_n}(\varepsilon_{n,\chi})$$

denotes the element defined in (13). Let $St^+_\infty := \varprojlim_n St^+_n$, and let $\varepsilon_{\infty,\chi} := \{\varepsilon_{n,\chi}\}_{n\geq 1}$. Since for $n \geq 1$, $c_n = \operatorname{cor}^{(r)}_{L_n,K_n}(\varepsilon_{n,\chi})$, it follows that

$$\begin{aligned} \operatorname{res}^{(r)}_{K_n,L_n}(c_n) &= \operatorname{res}^{(r)}_{K_n,L_n}(\operatorname{cor}^{(r)}_{L_n,K_n}(\varepsilon_{n,\chi})) \\ &= |\Delta|^{r-1} N_\Delta(\varepsilon_{n,\chi}), \end{aligned}$$

where $N_\Delta = \sum_{\sigma\in\Delta} \sigma$. Therefore, using the fact that the restriction map

$$\operatorname{res}_{K_n,L_n} : H^1(K_n, T) \longrightarrow H^1(L_n, T)^{\mathrm{Gal}(L_n/K_n)}$$

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is an isomorphism by (10), we obtain

$$|\Delta|^{r-1}N_\Delta((\widehat{St^+_\infty})_\chi) = \Lambda c,$$

where c is an element in the inverse image of $|\Delta|^{r-1}N_\Delta(\varepsilon_{\infty,\chi})$ under the composite map:

$$\bigwedge^r \varprojlim_n H^1_+(K_n, T) \longrightarrow \varprojlim_n \bigwedge^r H^1_+(K_n, T) \longrightarrow \varprojlim_n (\mathbb{Q}_2 \otimes_{\mathbb{Z}_2} \bigwedge^r H^1(K_n, T)).$$

Using Proposition 2.13 and isomorphisms (10) and (16), we get

$$H^1_{\mathcal{F}^+_{can}}(K_\infty, T) \cong (\widehat{\mathcal{E}'^{,+}_\infty} \otimes \mathcal{O}(\chi^{-1}))^{\mathrm{Gal}(L_\infty/K_\infty)}. \tag{17}$$

Proof of Theorem 1.1. On the one hand, the commutative exact diagram

$$\begin{array}{ccccccc}
 & & (\widehat{\mathrm{St}^+_\infty})_\chi & \longrightarrow & \bigwedge^r(\widehat{\mathcal{E}^+_\infty})_\chi & \twoheadrightarrow & (\bigwedge^r \widehat{\mathcal{E}^+_\infty}/\widehat{\mathrm{St}^+_\infty})_\chi \\
 & & \Big\downarrow |\Delta|^{r-1}N_\Delta & & \Big\downarrow N^{(r)}_\Delta & & \Big\downarrow \\
0 & \longrightarrow & \Lambda c & \longrightarrow & \bigwedge^r(\widehat{\mathcal{E}^+_\infty})^\chi & \twoheadrightarrow & (\bigwedge^r(\widehat{\mathcal{E}^+_\infty})^\chi)/\Lambda c \\
 & & & & \Big\downarrow & & \\
 & & & & \mathrm{coker}(N^{(r)}_\Delta) & &
\end{array}$$

shows that

$$\mathrm{char}((\bigwedge^r(\widehat{\mathcal{E}^+_\infty})^\chi)/\Lambda c) \quad \text{divides} \quad \mathrm{char}\left((\bigwedge^r \widehat{\mathcal{E}^+_\infty}/\widehat{\mathrm{St}^+_\infty})_\chi\right) \cdot \mathrm{char}(\mathrm{coker}(N^{(r)}_\Delta)),$$

where $(\widehat{\mathcal{E}^+_\infty})^\chi = (\widehat{\mathcal{E}^+_\infty} \otimes_{\mathbb{Z}_2} \mathcal{O}(\chi^{-1}))^\Delta$. On the other hand, isomorphism (17) and Propositions 3.3 and 2.12 show that

$$\mathrm{char}((A^+_\infty)_\chi) \quad \text{divides} \quad \mathrm{char}((\bigwedge^r(\widehat{\mathcal{E}'^{,+}_\infty})^\chi)/\Lambda \mathbf{c}) \cdot \mathrm{char}(\oplus_{v|\infty}(H^1_{Iw}(K_v, T))),$$

where $\mathbf{c}$ is the element appearing in Proposition 3.3. Since $\chi(D_v(L/K)) \neq 1$ for any 2-adic prime of K, we get

$$(\widehat{\mathcal{E}'^{,+}_\infty} \otimes_{\mathbb{Z}_2} \mathcal{O}(\chi^{-1}))^\Delta \cong (\widehat{\mathcal{E}^+_\infty} \otimes_{\mathbb{Z}_2} \mathcal{O}(\chi^{-1}))^\Delta.$$

Hence, using the fact that $\iota(\mathbf{c}) = \eta \cdot \iota(c)$ (ι is the map (14)), we obtain

$$\mathrm{char}((A^+_\infty)_\chi) \quad \text{divides} \quad \lambda \cdot \mathrm{char}\big(((\bigwedge^r \widehat{\mathcal{E}^+_\infty})/\widehat{\mathrm{St}^+_\infty})_\chi\big),$$

where

$$\lambda = \eta \cdot \mathrm{char}(\mathrm{coker}(N^{(r)}_\Delta)) \cdot \mathrm{char}(\oplus_{v|\infty}(H^1_{Iw}(K_v, T))), \tag{18}$$

and η is the lcm of the 2-adic numbers $1 - \chi(\mathrm{Frob}_\mathfrak{p})$, where $\mathfrak{p}$ runs through the set of 2-adic place of K.

REMARK 3.4. The cokernel of the morphism

$$(\widehat{\mathcal{E}_\infty^+})_\chi \xrightarrow{N_\Delta} (\widehat{\mathcal{E}_\infty^+})^\chi$$

is isomorphic to $\widehat{H}^0(\Delta, \widehat{\mathcal{E}_\infty^+} \otimes_{\mathbb{Z}_2} \mathcal{O}(\chi^{-1}))$, where $\widehat{H}^0(.,.)$ denotes the modified Tate cohomology group. The module $\mathrm{coker}(N_\Delta^{(r)})$ is then a finitely generated torsion Λ-module, annihilated by $|\Delta|$. Hence, the characteristic ideal $\mathrm{char}(\mathrm{coker}(N_\Delta^{(r)}))$ is a power of 2. By a standard argument (see, e.g., [**10**, Lemma 5.3.1]), we have

$$\oplus_{v|\infty} H^1_{Iw}(K_v, T) \cong \oplus_{v|\infty} H^1(K_v, T \otimes \Lambda).$$

Moreover, as the absolute Galois group of the field $K_v = \mathbb{R}$ is cyclic of order 2, using the cohomology of cyclic groups, we show that

$$H^1(K_v, T \otimes \Lambda) \cong \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} (T \otimes_{\mathbb{Z}_2} \Lambda).$$

Therefore,

$$\lambda = 2^r \cdot \eta \cdot \mathrm{char}(\mathrm{coker}(N_\Delta^{(r)}))$$

is a power of 2, where $r = [K : \mathbb{Q}]$.

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