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C^*-Algebras and Factorization Through Diagonal Operators

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Abstract. Let $\mathcal{A}$ be a C^*-algebra and E be a Banach space with the Radon-Nikodym property. We prove that if j is an embedding of E into an injective Banach space then for every absolutely summing operator $T\colon \mathcal{A} \to E$, the composition $j \circ T$ factors through a diagonal operator from l^2 into l^1. In particular, T factors through a Banach space with the Schur property. Similarly, we prove that for $2 < p < \infty$, any absolutely summing operator from $\mathcal{A}$ into E factors through a diagonal operator from l^p into l^2.

1 Introduction

Diagonal operators between l^p-spaces are probably the most well-understood among the many classes of operators involved in the theory of Banach spaces. Indeed, many ideals of operators in the literatures are based on factorizations through diagonal operators such as p-nuclear operators. It is well known that if E is a Banach space with the Radon Nikodyn property (RNP) then every absolutely summing operator from any $C(K)$-spaces into E is nuclear. Recognizing that $C(K)$-spaces are C^*-algebras, one may wonder if such permanent property extends to operators on C^*-algebras. This note is an attempt to isolate permanent properties of absolutely summing operators when the domain spaces are extended to include (non-commutative) C^*-algebras in general. Since absolutely summing operators from C^*-algebras are not necessarily integral operators, one should not expect that the result for $C(K)$-spaces would extend to non-commutative C^*-algebras. In fact, it is not even clear if such operators are p-nuclear for $p > 1$. The Hilbert space case was settled in [7] where it was shown that any absolutely summing operator from a C^*-algebra into a Hilbert space factors through a Hilbert space operator belonging to the 4-Schatten von Neumann class. Our main result is Theorem 3.1 below, which roughly states that any absolutely summing operator from a C^*-algebra into a Banach space with the (RNP) factors through a diagonal operators from l^2 into l^1 when viewed as an operator into an injective space. Our proof is based on a factorization technique proved in [8] along with basic properties of nuclear and integral operators.

Our terminology and notation are standard as may be found in [1, 3] for Banach spaces, [4, 11] for C^*-algebras and operator algebras.

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2 Preliminary Definitions

In this section, we recall some definitions.

Definition 2.1 Let X and Y be Banach spaces and $0 < p < \infty$. An operator $T\colon X \to Y$ is said to be *p-summing* if there is a constant C such that for any finite sequence $\{x_i\}_{i=1}^n$ of X, one has

$$\Big(\sum_{i=1}^n \|Tx_i\|^p\Big)^{\frac{1}{p}} \le C \sup\Big\{\Big(\sum_{i=1}^n |\langle x_i, x^*\rangle|^p\Big)^{\frac{1}{p}}\ ;\ x^* \in X^*, \|x^*\| \le 1\Big\}.$$

The smallest constant C for which the above inequality holds is denoted by $\pi_p(T)$ and is called the p-summing norm of T.

Definition 2.2 We say that an operator $T\colon X \to Y$ is an *integral* operator if it admits a factorization:

$$\begin{tikzcd} X \arrow[r, "iT"] \arrow[d, "\alpha"'] & Y^{**} \\ L^\infty(\mu) \arrow[r, "J"] & L^1(\mu) \arrow[u, "\beta"'] \end{tikzcd}$$

where i is the natural inclusion from Y into Y^{**}, μ is a probability measure on a compact space K, J is the natural inclusion and α and β are bounded linear operators.

We define the integral norm $i(T) := \inf\{\|\alpha\| \cdot \|\beta\|\}$ where the infimum is taken over all such factorizations.

Similarly, we shall say that T is *strictly integral* if T is integral and on the factorization above β takes its values in Y.

It is well known that integral operators are 1-summing but the converse is not true.

If $X = C(K)$ where K is a compact Hausdorff space then it is well known that every 1-summing operator from X into Y is integral. Similarly, if the range space Y is an injective Banach space, then every absolutely summing operator from X into Y is (strictly) integral.

Definition 2.3 ([6, p. 243]) Let $0 < r \le \infty$, $1 \le p, q \le \infty$ and $1+1/r \ge 1/q+1/p$. An operator $S\colon X \to Y$ is said to be (r, p, q)-nuclear if it admits a factorization:

$$\begin{tikzcd} X \arrow[r, "S"] \arrow[d, "A"'] & Y \\ l^{q'} \arrow[r, "D"] & l^p \arrow[u, "B"'] \end{tikzcd}$$

where $1/q + 1/q' = 1$ and D is a diagonal operator of the form $D((\xi_i)_i) = (\sigma_i \xi_i)_i$ with $(\sigma_i)_i \in l^r$.

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In this case, the (r, p, q)-nuclear norm is defined by

$$N_{(r,p,q)}(S) := \inf\{\|B\| \cdot \|(\sigma_i)\|_{l^r} \cdot \|A\|\}$$

where the infimum is taken over all such factorizations.

We remark that $(p, p, 1)$-nuclear corresponds to the usual p-nuclear operators. In this case, $N_{(p,p,1)}(\cdot)$ is denoted by $N_p(\cdot)$ and for $p = 1$, the nuclear norm will be denoted by $N(\cdot)$. For more details on the different properties of the ideals of operators involved, we refer to [2, 6].

The following simple fact will be needed in the sequel (see for instance, [3, Corollary 5, p. 174]).

Proposition 2.4 *If Y is a Banach space with the (RNP), then every absolutely summing operator T from any $C(K)$-space into Y is nuclear. In this case, $\pi_1(T) = N(T)$.*

More generally, the preceding proposition can be extended to strictly integral operators.

Proposition 2.5 *Let $T\colon X \to Y$ be a strictly integral operator. If Y has the (RNP) then T is nuclear with $i(T) = N(T)$.*

Proof The operator T has a factorization $T = \beta J\alpha$ where $\alpha\colon X \to L^\infty(\mu)$, $J\colon L^\infty(\mu) \to L^1(\mu)$ and $\beta\colon L^1(\mu) \to Y$ are as in the above definition. Note that J is 1-summing so $\beta J\colon L^\infty(\mu) \to Y$ is 1-summing and since $L^\infty(\mu)$ is a $C(K)$-space and Y has the (RNP), βJ (and hence T) is nuclear. ∎

We will now recall some basic facts about C^*-algebras and von Neumann algebras. Let $\mathcal{A}$ be a C^* algebra, we denote by $\mathcal{A}$ the set of Hermitian (self adjoint) elements of $\mathcal{A}$. For $x \in \mathcal{A}$ and $f \in \mathcal{A}^*$, as is customary, xf (resp., fx) denotes the element of $\mathcal{A}^*$ defined by $xf(y) = f(yx)$ (resp., $fx(y) = f(xy)$) for every $y \in \mathcal{A}$.

Definition 2.6 A von Neumann algebra is said to be *σ-finite* if it admits at most countably many orthogonal projections.

We refer to [4, 11] for some characterizations and examples of σ-finite von Neumann algebras. Of particular use in this paper is that a von Neumann algebra $\mathcal{M}$ is σ-finite if and only if there exists a faithful normal state $\varphi \in \mathcal{M}_*$.

3 The Results

The main result of this paper is Theorem 3.1 below which provides a factorization of absolutely summing operators through diagonal operators. The reader is referred to [3] for extensive exposition on Banach spaces with the Radon-Nikodym properties (RNP).

Theorem 3.1 *Let $\mathcal{A}$ be a C^*-algebra and E be a Banach space with the (RNP) and $T\colon \mathcal{A} \to E$ be an absolutely summing operator. Then for every operator $S\colon E \to L^\infty[0,1]$, the composition ST is $(2, 1, 2)$-nuclear with $N_{(2,1,2)}(ST) \le 2\|S\|\pi_1(T)$.*

For the proof, we will consider first the following particular case:

Proposition 3.2 *Let $\mathcal{M}$ be a σ-finite von Neumann algebra and E be a Banach space with the (RNP). If $T\colon \mathcal{M} \to E$ is absolutely summing and is weak* to weakly continuous and $S\colon E \to L^\infty[0,1]$ then ST is $(2,1,2)$-nuclear with $N_{(2,1,2)}(ST) \leq 2\|S\|\pi_1(T)$.*

Proof Let $\delta > 0$. From Lemma 2.3 of [8], there exists a faithful state f in $\mathcal{M}_*$ such that for every $x \in \mathcal{M}$,

$$\|Tx\| \leq 2(1+\delta)\pi_1(T)\|xf + fx\|_{\mathcal{M}_*}.$$

If $L^2(f)$ is the completion of the prehilbertian space $(\mathcal{M}, \langle\,\cdot\,,\,\cdot\,\rangle)$ where $\langle x, y\rangle = f((xy^\star + y^\star x)/2)$ then we have the following factorization:

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "T"] \arrow[d, "J"'] & E & \\
L^2(f) \arrow[r, "\theta"] & L^2(f)^* \arrow[r, "J^*"] & \mathcal{M}_* \arrow[ul, "L"']
\end{tikzcd}$$

where J is the inclusion map, $\theta(Jx) = \langle\,\cdot\,, J(x^\star)\rangle$ for every $x \in \mathcal{M}$ and

$$L((xf + fx)/2) = Tx.$$

Here we denote by $x^\star$ the adjoint of the element $x \in \mathcal{M}$. We recall that L is a well defined bounded linear map since $\{xf + fx\ ;\ x \in \mathcal{M}\}$ is dense in $\mathcal{M}_*$ and $\|L(xf + fx)\| \leq 4(1+\delta)\pi_1(T)\|xf + fx\|_{\mathcal{M}_*}$. By duality, the proposition will follow from the following lemma:

Lemma 3.3 *For every $S\colon E \to L^\infty[0,1]$, the composition JL^*S^* is 2-nuclear.*

Remark that $L^*S^*\colon L^\infty[0,1]^* \to \mathcal{M}$ so the composition $TL^*S^*\colon L^\infty[0,1]^* \to E$ is well-defined and is absolutely summing as T is absolutely summing.

Claim *$LT^*\colon E^* \to E$ is such that $LT^* = (LT^*)^* = TL^*$.*

In fact, for every e^* and f^* in E^*, we have:

$$\begin{aligned}
\langle LT^*(e^*), f^*\rangle &= \langle e^*, TL^*(f^*)\rangle \\
&= \langle e^*, LJ^*\theta JL^*(f^*)\rangle \\
&= \langle JL^*(e^*), \theta(JL^*)(f^*)\rangle \\
&= \langle JL^*(e^*), J((L^*(f^*))^\star)\rangle_{L^2(f)}.
\end{aligned}$$

By the definition of the scalar product on $L^2(f)$,

$$\langle LT^*(e^*), f^*\rangle = f\Big(\frac{L^*(e^*)L^*(f^*) + L^*(f^*)L^*(e^*)}{2}\Big)$$

which is symmetric on e^* and f^*. That is, $\langle LT^*(e^*), f^*\rangle = \langle LT^*(f^*), e^*\rangle$ and hence, $(LT^*)^* = LT^*$ and the claim is verified.

As $LT^* = TL^*$ is absolutely summing and $L^\infty[0,1]$ is injective, SLT^* is strictly integral. Consequently, its adjoint $TL^*S^*\colon L^\infty[0,1]^* \to E$ is strictly integral. Since E has the (RNP), Proposition 2.5 implies that TL^*S^* is nuclear. *A fortiori,*

$$STL^*S^*\colon L^\infty[0,1]^* \to L^\infty[0,1]$$

is nuclear.

To complete the proof of the lemma, fix $(f_n)_{n=1}^\infty$ and $(g_n)_{n=1}^\infty$ bounded sequences in $L^\infty[0,1]$ with $\lim_{n\to\infty}\|f_n\|_\infty = \lim_{n\to\infty}\|g_n\|_\infty = 0$ and $(\lambda_n)_{n=1}^\infty \in l^1$ such that

$$STL^*S^* = \sum_{n=1}^\infty \lambda_n f_n \otimes g_n.$$

For every $\xi \in L^\infty[0,1]^*$, $STL^*S^*(\xi) = \sum_{n=1}^\infty \lambda_n \langle f_n, \xi\rangle g_n$ and hence,

$$\langle STL^*S^*(\xi), \xi\rangle = \sum_{n=1}^\infty \lambda_n \langle f_n, \xi\rangle\langle g_n, \xi\rangle. \tag{3.1}$$

On the other hand, one can see that

$$\langle STL^*S^*(\xi), \xi\rangle = \|JL^*S^*(\xi)\|^2_{L^2(f)}. \tag{3.2}$$

Combining (3.1) and (3.2), we get that

$$\big\| JL^*S^*(\xi)\big\|^2_{L^2(f)} \le \frac{1}{2}\sum_{n=1}^\infty \big|\lambda_n\big| \cdot \big|\langle f_n, \xi\rangle\big|^2 + \frac{1}{2}\sum_{n=1}^\infty \big|\lambda_n\big| \cdot \big|\langle g_n, \xi\rangle\big|^2. \tag{3.3}$$

If we set $h_{2n} := f_n$ and $h_{2n+1} := g_n$, and $\alpha_{2n} = \alpha_{2n+1} = \sqrt{|\lambda_n|/2}$, then

$$\big\| JL^*S^*(\xi)\big\|^2_{L^2(f)} \le \sum_{n=1}^\infty \alpha_n^2 \big|\langle h_n, \xi\rangle\big|^2.$$

Define the operator $U\colon L^\infty[0,1]^* \to c_0$ by setting for $\xi \in L^\infty[0,1]^*$, $U(\xi) = (\langle h_n, \xi\rangle)_{n=1}^\infty$ and $D\colon c_0 \to l^2$ the diagonal operator $(a_n)_n \mapsto (\alpha_n a_n)_{n=1}^\infty$. Also if Z is the subspace of l^2 defined by $Z = \overline{\text{span}}\{(\alpha_n\langle h_n, \xi\rangle)_{n=1}^\infty, \xi \in L^\infty[0,1]^*\}$, then $(\alpha_n\langle h_n, \xi\rangle)_{n=1}^\infty \mapsto JL^*S^*(\xi)$ defines a bounded linear map from Z into $L^2(f)$. Since Z is a complemented subspace of l^2, it can be extended to a bounded linear map V from l^2 into $L^2(f)$. It is now clear that

$$\begin{tikzcd} L^\infty[0,1]^* \arrow[r, "JL^*S^*"] \arrow[d, "U"'] & L^2(f) \\ c_0 \arrow[r, "D"] & l^2 \arrow[u, "V"'] \end{tikzcd}$$

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is a commutative diagram that shows that JL^*S^* is 2-nuclear.

For the estimate of $N_{(2,1,2)}(ST)$, note that for $\varepsilon > 0$, the sequence $\{\lambda_n\}_{n=1}^{\infty}$ can be chosen so that $\sum_{n=1}^{\infty} |\lambda_n| \le N(STL^*S^*) + \varepsilon$. We now have the following estimate:

$$\begin{aligned}\sum_{n=1}^{\infty} |\lambda_n| - \varepsilon &\le \|S\| N(TL^*S^*) \\ &= \|S\| i(TL^*S^*) = \|S\| i(SLT^*) \\ &= \|S\| \pi_1(SLT^*) \le \|S\|^2 \pi_1(LT^*).\end{aligned}$$

As $LT^* = TL^*$ and $\|L\| \le 4(1+\delta)\pi_1(T)$, we deduce that

$$\sum_{n=1}^{\infty} |\lambda_n| - \varepsilon \le \|S\|^2 4(1+\delta)\pi_1(T)^2.$$

which shows that $\|(\alpha_n)\|_{\ell^2} \le \|S\| 2(1+\delta)^{1/2}\pi_1(T) + \varepsilon$. Taking infimum over δ and ε, we conclude that $N_{(2,1,2)}(ST) \le 2\|S\|\pi_1(T)$. The proof of the proposition is complete. ■

For the proof of Theorem 3.1, it is enough to reduce the general case to the particular case of Proposition 3.2. Recall that any C^*-algebra can be considered as a concrete C^*-algebra via its universal representation ([11, Theorem 2.4]) and its second dual can be identified with its universal enveloping von Neumann algebra. Denote by $\mathcal{M}$ the universal enveloping von Neumann algebra of $\mathcal{A}$.

Proposition 3.4 *There exists a countably decomposable projection p in $\mathcal{M}$ such that $T^{**}(x^{**}) = T^{**}(px^{**}p)$ for every $x^{**} \in \mathcal{A}^{**} = \mathcal{M}$.*

Since E has the (RNP), it has the compact range property. From [9], the operator T is compact and therefore $T^*(E^*)$ is separable. Let $\{\varphi_n\}_{n=1}^{\infty}$ be a countable dense subset of $T^*(E^*)$.

Lemma 3.5 *There exists a countably decomposable projection $p \in \mathcal{M}$ such that for all $n \ge 1$, $p\varphi_n = \varphi_n p = \varphi_n$.*

Fix an orthogonal family of cyclic projections $\{e_\alpha\}_{\alpha\in I}$ in $\mathcal{M}$ such that $\mathbf{1} = \bigvee_{\alpha \in I} e_\alpha$ (see for instance, [5, Proposition 5.5.9]). For each $n \in \mathbb{N}$ and $\varepsilon > 0$, set

$$E_{n,\varepsilon} := \{\alpha \in I\,;\, \|e_\alpha \varphi_n\| > \varepsilon\} \text{ and } E_n := \{\alpha \in I\,;\, \|e_\alpha\varphi_n\| \ne 0\}.$$

Claim *$E_{n,\varepsilon}$ is finite (hence E_n is countable).*

To see this, assume that $E_{n,\varepsilon}$ is infinite. Then there exists an infinite sequence $\{e_k\}_{k=1}^{\infty}$ in $\{e_\alpha\}_{\alpha\in I}$ such that $\|e_k\varphi_n\| \ge \varepsilon$ for all $k \in \mathbb{N}$. If J is a finite subset of $\mathbb{N}$, then

$$\begin{aligned}\Big\|\sum_{k\in J} e_k\varphi_n\Big\| &= \Big\|\Big(\sum_{k\in J} e_k\Big)\varphi_n\Big\| \\ &= \Big\|\Big(\bigvee_{k\in J} e_k\Big)\varphi_n\Big\| \le \|\varphi_n\|.\end{aligned}$$

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So $\|\sum_{k\in J} e_k\varphi_n\| \leq \|\varphi_n\|$ (a constant independent of J) which shows that $\sum_{k=1}^{\infty} e_k\varphi_n$ is a weakly unconditionally Cauchy (w.u.c.) series in $\mathcal{A}^* = \mathcal{M}_*$ but since $\mathcal{M}_*$ does not contain any copies of c_0, $\sum_{k=1}^{\infty} e_k\varphi_n$ is unconditionally convergent and hence $\lim_{k\to\infty} \|e_k\varphi_n\| = 0$ (see for instance, [1, p. 45]). This is in contradiction with the assumption that $\|e_k\varphi_n\| \geq \varepsilon$ for all $k \in \mathbb{N}$. We have proved that $E_{n,\varepsilon}$ is finite. It is clear that $E_n = \bigcup_{k\in\mathbb{N}} E_{n,k^{-1}}$ so it is at most countable. The claim is verified.

Similarly, if $R_n = \{\alpha \in I, \|\varphi_n e_\alpha\| \neq 0\}$ then R_n is at most countable.

Let $C = \bigcup_{n=1}^{\infty}(R_n \cup E_n)$. The set C is at most countable and if we set $p := \bigvee_{\alpha\in C} e_\alpha$, then p is a union of a countable family of disjoint cyclic projections in $\mathcal{M}$ so p is countably decomposable in $\mathcal{M}$ ([5, Proposition 5.5.19]). Moreover, the construction of p implies that $p\varphi_n = \varphi_n p = \varphi_n$ for all $n \geq 1$. The lemma is proved.

To complete the proof of the proposition, let $x^{**} \in \mathcal{A}^{**} = \mathcal{M}$ and $e^* \in E^*$. Fix an ultrafilter $\mathcal{U}$ in $\mathbb{N}$ such that $T^*(e^*) = \lim_{n,\mathcal{U}} \varphi_n$.

$$\begin{aligned}\langle T^{**}(x^{**}), e^*\rangle &= \langle x^{**}, T^*(e^*)\rangle \\ &= \lim_{n,\mathcal{U}} \langle x^{**}, \varphi_n\rangle.\end{aligned}$$

As $\varphi_n p = p\varphi_n = \varphi_n$ for all $n \geq 1$, we have

$$\begin{aligned}\langle T^{**}(x^{**}), e^*\rangle &= \lim_{n,\mathcal{U}} \langle x^{**}, p\varphi_n p\rangle \\ &= \lim_{n,\mathcal{U}} \langle p x^{**} p, \varphi_n\rangle \\ &= \langle p x^{**} p, T^*(e^*)\rangle\end{aligned}$$

which shows that $\langle T^{**}(x^{**}), e^*\rangle = \langle T^{**}(p x^{**} p), e^*\rangle$ and as e^* is arbitrary, the proposition follows.

To complete the proof, note that since p is countably decomposable, the von Neumann algebra $p\mathcal{M}p$ is σ-finite and the following commutes:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\ T\ } & E \\ {\scriptstyle j}\downarrow & & \uparrow{\scriptstyle T^{**}|_{p\mathcal{M}p}} \\ \mathcal{A}^{**} = \mathcal{M} & \xrightarrow{\ Q\ } & p\mathcal{M}p \end{array}$$

where j is the natural inclusion and $Q(x^{**}) = p x^{**} p$ for all $x^{**} \in \mathcal{A}^{**}$. It is clear that $T^{**}|_{p\mathcal{M}p}$ satisfies the conditions of Proposition 3.2. ■

For the next extension, we refer to [12] for definitions and examples of JB^*-triples and JBW^*-triples.

Corollary 3.6 *Let $\mathcal{A}$ be a JB^*-triple and E be a Banach space with the (RNP). If $T\colon \mathcal{A} \to E$ is absolutely summing operator then for every $S\colon E \to L^\infty[0,1]$, ST is $(2,1,2)$-nuclear.*

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Proof Let $T\colon \mathcal{A} \to E$ and $S\colon E \to L^\infty[0,1]$ as in the statement, $\mathcal{A}^{**}$ is a JBW^*-triples. Since JBW^*-triples are (as Banach space) isometric to complemented subspaces of von Neumann algebras, it follows that ST is $(2,1,2)$-nuclear. ∎

Remarks 3.7

(i) In Theorem 3.1, the space $L^\infty[0,1]$ can be replaced by any injective Banach spaces.

(ii) In general, the operator T itself cannot be $(2,1,2)$-nuclear. In fact, it is enough to consider the Hilbert space. If every absolutely summing operator from $\mathcal{A}$ to l^2 were to be $(2,1,2)$-nuclear then it factors through a Hilbert-Schmidt operator. An example was provided in [7] to show that this is not the case in general.

It was noted in [7] that in general, absolutely summing operators from C^*-algebras into Hilbert space are not L_1-factorable. Theorem 3.1 also implies the following:

Corollary 3.8 *Let $\mathcal{A}$ be a C^*-algebra and E be a Banach space with the (RNP). Every absolutely summing operator from $\mathcal{A}$ into E factors through a subspace of l^1.*

One can refine the argument given in the proof of Proposition 3.2 to get the following stronger result:

Theorem 3.9 *Let $\mathcal{A}$ be a C^*-algebra and E be a Banach space with the (RNP) and $T\colon \mathcal{A} \to E$ be an absolutely summing operator. Then for every operator $S\colon E \to L^\infty[0,1]$ and every $1 < p < \infty$, the composition ST is $(p,1,p)$-nuclear.*

The theorem can be deduced from the following two results due to Saphar [10] and Lewis-Stegall [3, p. 66], respectively.

Theorem 3.10 ([10, Theorem 8]) *For $1 < p < \infty$, every compact operator from l^1 into l^2 is p-nuclear.*

Theorem 3.11 *Every representable operator from $L^1[0,1]$ into any given Banach space factors through l^1.*

Sketch of the proof of Theorem 3.9 It is enough to verify that for $1 < p < \infty$, $(ST)^*$ is p-nuclear. Since $(ST)^*$ is 2-nuclear, there exist compact operators $U\colon L^\infty[0,1]^* \to c_0$, a diagonal operator $D\colon c_0 \to l^2$ and $V: l^2 \to \mathcal{A}^*$ such that $(ST)^* = VDU$. Since U is compact, by Theorem 3.11, it factors through l^1, that is there are $U_1\colon L^\infty[0,1]^* \to l^1$ and $U_2: l^1 \to c_0$ such that $U = U_2U_1$. By Theorem 3.10, VDU_2 is p-nuclear and hence $(ST)^*$ is p-nuclear. ∎

Our final result provides a factorization of absolutely summing operators without embedding the range space into an injective space.

Corollary 3.12 *Let $\mathcal{A}$ be a C^*-algebra and E be a Banach space with the (RNP) and $T\colon \mathcal{A} \to E$ be an absolutely summing operator. Then for every $2 < p < \infty$, T factors through a diagonal operator from l^p into l^2.*

Proof We can assume that E is separable. Let j be an embedding of E into $L^\infty[0,1]$. Let $2 < p < \infty$. From Theorem 3.9, there exists a diagonal operator $D\colon l^p \to l^1$ such that jT factors through D. That is, there exist operators $A\colon \mathcal{A} \to$ and $B\colon l^1 \to L^\infty[0,1]$ such that $jT = BDA$. Fix $(\alpha_n)_{n=1}^\infty$ such that $D((a_n)_n) = (\alpha_n a_n)_n$ for all $(a_n)_n \in l^p$. Clearly, $(\alpha_n)_n \in l^q$ where $1/p + 1/q = 1$. For $1/2 = 1/p + 1/r$, let $\gamma_n := |\alpha_n|^{q/r}$ for all $n \geq 1$. It is clear that $(\gamma_n)_n \in l^r$ so a diagonal operator $D_0 \to l^p \to l^2$ defined by $D((a_n)_n) = (\gamma_n a_n)_n$ for all $(a_n)_n \in l^p$ is a well defined bounded operator and $jT = CD_0A$ for some operator $C \to l^2 \to L^\infty[0,1]$. Since $X = C^{-1}(j(E))$ is complemented in l^2 and $C|_X$ has its range in $j(E)$, one can define an operator $S \to l^2 \to E$ such that $T = SD_0A$. The proof is complete. ■

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