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A report on an ergodic dichotomy

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Abstract. We establish (some directions of) a *Ledrappier correspondence* between Hölder cocycles, Patterson–Sullivan measures, etc for word-hyperbolic groups with metric-Anosov Mineyev flow. We then study Patterson–Sullivan measures for ϑ-Anosov representations over a local field and show that these are parameterized by the ϑ-*critical hypersurface* of the representation. We use these Patterson–Sullivan measures to establish a dichotomy concerning directions in the interior of the ϑ-limit cone of the representation in question: if u is such a half-line, then the subset of u-*conical limit points* has either total mass if $|\vartheta| \leq 2$ or zero mass if $|\vartheta| \geq 4$. The case $|\vartheta| = 3$ remains unsettled.

Key words: Anosov representations, Patterson–Sullivan measures, skew-products over subshifts
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1. *Introduction*

Let G be the real points of a semi-simple real-algebraic group of the non-compact type. The (Riemannian) globally symmetric space X associated to G is non-positively curved, its visual boundary $\partial_\infty\mathsf{X}$ is a union of compact G-orbits, parameterized by directions in a (fixed beforehand) closed Weyl chamber $\mathfrak{a}^+$ of $\mathfrak{g}$. The G-orbit associated to a direction $\mathsf{u} \in \mathbb{P}(\mathfrak{a}^+)$ is G-equivariantly identified with the flag space $\mathcal{F}_{\vartheta_\mathsf{u}}$ of G, where ϑ_u is the subset of simple roots that do not vanish on u.

Let now Γ be a finitely generated group and $\rho : \Gamma \to \mathsf{G}$ a representation with discrete image. A fundamental object of study is the *limit set* L_ρ of $\rho(\Gamma)$ on the visual boundary $\partial_\infty\mathsf{X}$, defined as the set of accumulation points of an (any) orbit $\rho(\Gamma) \cdot o$ on the natural compactification $\mathsf{X} \cup \partial_\infty\mathsf{X}$.

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When $\rho(\Gamma)$ is Zariski-dense, this object has the following topological description by Benoist [7]: the action of $\rho(\Gamma)$ on each flag space $\mathcal{F}_\vartheta$ has a smallest closed invariant set, called the *limit set on* $\mathcal{F}_\vartheta$ and denoted by $\mathrm{L}_\rho^\vartheta$; additionally, one has the *limit cone* $\mathscr{L}_\rho \subset \mathfrak{a}^+$ of $\rho(\Gamma)$, defined as the subset of $\mathfrak{a}^+$ of accumulation points of sequences of the form

$$t_n a(\rho(\gamma_n)),$$

where $t_n \in \mathbb{R}_+$ converges to 0, $\gamma_n \in \Gamma$ goes to infinity, and $a : \mathsf{G} \to \mathfrak{a}^+$ is the *Cartan projection*. It is a convex cone with non-empty interior and the limit set $\mathrm{L}_{\rho(\Gamma)}$ on $\partial_\infty \mathsf{X}$ is the 'fibration' over $\mathbb{P}(\mathscr{L}_\rho)$, whose fiber over a given direction $\mathsf{u} \in \mathbb{P}(\mathscr{L}_\rho)$ is the limit set $\mathrm{L}_\rho^{\vartheta_\mathsf{u}}$ of $\rho(\Gamma)$ on $\mathcal{F}_{\vartheta_\mathsf{u}}$.

Inspired by Sullivan's [68] work on the rank-1 case, one may seek to distinguish the subset of *conical points* of L_ρ, that is, points on the limit set that are approached in a uniform manner by elements of the orbit $\rho(\Gamma) \cdot o$. However, the definition of uniform depends on:

- the type of G-orbit the point lies in—a point $x \in \mathrm{L}_\rho^\vartheta$ is *conical* if there exists a (to be called *conical*) sequence $\{\gamma_n\} \subset \Gamma$ converging to x such that for every $y \in \mathrm{L}_\rho^{\mathrm{i}\,\vartheta}$ (here we let $\mathrm{i} : \mathfrak{a} \to \mathfrak{a}$ be the opposition involution and $\mathrm{i}\,\vartheta := \vartheta \circ \mathrm{i}$) in general position with x, the sequence $\gamma_n^{-1}(y, x)$ has compact closure on the space of pairs of flags in general position $\mathcal{F}_\vartheta^{(2)}$;
- the specific direction $\mathsf{u} \in \mathbb{P}(\mathscr{L}_\rho)$, associated to the given point—fix a norm $\| \ \|$ on $\mathfrak{a}$ and define the *tube of size* $r > 0$ as the r-tubular neighborhood

  $$\mathbb{T}_r(\mathsf{u}) = \{v \in \mathfrak{a} : B(v, r) \cap \mathsf{u} \neq \emptyset\},$$

  then $x \in \mathrm{L}_\rho^{\vartheta_\mathsf{u}}$ is u-*conical* if there exists $r > 0$ and a conical sequence $\gamma_n \to x$ such that for all n, one has

  $$a(\rho(\gamma_n)) \in \mathbb{T}_r(\mathsf{u}).$$

A measurable description has been recently established by Burger *et al* [20] for u-conical points of Zariski-dense subgroups: under some extra assumptions, the Patterson–Sullivan measure associated to the direction u charges totally the subset of u-conical points if and only if G has rank ≤ 3; if rank $\mathsf{G} \geq 4$, then the subset of u-conical points has zero mass.

In this paper, we will also study a measurable description of u-conical limit points, but for general Anosov representations, a class introduced by Labourie [43] for fundamental groups of closed negatively curved manifolds and generalized by Guichard and Wienhard [35] for arbitrary (finitely generated) word-hyperbolic groups. Thanks to the recent work by Kapovich, Leeb, and Porti [40] (see also those of Bochi, Potrie, and Sambarino [10] and Guéritaud *et al* [34]), we can define them as follows, see §5.2.

Definition 1.0.1. Let $\vartheta \subset \Delta$ be a non-empty subset of simple roots and denote by $| \ |$ the word length on Γ for some (fixed) symmetric generating set. A representation $\rho : \Gamma \to \mathsf{G}$ is ϑ-*Anosov* if there exist positive constants c, μ such that for all $\gamma \in \Gamma$ and $\sigma \in \vartheta$, one has

$$\sigma(a(\rho(\gamma))) \geq \mu|\gamma| - c.$$

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A key feature of a ϑ-Anosov representation ρ is that Γ is necessarily word-hyperbolic and there exist continuous ρ-equivariant limit maps (Proposition 5.2.3) defined on its Gromov boundary,

$$\xi^{\vartheta} : \partial\Gamma \to \mathcal{F}_{\vartheta},$$
$$\xi^{\mathrm{i}\,\vartheta} : \partial\Gamma \to \mathcal{F}_{\mathrm{i}\,\vartheta},$$

such that the flags $\xi^{\mathrm{i}\,\vartheta}(x)$ and $\xi^{\vartheta}(y)$ are in general position whenever $x \neq y$.

We begin by studying the Patterson–Sullivan theory for these groups. Fix then $\vartheta \subset \Delta$, let

$$\mathfrak{a}_{\vartheta} = \bigcap_{\sigma \in \Delta - \vartheta} \ker \sigma$$

be the center of the associated Levi group and let $p_{\vartheta} : \mathfrak{a} \to \mathfrak{a}_{\vartheta}$ be the projection invariant under the subgroup of the Weyl group point-wise fixing $\mathfrak{a}_{\vartheta}$ (see §4.2). The dual space $(\mathfrak{a}_{\vartheta})^*$ sits naturally as the subspace of $\mathfrak{a}^*$ of p_{ϑ}-invariant linear forms. It is spanned by the fundamental weights of the elements in ϑ:

$$(\mathfrak{a}_{\vartheta})^* = \langle \{\varpi_{\sigma}|\mathfrak{a}_{\vartheta} : \sigma \in \vartheta\}\rangle.$$

Let us write a_{ϑ} for the composition $a_{\vartheta} = p_{\vartheta} \circ a : \mathsf{G} \to \mathfrak{a}_{\vartheta}$.

Let $\beta : \mathsf{G} \times \mathcal{F}_{\Delta} \to \mathfrak{a}$ be the *Buseman–Iwasawa cocycle* of G introduced by Quint [**62**] (see §4.6). The map $\beta_{\vartheta} = p_{\vartheta} \circ \beta$ factors as a cocycle $\beta_{\vartheta} : \mathsf{G} \times \mathcal{F}_{\vartheta} \to \mathfrak{a}_{\vartheta}$.

Definition 1.0.2. A *Patterson–Sullivan measure for ρ on $\mathcal{F}_{\vartheta}$* is a probability measure ν on $\mathcal{F}_{\vartheta}$ such that there exists $\varphi \in \mathfrak{a}_{\vartheta}^*$ with, for every $\gamma \in \Gamma$,

$$\frac{d\rho(\gamma)_*\nu}{d\nu}(\cdot) = q^{-\varphi(\beta_{\vartheta}(\rho(\gamma)^{-1},\cdot))}.$$

For $\varphi \in (\mathfrak{a}_{\vartheta})^*$, denote

$$\delta^{\varphi} = \lim_{t\to\infty} \frac{1}{t} \log \#\{\gamma \in \Gamma : \varphi(a(\rho(\gamma))) \leq t\} \in [0, \infty]$$

and, inspired by Quint's growth indicator [**61**], consider the *ϑ-critical hypersurface*

$$\mathcal{Q}_{\vartheta,\rho} = \{\varphi \in (\mathfrak{a}_{\vartheta})^* : \delta^{\varphi} = 1\}.$$

Let us define the *ϑ-limit cone* of ρ, denoted by $\mathscr{L}_{\vartheta,\rho}$, as the asymptotic cone of the projections

$$\{a_{\vartheta}(\rho(\gamma)) : \gamma \in \Gamma\},$$

that is, all limits of sequences of the form $t_n a_{\vartheta}(\rho(\gamma_n))$, where $\gamma_n \to \infty$ in Γ and $t_n \to 0$ in $\mathbb{R}_+$.

In the real case, if $\rho(\Gamma)$ is Zariski-dense, then Benoist's aforementioned result implies that $\mathscr{L}_{\vartheta,\rho}$ has non-empty interior. However, for arbitrary local fields, this is no longer the case (even assuming Zariski-density and Anosov). We aim to work on this more general context, so let us assume now that G is (the $\mathbb{K}$-points of) a semi-simple algebraic group over a local field $\mathbb{K}$. We refer the reader to §4 for the analogous definitions, where $\mathfrak{a}_{\vartheta}$ is replaced by the real vector space E_{ϑ}, etc.

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Let $\operatorname{Ann}(\mathscr{L}_{\vartheta,\rho})$ be the annihilator of the ϑ-limit cone and denote by $\pi_\rho^\vartheta : (\mathsf{E}_\vartheta)^* \to (\mathsf{E}_\vartheta)^*/\operatorname{Ann}(\mathscr{L}_{\vartheta,\rho})$ the quotient projection.

THEOREM A. *Let $\rho : \Gamma \to \mathsf{G}$ be ϑ-Anosov. Then, $\mathcal{Q}_{\vartheta,\rho}$ is a closed co-dimension-one analytic sub-manifold of $(\mathsf{E}_\vartheta)^*$ that bounds a convex set; moreover, the projection $\pi_\rho^\vartheta(\mathcal{Q}_{\vartheta,\rho})$ is also a closed co-dimension-one analytic sub-manifold boundary of a strictly convex set. For each $\varphi \in \mathcal{Q}_{\vartheta,\rho}$, there exists a unique Patterson–Sullivan measure ν^φ with support on $\xi^\vartheta(\partial\Gamma)$. The map $\varphi \mapsto \nu^\varphi$ is an analytic homeomorphism between the projection $\pi_\rho^\vartheta(\mathcal{Q}_{\vartheta,\rho})$ and the space of Patterson–Sullivan measures on $\mathcal{F}_\vartheta$ whose support is contained in $\xi^\vartheta(\partial\Gamma)$. Such Patterson–Sullivan measures are ergodic and pairwise mutually singular.*

We refer the reader to Corollary 5.5.3 and Proposition 5.9.2 for the proofs of the above statements.

The fact that both $\mathcal{Q}_{\vartheta,\rho}$ and $\pi_\rho^\vartheta(\mathcal{Q}_{\vartheta,\rho})$ are closed analytic hypersurfaces is stated by Potrie and Sambarino [57, Proposition 4.11] for $\mathbb{K} = \mathbb{R}$ with essentially the same arguments. The parameterization of Patterson–Sullivan measures by $\pi_\rho^\vartheta(\mathcal{Q}_{\vartheta,\rho})$ was previously established by Lee and Oh [46, Theorem 1.3] for $\mathbb{K} = \mathbb{R}$, $\vartheta = \Delta$, and assuming Zariski-density of $\rho(\Gamma)$. Existence and ergodicity was previously established, for $\mathbb{K} = \mathbb{R}$, by Dey and Kapovich [29, Main Theorem] for i-invariant functionals $\varphi \in (\mathfrak{a}^+)^* \cap (\mathfrak{a}_\vartheta)^*$ and i-invariant subsets ϑ; and by Sambarino [66, Corollary 4.22] for arbitrary functionals but Zariski-dense representations of fundamental groups of negatively curved manifolds. Existence of Patterson–Sullivan measures has also been established by Canary, Zhang, and Zimmer [21] in the real case for *relative* Anosov representations.

We keep the discussion for $\mathbb{K} = \mathbb{R}$ since this is essential in the following result. Consider $\varphi \in \mathcal{Q}_{\vartheta,\rho}$ with associated Patterson–Sullivan measure μ^φ. Via the duality

$$\mathrm{Gr}_{\dim \mathfrak{a}_\vartheta - 1}((\mathfrak{a}_\vartheta)^*) \to \mathbb{P}(\mathfrak{a}_\vartheta),$$

the tangent space $\mathsf{T}_\varphi \mathcal{Q}_{\vartheta,\rho}$ gives a direction u_φ of $\mathbb{P}(\mathfrak{a}_\vartheta)$ contained in the relative interior of the limit cone $\mathscr{L}_{\vartheta,\rho}$ (Corollary 5.9.1). We then further investigate the μ^φ-mass of u_φ-conical points on $\xi^\vartheta(\partial\Gamma)$.

Since we are dealing with the limit cone on $\mathfrak{a}_\vartheta$ (and not on $\mathfrak{a}$ as before), u_φ-conical points are yet to be defined. It is standard that every point $\xi^\vartheta(x) \in \xi^\vartheta(\partial\Gamma)$ is conical. (This follows from the fact that every point $x \in \partial\Gamma$ is conical and the existence of the equivariant limit maps for ρ). Let us say it is further u_φ-*conical* if there exists a conical sequence (for x) as above and $r > 0$ such that $a_\vartheta(\rho(\gamma_n)) \in \mathbb{T}_r(\mathsf{u}_\varphi)$. Denote by $\partial_\varphi\Gamma \subset \partial\Gamma$ the subset

$$\partial_\varphi\Gamma = \{x \in \partial\Gamma : \xi^\vartheta(x) \text{ is } \mathsf{u}_\varphi\text{-conical}\}.$$

THEOREM B. (Theorem 5.13.3) *Let $\mathbb{K} = \mathbb{R}$ and assume ρ is ϑ-Anosov and Zariski-dense. If $|\vartheta| \leq 2$, then $\mu^\varphi(\xi^\vartheta(\partial_\varphi\Gamma)) = 1$; if $|\vartheta| \geq 4$, then $\mu^\varphi(\xi^\vartheta(\partial_\varphi\Gamma)) = 0$.*

The case $|\vartheta| = 3$ is sadly presently untreated. The missing fact that would make our technique directly apply is an ergodicity result for translation skew-products over metric-Anosov flows where the abelian group is isomorphic to $\mathbb{R}^2 = \mathbb{R}^{|\vartheta|-1}$, more precisely, we need equivalence between ergodicity and $\dim V \leq 2$ in Corollary 2.5.5.

When $\vartheta = \Delta$, a stronger version of Theorem B dealing also with the case $|\Delta| = 3$ was previously established by Burger *et al* [20, Theorem 1.6]. It is likely that the combination of their techniques and ours settles the missing $|\vartheta| = 3$ case.

1.1. *General strategy for Theorem B.* Let us briefly explain the proof of Theorem B, which we believe is the main contribution of this work. The main ingredient is a precise description of the *ϑ-parallel sets dynamics* of G. If $(x, y) \in \mathcal{F}_{\mathrm{i}\,\vartheta} \times \mathcal{F}_\vartheta$ are in general position, the associated *parallel set* is a subset of X consisting of the union of totally geodesic maximal flats p of X whose associated complete flags in the Furstenberg boundary $\mathsf{p}(-\mathfrak{a}^+)$ and $\mathsf{p}(\mathfrak{a}^+)$ contain respectively x and y as a partial flag. This parallel set is a reductive symmetric space and the associated dynamical system consists of moving along its center.

More concisely, if one considers the space $\mathcal{F}_\vartheta^{(2)} \subset \mathcal{F}_{\mathrm{i}\,\vartheta} \times \mathcal{F}_\vartheta$ of transverse flags, then the space $\mathcal{F}_\vartheta^{(2)} \times \mathfrak{a}_\vartheta$ carries a G-action (on the left) given by

$$g(x, y, v) = (gx, gy, v - \beta_\vartheta(g, y)),$$

and an $\mathfrak{a}_\vartheta$-action (on the right) by translation on the last coordinate.

Observe however that the left-action of $\rho(\Gamma)$ on $\mathcal{F}_\vartheta^{(2)} \times \mathfrak{a}_\vartheta$ need not be proper. For ϑ-Anosov groups though, one finds an $\mathfrak{a}_\vartheta$-invariant subset which is also $\rho(\Gamma)$-invariant and on which this latter action is proper (§§3.5.2 and 5.3.2). Its quotient by $\rho(\Gamma)$ will be denoted, throughout this introduction, by $\mathcal{O}_{\vartheta,\rho}$.

For each $\varphi \in \mathcal{Q}_{\vartheta,\rho}$, the space $\mathcal{O}_{\vartheta,\rho}$ will carry a *φ-Bowen–Margulis measure* Ω^φ invariant under the *directional flow* $\omega^\varphi : \mathcal{O}_{\vartheta,\rho} \to \mathcal{O}_{\vartheta,\rho}$ along $u_\varphi \in \mathsf{u}_\varphi$, defined by (the induction on the quotient by $\rho(\Gamma)$ of)

$$(x, y, v) \mapsto (x, y, v - tu_\varphi).$$

The idea generalizing that of Sambarino [66] is that ω^φ is conjugated to a skew-product over a metric-Anosov flow $\phi^\varphi = (\phi_t^\varphi : \chi^\varphi \to \chi^\varphi)_{t\in\mathbb{R}}$ on a compact metric space χ^φ. This is established in §5.12 and was previously established by Carvajales [23, Appendix] for $\vartheta = \Delta$.

Remark 1.1.1. The flow ϕ^φ plays a central role in this work. We propose to name it *the φ-refraction flow* of ρ, because one may think that the projection on the base χ^φ *refracts* the orbits of ω^φ (almost all of them wondering when $|\vartheta| \geq 4$) to bind them in a compact space χ^φ and obtain non-trivial dynamical behavior. (In spite of being topologically mixing, these flows are wondering in a measurable sense, that is, almost every point belongs to a subset of positive measure with bounded return times.) Also, the term *geodesic flow* has too many meanings on this setting (the geodesic flow of Γ, the geodesic flow of the locally symmetric space $\rho(\Gamma)\backslash\mathsf{X}$, the geodesic flow of a projective-Anosov representation associated to ρ by Plucker embeddings, and so forth).

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An ergodicity result for skew-products over metric-Anosov flows (see §2.5) gives an *ergodic versus totally dissipative* dichotomy for ω^{φ} according to $|\vartheta| \leq 2$ or $|\vartheta| \geq 4$. Here, the base field $\mathbb{K} = \mathbb{R}$ and Zariski-density of ρ are essential, since Benoist's [8] density of Jordan projections does not hold for non-Archimedean $\mathbb{K}$. This dichotomy is reminiscent of Sullivan's [68] *conservative versus totally dissipative* dichotomy in rank 1. Observe again the untreated case $|\vartheta| = 3$.

These dynamical properties of ω^{φ} imply the following. The set $\mathcal{K}(\omega^{\varphi})$ of points in $O_{\vartheta,\rho}$, whose future orbit returns unboundedly to some open bounded set, has either zero Ω^{φ}-mass if $|\vartheta| \geq 4$ or its complement has zero Ω^{φ}-mass if $|\vartheta| \leq 2$.

The key feature now is to relate u_{φ}-conical points with the set $\mathcal{K}(\omega^{\varphi})$. This is attained in Lemma 5.13.3 where it is shown that a triple $(x, y, v) \in \mathcal{F}_{\vartheta}^{(2)} \times \mathfrak{a}_{\vartheta}$ projects to $\mathcal{K}(\omega^{\varphi})$ *if and only if* y is u_{φ}-conical. The previous dynamical dichotomy gives then the dichotomy on the μ^{φ}-measure on conical points:

$$|\vartheta| \leq 2 \Rightarrow \mu^{\varphi}(\partial_{\varphi}\Gamma) = 1, \quad |\vartheta| \geq 4 \Rightarrow \mu^{\varphi}(\partial_{\varphi}\Gamma) = 0. \tag{1}$$

The global strategy of our proof is different from the analog result of Burger *et al* [20]. While, inspired by them, we also use a mixing result. The use of a Dirichlet–Poincaré series along tubes does not play any role in the proof of Theorem B, nor on the ergodicity dichotomy for directional flows.

Let us end §1 by observing that both Burger *et al* [20] and Chow and Sarkar [25] prove dynamical statements on $\rho(\Gamma)\backslash\mathsf{G}$ (as opposed to $\rho(\Gamma)\backslash\mathsf{G}/\mathsf{M}$).

1.2. *Plan of the paper.* In §2, we recall some basic facts about the ergodic theory of metric-Anosov flows, and then study translation cocycles over them. Section 3 deals with a Ledrappier correspondence for word-hyperbolic groups whose Gromov–Mineyev geodesic flow is metric-Anosov. We will mainly apply these results to the Buseman–Iwasawa cocycle of G. For applications to other cocycles, the reader may check the work of Carvajales [22, 23].

We then recall in §4 necessary definitions on semi-simple algebraic groups over a local field and deal with Anosov representations in §5. We explain in this section how the Ledrappier correspondence applies in this setting to give, mainly:

- uniqueness results on the Patterson–Sullivan measures;
- precise dynamical information on the directional flows ω^{φ}.

The proof of Theorem B can be found in §5.13.

2. *Skew-products over metric-Anosov flows*

Throughout this section, we let X be a compact metric space and V a finite dimension real vector space.

2.1. *Thermodynamic formalism and reparameterizations.* Let $\phi = (\phi_t : X \to X)_{t\in\mathbb{R}}$ be a continuous flow without fixed points. The space of ϕ-invariant probability measures on X is denoted by $\mathcal{M}^{\phi}$. It is a convex, weakly compact subset of $C^*(X)$, the dual space to the space of continuous functions equipped with the uniform topology. The *metric entropy* of

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$m \in \mathcal{M}^\phi$ will be denoted by $h(\phi, m)$. Its definition can be found in Aaronson's book [1]. Via the variational principle, we will define the *topological pressure* (or just *pressure*) of a function $f : X \to \mathbb{R}$ as the quantity

$$P(\phi, f) = \sup_{m \in \mathcal{M}^\phi} \bigg(h(\phi, m) + \int_X f \, dm \bigg). \tag{2}$$

A probability measure m realizing the least upper bound is called an *equilibrium state* of f. An equilibrium state for $f \equiv 0$ is called a *measure of maximal entropy*, and its entropy is called the *topological entropy* of ϕ, denoted by $h(\phi)$.

Let $f : X \to \mathbb{R}_{>0}$ be continuous. For every $x \in X$, the function $k_f : X \times \mathbb{R} \to \mathbb{R}$, defined by $k_f(x, t) = \int_0^t f(\phi_s x) \, ds$, is an increasing homeomorphism of $\mathbb{R}$. There is thus a continuous function $\alpha_f : X \times \mathbb{R} \to \mathbb{R}$ such that for all $(x, t) \in X \times \mathbb{R}$,

$$\alpha_f(x, k_f(x, t)) = k_f(x, \alpha_f(x, t)) = t.$$

The *reparameterization* of ϕ by $f : X \to \mathbb{R}_{>0}$ is the flow $\phi^f = (\phi^f_t : X \to X)_{t \in \mathbb{R}}$ defined, for all $(x, t) \in X \times \mathbb{R}$, by

$$\phi^f_t(x) = \phi_{\alpha_f(x,t)}(x).$$

The *Abramov transform* of $m \in \mathcal{M}^\phi$ is the probability measure $m^\# \in \mathcal{M}^{\phi^f}$ defined by

$$m^\# = \frac{f \cdot m}{\int f \, dm}. \tag{3}$$

One has the following lemma.

LEMMA 2.1.1. (Sambarino [65, Lemma 2.4]) *Let $f : X \to \mathbb{R}_{>0}$ be a continuous function. Assume the equation*

$$P(\phi, -sf) = 0 \quad s \in \mathbb{R}$$

has a finite positive solution h, then h is the topological entropy of ϕ^f. Conversely, if $h(\phi^f)$ is finite, then it is a solution to the last equation. In this situation, the Abramov transform induces a bijection between the set of equilibrium states of $-hf$ and the set of probability measures maximizing entropy for ϕ^f.

Two continuous maps $f, g : X \to V$ are *Livšic-cohomologous* if there exists a $U : X \to V$ of class C^1 in the direction of the flow (that is, such that if for every $x \in X$, the map $t \mapsto U(\phi_t x)$ is of class C^1 and the map $x \mapsto \partial/\partial t|_{t=0} U(\phi_t x)$ is continuous), such that for all $x \in X$, one has

$$f(x) - g(x) = \frac{\partial}{\partial t}\bigg|_{t=0} U(\phi_t x).$$

Remark 2.1.2. If f and g are real-valued and Livšic-cohomologous, then $P(\phi, f) = P(\phi, g)$.

2.2. *Metric-Anosov flows I: Livšic-cohomology.* *Metric-Anosov flows* are a metric version of what is commonly known as *hyperbolic flows*. The former are called Smale flows

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by Pollicott [56], who transferred to this more general setting the classical theory carried out for the latter. We recall here their definition and some well-known facts on their ergodic theory needed in what follows. Throughout this subsection, we will further assume that ϕ is Hölder-continuous with an exponent independent of t, that it is *transitive*, that it has a dense orbit, and that it is metric-Anosov.

For $\varepsilon > 0$, the *local stable/unstable set* of x are (respectively)

$$W^{\mathrm{s}}_{\varepsilon}(x) = \{y \in X : d(\phi_t x, \phi_t y) \leq \varepsilon \text{ for all } t > 0 \text{ and } d(\phi_t x, \phi_t y) \to 0 \text{ as } t \to \infty\},$$
$$W^{\mathrm{u}}_{\varepsilon}(x) = \{y \in X : d(\phi_{-t} x, \phi_{-t} y) \leq \varepsilon \text{ for all } t > 0 \text{ and } d(\phi_{-t} x, \phi_{-t} y) \to 0 \text{ as } t \to \infty\}.$$

Definition 2.2.1. (Metric-Anosov) The flow ϕ is *metric-Anosov* if the following hold.

- (Exponential decay) There exist positive constants C, λ and ε such that for every $x \in X$, every $y \in W^{\mathrm{s}}_{\varepsilon}(x)$, and every $t > 0$, one has
$$d(\phi_t x, \phi_t y) \leq Ce^{-\lambda t},$$
and such that for every $y \in W^{\mathrm{u}}_{\varepsilon}(x)$, one has $d(\phi_{-t} x, \phi_{-t} y) \leq Ce^{-\lambda t}$.
- (Local product structure) There exist $\delta, \varepsilon > 0$ and a Hölder-continuous map
$$\nu : \{(x, y) \in X \times X : d(x, y) < \delta\} \to \mathbb{R}$$
such that $\nu(x, y)$ is the unique value ν such that $W^{\mathrm{u}}_{\varepsilon}(\phi_{\nu} x) \cap W^{\mathrm{s}}_{\varepsilon}(y)$ is non-empty, and consists of exactly one point, called $\langle x, y\rangle$; and for every $x \in X$, the map
$$W^{\mathrm{s}}_{\varepsilon}(x) \times W^{\mathrm{u}}_{\varepsilon}(x) \times (-\delta, \delta) \to X,$$
given by $(y, z, t) \mapsto \phi_t(\langle y, z\rangle)$, is a Hölder-homeomorphism onto an open neighborhood of x.

A *translation cocycle* over ϕ is a map $k : X \times \mathbb{R} \to V$ such that for every $x \in X$ and $t, s \in \mathbb{R}$, one has

$$k(x, t + s) = k(\phi_s x, t) + k(x, s),$$

and such that the map $k(\cdot, t)$ is Hölder-continuous with exponent independent of t and with bounded multiplicative constant when t remains on a bounded set. Two translation cocycles k_1 and k_2 are *Livšic-cohomologous* if there exists a continuous map $U : X \to V$, such that for all $x \in X$ and $t \in \mathbb{R}$, one has

$$k_1(x, t) - k_2(x, t) = U(\phi_t x) - U(x). \tag{4}$$

If k is a translation cocycle, then the *period for* k of a periodic orbit τ is

$$\ell_{\tau}(k) = k(x, p(\tau))$$

for any $x \in \tau$. The *marked spectrum* $\tau \mapsto \ell_k(\tau)$ is a cohomological invariant that uniquely determines its class.

THEOREM 2.2.2. (Livšic [47]) *Let $k : X \times \mathbb{R} \to V$ be a translation cocycle. If $\ell_k(\tau) = 0$ for every periodic orbit τ, then k is Livšic-cohomologous to* 0.

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Observe that if $f : X \to V$ is Hölder-continuous, then the map

$$k_f(x,t) = \int_0^t f(\phi_s x)\, ds$$

is a translation cocycle. Two such functions are Livšic-cohomologous if and only if the associated cocycles are, and the *period* of f on τ is, for any $x \in \tau$,

$$\ell_\tau(f) = \int_\tau f = k_f(x, p(\tau)).$$

It turns out that every cocycle is Livšic-cohomologous to a cocycle of the form k_f.

COROLLARY 2.2.3. (Sambarino [66, Lemma 2.6]) *If $k : X \times \mathbb{R} \to V$ is a translation cocycle, then there exists a Hölder-continuous $f : X \to V$ such that k and k_f are Livšic-cohomologous.*

Proof. For any $\kappa > 0$, the function $j(x,t) = 1/2\kappa \int_{-\kappa}^{\kappa} k(x, t+s)\, ds$ is differentiable on the second variable. Let $f(x) = (\partial/\partial s)|_{s=0} j(x,s)$. Then,

$$\begin{aligned} k_f(x,t) &= \int_0^t f(\phi_u x)\, du = \int_0^t \frac{\partial}{\partial s}\bigg|_{s=0} j(\phi_u x, s)\, du \\ &= \int_0^t \frac{\partial}{\partial s}\bigg|_{s=0} j(x, s+u)\, du = j(x,t) - j(x,0), \end{aligned}$$

so the period $k_f(x, p(\tau)) = j(x, p(\tau)) - j(x,0) = k(x, p(\tau))$. By Theorem 2.2.2, the cocycles k and k_f are thus Livšic-cohomologous. □

We record also the following immediate consequence of Livšic's theorem.

Remark 2.2.4. The space of functions Livšic-cohomologous to a strictly positive function is an (open cone on an) infinite dimensional space.

In this context, much more information can be stated about the pressure function. Recall that the space $\mathrm{Holder}^\alpha(X)$ of real valued α-Hölder functions is naturally a Banach space when equipped with the norm

$$\|f\|_\alpha = \|f\|_\infty + \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)^\alpha}.$$

PROPOSITION 2.2.5. (Bowen and Ruelle [16] and Parry and Pollicott [53, Proposition 4.10]) *The function $P(\phi, \cdot)$ is analytic on* $\mathrm{Holder}^\alpha(X)$. *If $f, g \in$* $\mathrm{Holder}^\alpha(X)$, *then*

$$\frac{\partial}{\partial t}\bigg|_{t=0} P(\phi, f + tg) = \int g\, dm_f,$$

where m_f is the equilibrium state of f, and the function $t \mapsto P(\phi, f+tg)$ is strictly convex unless g is Livšic-cohomologous to a constant. Finally, one also has

$$P(\phi, f) = \limsup_{t\to\infty} \frac{1}{t} \log \sum_{\tau : p(\tau) \le t} e^{\ell_\tau(f)}. \tag{5}$$

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Let $f \in \mathrm{Holder}^{\alpha}(X)$ have non-negative (and not all vanishing) periods and define its *entropy* by

$$\hbar_f = \limsup_{s\to\infty} \frac{1}{s} \log \#\left\{\tau \text{ periodic} : \int_{\tau} f \leq s\right\} \in (0, \infty].$$

Remark 2.2.6. Observe that $\hbar_f$ is necessarily > 0 since f must have a positive maximum and $h(\phi) > 0$.

One has the following lemma.

LEMMA 2.2.7. (Ledrappier [44, Lemma 1] and Sambarino [65, Lemma 3.8]) *Consider a Hölder-continuous function $f : X \to \mathbb{R}$ with non-negative periods. Then the following statements are equivalent:*

- *the function f is Livšic-cohomologous to a positive Hölder-continuous function;*
- *there exists $\kappa > 0$ such that $\int_{\tau} f > \kappa p(\tau)$ for every periodic orbit τ;*
- *the entropy $\hbar_f$ is finite;*
- *the function $t \mapsto P(\phi, -tf)$ has a positive zero, in which case is $\hbar_f$.*

Let us fix an exponent α and consider the cone $\mathrm{Holder}^{\alpha}_{+}(X, \mathbb{R})$ of Hölder-continuous functions that are Livšic-cohomologous to a strictly positive function. The implicit function theorem for Banach spaces (see [2]) and the explicit formula for the derivative of pressure (Proposition 2.2.5) give the following corollary.

LEMMA 2.2.8. *The entropy map $\hbar : \mathrm{Holder}^{\alpha}_{+}(X, \mathbb{R}) \to \mathbb{R}_{+}$ is analytic.*

Proof. Indeed, Lemma 2.2.7 gives the equation $P(\phi, -\hbar_f f) = 0$ and equation (2.2.5) gives the non-vanishing derivative

$$d_{-\hbar_f f} P(\phi, f) = \int f \, dm_{-h_f f} > 0,$$

so the implicit function theorem completes the result. □

2.3. *Metric-Anosov flows II: ergodic theory.* A fundamental tool for studying the ergodic theory of metric-Anosov systems is the existence of a Markov coding.

Let Σ be an irreducible sub-shift of finite type equipped with its shift transformation $\sigma : \Sigma \to \Sigma$, and $r : \Sigma \to \mathbb{R}_{>0}$ be Hölder-continuous. Let $\hat{r} : \Sigma \times \mathbb{R} \to \Sigma \times \mathbb{R}$ be defined as

$$\hat{r}(x, s) = (\sigma x, s - r(x)),$$

and consider the quotient space $\Sigma_r = \Sigma \times \mathbb{R}/\langle \hat{r} \rangle$. It is equipped with the flow $\sigma^r = (\sigma^r_t : \Sigma_r \to \Sigma_r)_{t\in\mathbb{R}}$ induced on the quotient by the translation flow.

Definition 2.3.1. (Markov coding) A triplet (Σ, π, r) is a *Markov coding* for ϕ if Σ and r are as above, $\pi : \Sigma \to X$ is Hölder-continuous, and the function $\pi_r : \Sigma \times \mathbb{R} \to X$ defined as

$$\pi_r(x, t) = \phi_t \pi(x)$$

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verifies the following conditions:

(i) π_r is Hölder-continuous, surjective, and $\hat{r}$-invariant, it passes then to the quotient Σ_r;

(ii) $\pi_r : \Sigma_r \to X$ is bounded-to-one and injective on a residual set which is of full measure for every ergodic σ^r-invariant measure of total support;

(iii) for every $t \in \mathbb{R}$, one has $\pi_r \sigma_t^r = \phi_t \pi_r$.

The following result has a long history, see for example the works of Bowen [14, 15], Ratner [63], Pollicott [56], and more recently Constantine, Lafont, and Thompson [27].

THEOREM 2.3.2. (Existence of coding) *A transitive metric-Anosov flow admits a Markov coding.*

The above is a fundamental tool to obtain the following, see for example the works of Bowen and Ruelle [16], Parry and Pollicott [53], and more recently Giulietti *et al* [31]. Recall from §2.1 the definition of equilibrium state.

THEOREM 2.3.3. (Uniqueness of equilibrium states) *Let $f : X \to \mathbb{R}$ be Hölder-continuous. Then there exists a unique equilibrium state for f, denoted by m_f; it is an ergodic measure. If $g : X \to \mathbb{R}$ is also Hölder, then $m_g \ll m_f$ if and only if $f - g$ is Livšic-cohomologous to a constant function, in which case, $m_g = m_f$. The function $f \mapsto m_f$, defined on the space of Hölder-continuous functions with fixed exponent, is analytic. (We emphasize that the space of measures is endowed with the differentiable structure induced by being the dual space of continuous functions.)*

A final fact we will require in this setting (introduced by Margulis [48]) is the decomposition of the measure of maximal entropy along the stable/central-unstable sets of ϕ.

The *stable/unstable leaf* of x is

$$W^{\mathrm{s}}(x) = \bigcup_{t \in \mathbb{R}_+} \phi_{-t}(W^{\mathrm{s}}_{\varepsilon}(\phi_t x)),$$

$$W^{\mathrm{u}}(x) = \bigcup_{t \in \mathbb{R}_+} \phi_{t}(W^{\mathrm{u}}_{\varepsilon}(\phi_{-t} x)),$$

and the *central* stable/unstable leaf is (respectively) the ϕ-orbit of $W^{\mathrm{s}}(x)$ (respectively $W^{\mathrm{u}}(x)$). These sets are independent of (any small enough) ε (that is smaller than the ε given by Definition 2.2.1).

One has the following, see for example the works of Margulis [49], Pollicott [55] for a construction via Markov codings or Katok and Hasselblatt's book [41, §5 of Ch. 20] for the discrete-time case.

THEOREM 2.3.4. (Margulis description) *For each $x \in X$, there exists a measure μ^{s}_x on the stable leaf $W^{\mathrm{s}}(x)$ and a measure μ^{cu}_x on the central unstable leaf such that:*

- *for all $t > 0$ and all measurable $U \subset W^{\mathrm{s}}(x)$, one has*

$$\mu^{\mathrm{s}}_{\phi_t x}(\phi_t U) = e^{-h(\phi)t} \mu^{\mathrm{s}}_x(U); \tag{6}$$

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- *the local product structure in Definition 2.2 induces a local isomorphism between the measure $\mu_x^{\mathrm{s}} \otimes \mu_x^{\mathrm{cu}}$ and the measure of maximal entropy of ϕ.*

The family of measures is unique in the following sense. If ν_x^{s} and ν_x^{cu} are also a family of measures along the stable and central-unstable leafs of x such that the product $\nu_x^{\mathrm{s}} \otimes \nu_x^{\mathrm{cu}}$ is locally isomorphic to a ϕ-invariant measure and ν_x^{s} verifies equation (6) with some arbitrary fixed $\delta > 0$ (instead of $h(\phi)$), then $\delta = h(\phi)$, $\nu_x^{\mathrm{s}} = \mu_x^{\mathrm{s}}$, and $\mu_x^{\mathrm{cu}} = \nu_x^{\mathrm{cu}}$.

2.4. *Skew-products over sub-shifts.* Consider now the two-sided subshift Σ and let $\mathsf{K} : \Sigma \to V$ be Hölder-continuous.

Definition 2.4.1. We say that K is *non-arithmetic* if the group spanned by the periods of K is dense in V.

The *skew-product system* is defined by $f = f^{\mathsf{K}} : \Sigma \times V \to \Sigma \times V$

$$f(p, v) = (\sigma(p), v - \mathsf{K}(p)). \tag{7}$$

If ν is a σ-invariant probability measure on Σ, then the measure $\Omega = \Omega_\nu = \nu \otimes \mathrm{Leb}$ is f-invariant.

The following proposition seems to be well known but we have not been able to find a specific reference in the literature, for completeness, we added a short proof in Appendix A.

PROPOSITION 2.4.2. *Let Σ be a two-sided sub-shift, ν be an equilibrium state of some Hölder potential, and $\mathsf{K} : \Sigma \to \mathbb{R}$ a non-arithmetic Hölder-continuous function with $\int \mathsf{K} d\nu = 0$. Then the skew-product $f^{\mathsf{K}} : \Sigma \times \mathbb{R} \to \Sigma \times \mathbb{R}$ is ergodic with respect to Ω_ν.*

We record also the following classical lemma. Let us say that a subset of $\Sigma \times V$ is *bounded* if it has compact closure, and that it has *total mass* (with respect to Ω) if its complement has measure zero. As the space $\Sigma \times V$ is non-compact, it is natural to study the subset of points of $\Sigma \times V$ whose future orbit returns infinitely many times to a fixed open bounded set:

$$\mathcal{K}(f) = \{p \in \Sigma \times V : \text{ there exists B open bounded set and } n_k \to \infty \text{ with } f^{n_k}(p) \in \mathrm{B}\}.$$

One can be more specific. If $\mathrm{B}_1, \mathrm{B}_2 \subset \Sigma \times W$, we want to understand the measure of

$$\mathcal{K}(\mathrm{B}_1, \mathrm{B}_2) = \{p \in \mathrm{B}_1 : \text{ there exists } n_k \to \infty \text{ with } f^{n_k}(p) \in \mathrm{B}_2\}.$$

To this end, one considers the sum $\sum_0^\infty \Omega(\mathbf{1}_{\mathrm{B}_1} \cdot \mathbf{1}_{\mathrm{B}_2} \circ f^n)$.

LEMMA 2.4.3. *If $\sum_{n=0}^\infty \Omega(\mathbf{1}_{\mathrm{B}_1} \cdot \mathbf{1}_{\mathrm{B}_2} \circ f^n) < \infty$, then $\Omega(\mathcal{K}(\mathrm{B}_1, \mathrm{B}_2)) = 0$. However, if ν has no atoms and f is ergodic with respect to Ω, then $\mathcal{K}(f)$ has total mass and for every pair $\mathrm{B}_1, \mathrm{B}_2$, one has $\mathcal{K}(\mathrm{B}_1, \mathrm{B}_2)$ has total mass on B_1.*

Proof. This is a standard argument valid for any measure-preserving transformation. The first assertion follows by looking at the tail of the series in question:

$$\sum_{n=k}^{\infty} \Omega(\mathbf{1}_{\mathrm{B}_1} \cdot \mathbf{1}_{\mathrm{B}_2} \circ f^n) \geq \Omega(E_k),$$

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where, for each $k \in \mathbb{N}$, $E_k = \{p \in \mathrm{B}_1 : \text{there exists } N \geq k \text{ with } f^N(p) \in \mathrm{B}_2\}$. The second assertion can be found in, for example, Aaronson's book [1, p. 22]. □

2.5. *An ergodic dichotomy.* As in §2.3, let σ^r be the suspension of the shift on Σ by the function r. If ν is a σ-invariant probability measure, then $\nu \otimes dt / \int r d\nu$ is invariant under the translation flow $\Sigma \times \mathbb{R}$ and induces thus a σ^r-invariant probability measure on Σ_r, denoted by $\hat{\nu}$.

Remark 2.5.1. It is a classical fact, stated for example by Bowen and Ruelle [16], that if ν is the equilibrium state of $-h(\sigma^r)r$, then $\hat{\nu}$ realizes the topological entropy of σ_r.

Let $K : \Sigma \times \mathbb{R} \to V$ be a $\langle \hat{r} \rangle$-invariant Hölder-continuous function and consider the flow

$$\psi = (\psi_t : \Sigma_r \times V \to \Sigma_r \times V)_{t \in \mathbb{R}}$$
$$\psi_t(p, v) = \left(\sigma_t^r(p), v - \int_0^t K(\sigma_s^r p)\, ds\right).$$

Consider the measure on $\Sigma_r \times V$ defined by $\bar{\Omega}_\nu = \hat{\nu} \otimes \mathrm{Leb}$.

THEOREM 2.5.2. (Sambarino [66, Theorem 3.8]) *Assume the group generated by the periods of* (r, K) *is dense in* $\mathbb{R} \times V$ *and that* $\int K d\hat{\nu} = 0$ *for the equilibrium state* ν *of* $-h(\sigma^r)r$. *Then there exists* $\kappa > 0$ *such that given two compactly supported continuous functions* $g_1, g_2 : \Sigma_r \times V \to \mathbb{R}$, *one has*

$$t^{\dim V/2} \bar{\Omega}_\nu(g_1 \cdot g_2 \circ \psi_t) \to \kappa \bar{\Omega}_\nu(g_1) \bar{\Omega} \nu(g_2),$$

as $t \to \infty$.

We include the main outline of its proof in Appendix B. As it is classical, the above result holds for characteristic functions of open bounded sets whose boundary has measure zero.

COROLLARY 2.5.3. *Under the same assumptions of Theorem 2.5.2, if* $\dim V = 1$, *then* ψ *is ergodic with respect to* $\bar{\Omega}_\nu$. *If* $\dim V \geq 3$, *then* $\bar{\Omega}_\nu(\mathcal{K}(\psi)) = 0$.

Proof. The skew-product system $f^{\mathsf{K}} : \Sigma \times V \to \Sigma \times V$ of equation (7), where $\mathsf{K} : \Sigma \to V$ is defined by

$$\mathsf{K}(x) = \int_0^{r(x)} K(x, s)\, ds, \tag{8}$$

is the first-return map of ψ to its global section $(\Sigma \times \{0\})/_\sim \times V$. Consequently, following the flow-lines until reaching the section, one finds a natural (measurable) bijection between ψ-invariant subsets and f-invariant subsets. If μ is a σ-invariant probability measure on Σ, then this bijection preserves the class of invariant-zero-sets between the measures $\hat{\mu} \otimes \mathrm{Leb}$ on $\Sigma_r \times V$ and $\mu \otimes \mathrm{Leb}$ on $\Sigma \times V$. Thus, we can translate ergodicity results from f to the flow ψ and *vice versa*.

The case $\dim V = 1$ is hence settled by Proposition 2.4.2.

For $\dim V \geq 3$, one considers an open $A \subset \Sigma$ with $\nu(\partial A) = 0$, an open interval $I \subset \mathbb{R}$ with length $< \min r$, and $B \subset V$ an open ball. Applying Theorem 2.5.2 to

$$\bar{\mathrm{B}} = \bar{\mathrm{B}}_1 = \bar{\mathrm{B}}_2 = A \times I \times B$$

gives a positive C such that for large t, one has $t^{\dim V/2}\bar{\Omega}(\mathbf{1}_{\bar{\mathrm{B}}} \cdot \mathbf{1}_{\bar{\mathrm{B}}} \circ \psi_t) \leq C$. Thus, for a fixed $t_0 > 0$, one has that

$$\int_{t_0}^{\infty} \bar{\Omega}(\mathbf{1}_{\bar{\mathrm{B}}} \cdot \mathbf{1}_{\bar{\mathrm{B}}} \circ \psi_t)\, dt \leq C \int_{t_0}^{\infty} \frac{1}{t^{\dim V/2}}.$$

If $\dim V \geq 3$, then $\int_{t_0}^{\infty} \bar{\Omega}(\mathbf{1}_{\bar{\mathrm{B}}} \cdot \mathbf{1}_{\bar{\mathrm{B}}} \circ \psi_t)\, dt < \infty$ and Lemma 2.4.3 gives $\bar{\Omega}(\mathcal{K}(\psi)) = 0$, in particular, the system is not ergodic. □

Remark 2.5.4. An ergodicity dichotomy for f^{K} has been previously established by Guivarc'h [**36**, Corollaire 3 on p. 443] under the stronger assumption that K is *aperiodic*.

By means of Markov partitions (Theorem 2.3.2), the above corollary immediately translates to the following. As in the previous section, let X be a compact metric space equipped with a topologically transitive, Hölder-continuous, metric-Anosov flow ϕ. Let $F : X \to V$ be Hölder-continuous and consider the flow $\Phi = (\Phi_t : X \times V \to X \times V)_{t\in\mathbb{R}}$

$$\Phi_t(p, v) = \left(\phi_t p, v - \int_0^s F(\phi_s p)\, ds\right).$$

It is convenient to call the flow Φ by *the skew product of ϕ by F.*

COROLLARY 2.5.5. (Dichotomy) *Assume the group spanned by the periods of* $(1, F)$ *is dense in* $\mathbb{R} \times V$ *and that* $\int F\, dm = 0$ *for the measure of maximal entropy m of ϕ. Then, Φ is mixing as in Theorem 2.5.2, moreover*

$$\dim V \leq 1 \Rightarrow \Phi \textit{ is ergodic with respect to } m \otimes \mathrm{Leb} \Rightarrow \dim V \leq 2.$$

If $\dim V \geq 3$, *then* $\mathcal{K}(\Phi)$ *has zero measure.*

Proof. The proof follows from the corresponding results for subshifts and Remark 2.5.1 describing the measure of maximal entropy of ϕ. □

2.6. *The critical hypersurface.* We recall here two results of Babillot and Ledrappier [**5**]. Their paper concerns differentiable Anosov flows but, as one checks the proofs, only the existence of a Markov coding is required for both their results below. We take the liberty to state them in our broader generality and refer the reader to *loc. cit.* whose proofs work verbatim.

As before, let $F : X \to V$ be Hölder-continuous.

Assumption A. We will assume throughout the remainder of §2 that the closed group Δ spanned by the periods of F has rank $\dim V$, (that is, $\Delta \simeq \mathbb{R}^k \times \mathbb{Z}^{\dim V-k}$ for some $k \in [\![0, \dim V]\!]$) and that, moreover, the group spanned by

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$$\left\{(p(\tau), \int_\tau F) : \tau \text{ periodic}\right\}$$

is isomorphic to $\mathbb{R} \times \Delta$.

The compact convex subset of V,

$$\mathcal{M}^\phi(F) = \left\{ \int_X F d\mu : \mu \in \mathcal{M}^\phi \right\},$$

has hence non-empty interior. Also, for each $\varphi \in V^*$, one can consider the pressure of the function $\varphi(F) : X \to \mathbb{R}$:

$$\mathbf{P}(\varphi) = P(\phi, -\varphi \circ F).$$

By Assumption A, the function $\mathbf{P} : V^* \to \mathbb{R}$ is analytic and strictly convex (Proposition 2.2.5). Using the formula for the derivative of pressure (Proposition 2.2.5), and the natural identification $\mathrm{Gr}_{\dim V-1}(V^*) \to \mathbb{P}(V)$, one has, for $\varphi \in V^*$, that

$$d_\varphi \mathbf{P} = \int F dm_{-\varphi(F)}, \tag{9}$$

where $m_{-\varphi(F)}$ is the equilibrium state of $-\varphi(F)$. One has the following proposition.

PROPOSITION 2.6.1. (Babillot and Ledrappier [5, Proposition 1.1]) *The map $\wp : V^* \to V$ defined by $\varphi \mapsto d_\varphi \mathbf{P}$ is a diffeomorphism between V^* and the interior of $\mathcal{M}^\phi(F)$.*

Let us denote by $\mathscr{L}_F = \mathbb{R}_+ \cdot \mathcal{M}^\varphi(F)$ the closed cone generated by the periods of F. If 0does not belong to $\mathcal{M}^\phi(F)$, then $\mathscr{L}_F$ is a sharp cone (that is, does not contain a hyperplane of V) and its interior is

$$\operatorname{int} \mathscr{L}_F = \mathbb{R}_+ \cdot \operatorname{int}(\mathcal{M}^\phi(F)).$$

One has moreover the following proposition.

PROPOSITION 2.6.2. (Babillot and Ledrappier [5, Proposition 3.1]) *Assume* $0 \notin \mathcal{M}^\phi(F)$. *Then the set* $\wp(\{\varphi \in V^* : \mathbf{P}(\varphi) = 0\})$ *generates the cone* $\operatorname{int} \mathscr{L}_F$.

Remark 2.6.3. Observe that if $\varphi \in V^*$ is such that $\sum_{\tau \text{ periodic}} e^{-\ell_\tau(\varphi\circ F)} < \infty$, then necessarily φ is strictly positive on the cone $\mathscr{L}_F$, that is, $\varphi \in \operatorname{int}(\mathscr{L}_F^*)$. Indeed, if $\sum_\tau e^{-\ell_\tau(\varphi\circ F)}$ is convergent, then the formula for pressure on Proposition 2.2.5 gives that $\mathbf{P}(\varphi) \leq 0$. Since $\mathbf{P}(0) > 0$, there exists $s \in (0, 1]$ such that $\mathbf{P}(s\varphi) = 0$. The variational principle (equation (2)) implies that

$$\int_\tau \varphi(F) \geq 0$$

for every periodic orbit τ, and thus Lemma 2.2.7 applies to give that $\varphi(F)$ is Livšic-cohomologous to a strictly positive function and $\hbar_{\varphi(F)} \in (0, 1]$.

We are thus interested in the *convergence domain* of F

$$\begin{aligned}\mathcal{D}_F &= \{\varphi \in \operatorname{int}(\mathscr{L}_F^*) : \hbar_{\varphi(F)} \in (0, 1)\} \\ &\subset \left\{\varphi \in V^* : \sum_{\tau \text{ periodic}} e^{-\ell_\tau(\varphi\circ F)} < \infty\right\},\end{aligned}$$

and the *critical hypersurface* (also usually called the *entropy-one set* or *the Manhattan curve*)

$$Q_F = \{\varphi \in \operatorname{int}(\mathscr{L}_F^*) : \hbar_{\varphi(F)} = 1\}$$

whose name is justified in the next corollary.

COROLLARY 2.6.4. *Assume $0 \notin \mathcal{M}^\phi(F)$. Then $\mathcal{D}_F = \mathbf{P}^{-1}(-\infty, 0)$ and $Q_F = \mathbf{P}^{-1}(0)$. Consequently, $\mathcal{D}_F$ is a strictly convex set whose boundary coincides with Q_F. The latter is a closed analytic co-dimension-one submanifold of V^*. The map*

$$\varphi \in Q_F \mapsto \mathsf{T}_\varphi Q_F$$

induces a diffeomorphism between Q_F and directions in the interior of the cone $\mathscr{L}_F$.

Proof. We have already shown the inclusions $\mathbf{P}^{-1}(-\infty, 0) \subset \mathcal{D}_F$ and $\mathbf{P}^{-1}(0) \subset Q_F$. The other ones follow at once from Lemmas 2.2.7 and 2.1.1. Since $0 \notin \mathcal{M}^\varphi(F)$, Proposition 2.6.1 implies that $\mathbf{P}$ has no critical points, thus $\mathbf{P}^{-1}(0) = Q_F$ is an analytic sub-manifold of V^*. Strict convexity follows from that of $\mathbf{P}$, and the last assertion follows by observing that the tangent space $\mathsf{T}_\varphi Q_F$ equals $\ker d_\varphi \mathbf{P}$. □

We now focus on the variation of the critical hypersurface when F varies. To this end, consider the Banach space $\operatorname{Holder}^\alpha(X, V)$ of V-valued Hölder continuous functions with exponent α. The pressure can be considered as an analytic map $\mathbf{P} : \operatorname{Holder}^\alpha(X, V) \times V^* \to \mathbb{R}$ defined as $\mathbf{P}(G, \psi) := P(\phi, -\psi(G))$. Its differential at the point (F, φ) on the vector (G, ψ) is

$$d_{(F,\varphi)}\mathbf{P}(G, \psi) = -\int (\psi(F) + \varphi(G))\, dm_{-\varphi(F)},$$

and vanishes identically only if $(F, \varphi) = (0, 0)$. The pre-image $\mathbf{P}^{-1}(0)$ is thus a Banach-manifold.

If $F \in \operatorname{Holder}^\alpha(X, V)$ is such that $0 \notin \mathcal{M}^\phi(F)$, then its critical hypersurface

$$Q_F = \{\varphi \in V^* : (F, \varphi) \in \mathbf{P}^{-1}(0)\}$$

is the intersection of $\{F\} \times V^*$ with the level set $\mathbf{P}^{-1}(0)$. This intersection will vary analytically on compact sets with F as long as the tangent space $\ker d_{(F,\varphi)}\mathbf{P}$ (for fixed (F, φ) with $\mathbf{P}(F, \varphi) = 0$) is transverse to the vector space $\{0\} \times V^*$. Since $\ker d_{(F,\varphi)}\mathbf{P}$ has co-dimension 1, transversality is implied by $\ker d_{(F,\varphi)}\mathbf{P} \cap \{0\} \times V^*$ being co-dimension 1 on V^*. However, by Corollary 2.6.4, this latter intersection is, as long as F verifies assumption A and $0 \notin \mathcal{M}^\phi(F)$, the tangent space $\mathsf{T}_\varphi Q_F$, which has co-dimension 1. We have thus established the following corollary.

COROLLARY 2.6.5. *The critical hypersurface Q_F varies analytically on compact sets when varying the function F among Hölder functions verifying the hypothesis of Corollary 2.6.4.*

2.7. *Dynamical intersection and the critical hypersurface.* We recall here the concept of *dynamical intersection* of Bridgeman *et al* [17]. Similar concepts have been previously

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studied by Bonahon [11], Burger [19], Croke and Fathi [28], and Knieper [42], among others.

Let $f : X \to \mathbb{R}_+$ be a positive Hölder-continuous function and, for $t > 0$, consider the finite set $\mathsf{R}_t(f) = \{\tau \text{ periodic} : \ell_\tau(f) \leq t\}$. Let $g : X \to \mathbb{R}$ be Hölder-continuous (but not necessarily positive). Then the *dynamical intersection* between f and g is defined by

$$\mathbf{I}(f, g) = \lim_{t\to\infty} \frac{1}{\#\mathsf{R}_t(f)} \sum_{\tau\in\mathsf{R}_t(f)} \frac{\ell_\tau(g)}{\ell_\tau(f)}.$$

Then one has the following proposition.

PROPOSITION 2.7.1. [17, §3.4] *One has*

$$\mathbf{I}(f, g) = \frac{\int g dm_{-\hbar_f f}}{\int f\, dm_{-\hbar_f f}},$$

in particular, $\mathbf{I}$ *is well defined and varies analytically with* f *and* g *among Hölder-continuous functions with fixed Hölder exponent. If* g *is moreover positive, then one has* $\mathbf{I}(f, g) \geq \hbar_f/\hbar_g$.

We now place ourselves in the context of the previous subsection, that is, we consider a Hölder-continuous $F : X \to V$ and we assume moreover that $0 \notin \mathcal{M}^\phi(F)$. For $\varphi \in \mathcal{Q}_F$, we consider the map $\mathbf{I}_\varphi : V^* \to \mathbb{R}$ defined by

$$\mathbf{I}_\varphi(\psi) := \mathbf{I}(\varphi(F), \psi(F)) = \frac{\int \psi(F)\, dm_{-\varphi(F)}}{\int \varphi(F)\, dm_{-\varphi(F)}}, \tag{10}$$

where the last equality comes from Proposition 2.7.1 and the fact that $\hbar_{\varphi(F)} = 1$. Observe that it is a linear map. We then have the following explicit interpretation of the tangent space to the critical hypersurface purely in terms of periods.

COROLLARY 2.7.2. *Let* $F : X \to V$ *be as in §2.6 and such that* $0 \notin \mathcal{M}^\phi(F)$. *Then for* $\varphi \in \mathcal{Q}_F$, *one has* $\mathsf{T}_\varphi\mathcal{Q}_F = \ker \mathbf{I}_\varphi$.

Proof. Since by Corollary 2.6.4 one has $\mathcal{Q}_F = \mathbf{P}^{-1}(0)$, the tangent space $\mathsf{T}_\varphi\mathcal{Q}_F = \ker d_\varphi\mathbf{P} = \ker \mathbf{I}_\varphi$, where the last equality comes from the combination of equations (9) and (10). □

3. *A Ledrappier correspondence*

In [44], Ledrappier establishes, for a closed negatively curved manifold M, bijections between Livšic-cohomology classes of *pressure zero functions* on T^1M, *normalized Hölder cocycles* for the action of $\pi_1 M$ on the visual boundary $\partial_\infty\tilde{M}$ of the universal cover of M, *quasi-invariant measures* on $\partial_\infty\tilde{M}$, among other objects. The purpose of this section is to establish, in the context of word-hyperbolic groups with metric-Anosov geodesic flow, some of these correspondences. We also extend results of Sambarino [64–66] to this setting. Some ideas of Bridgeman *et al* [17, 18] and Carvajales [23, Appendix] are used. The reader can also check the work of Paulin, Pollicott, and Schapira [54] for similar results in situations allowing cusps.

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Let Γ be a finitely generated, non-elementary, word-hyperbolic group (see [30] for a definition). Denote by $\mathrm{g} = (\mathrm{g}_t : \mathsf{U}\Gamma \to \mathsf{U}\Gamma)_{t\in\mathbb{R}}$ the *Gromov–Mineyev geodesic* flow of Γ (see [33] and [51]). The total space $\mathsf{U}\Gamma$ is the quotient of $\partial^2\Gamma \times \mathbb{R}$ by a properly discontinuous co-compact Γ-action (defined in *loc. cit.*). This action restricted to $\partial^2\Gamma$ coincides with the induced Γ-action on its Gromov boundary, and commutes with the $\mathbb{R}$-action by translations, giving on the quotient the desired flow g. We will save the notation

$$\widetilde{\mathsf{U}\Gamma}$$

for the pair consisting on the space $\partial^2\Gamma \times \mathbb{R}$ equipped with the above Γ-action.

Assumption B. We will assume throughout §3 that g is metric-Anosov (recall that, in general, g is transitive (see Remark 3.0.1)) and that the lamination induced on the quotient by $\widetilde{\mathscr{W}}^{cu} = \{(x, \cdot, \cdot) \in \widetilde{\mathsf{U}\Gamma}\}$ is the central-unstable lamination of g.

Let us emphasize that, in what follows, the Gromov–Mineyev geodesic flow is merely an auxiliary object. The whole discussion works verbatim replacing g by the following flows known to satisfy Assumption B:

- the non-wandering set of the geodesic flow of a convex co-compact action of Γ on a CAT(-1) space (if this is known to exist), see [27];
- the geodesic flow of a projective-Anosov representation of ρ (again if this is known to exist) as introduced by Bridgeman *et al* [17], see also §5.1.

Recall that every *hyperbolic element* (that is, an infinite order element) $\gamma \in \Gamma$ has two fixed points on $\partial\Gamma$, the attracting γ_+ and the repelling γ_-. If $x \in \partial\Gamma - \{\gamma_-\}$, then $\gamma^n x \to \gamma_+$ as $n \to \infty$. The axis $(\gamma_-, \gamma_+) \times \mathbb{R}$ projects then to a periodic orbit of g, denoted by $[\gamma]$. If $l(\gamma)$ denotes the translation length of γ along this axis, then $l(\gamma)$ is an integer multiple of the period of the periodic orbit $[\gamma]$. Observe that we allow $[\gamma]$ to tour several times along the orbits it surjects to, so at least formally, we let $[\gamma^n]$ be the orbit $[\gamma]$ toured n-times.

Remark 3.0.1. We briefly justify why g is transitive. It suffices to show that given two open sets U, V, there exists $t \in \mathbb{R}$ such that $\mathrm{g}_t(U) \cap V \neq \emptyset$, so the question is reduced to the same question for the action of Γ on $\partial^2\Gamma$; the open sets to be considered can be reduced to be of the form $U_1 \times U_2$ and $V_1 \times V_2$, where $U_i, V_i \subset \partial\Gamma$ are open and $\overline{U_1} \cap U_2 = \overline{V_1} \cap V_2 = \emptyset$; an element $\gamma \in \Gamma$ with $\gamma_- \in U_2$ and $\gamma_+ \in V_1$ has a positive power such that $\gamma^n(U_1 \times U_2) \cap V_1 \times V_2 \neq \emptyset$.

3.1. *The Ledrappier potential of a Hölder cocycle.* Let V be a finite dimensional real vector space. A *Hölder cocycle* is a function $c : \Gamma \times \partial^2\Gamma \to V$ such that:

- for all $\gamma, h \in \Gamma$, one has $c(\gamma h, (x, y)) = c(h, (x, y)) + c(\gamma, h(x, y))$;
- there exists $\alpha \in (0, 1]$ such that for every $\gamma \in \Gamma$, the map $c(\gamma, \cdot)$ is α-Hölder continuous.

The *period* of a Hölder cocycle for a hyperbolic $\gamma \in \Gamma$ is $\ell_c(\gamma) := c(\gamma, (\gamma^-, \gamma^+))$. Two cocycles c, c' are *cohomologous* if there exists a Hölder-continuous function $U : \partial^2\Gamma \to V$

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such that for all $\gamma \in \Gamma$ and $(x, y) \in \partial^2\Gamma$, one has

$$c(\gamma, (x, y)) - c'(\gamma, (x, y)) = U(\gamma(x, y)) - U(x, y).$$

Two cohomologous cocycles have the same marked spectrum $\gamma \mapsto \ell_c(\gamma)$. The following should be compared with [**44**, Théorème 3].

PROPOSITION 3.1.1. *For every Hölder cocycle c, there exists a Hölder-continuous function $\mathcal{J}_c : \mathsf{U}\Gamma \to V$ such that for every hyperbolic $\gamma \in \Gamma$, one has*

$$\int_{[\gamma]} \mathcal{J}_c = \ell_c(\gamma).$$

Cohomologous cocycles induce Livšic-cohomolgous functions.

Proof. The general case follows from the case $V = \mathbb{R}$ by the Riesz representation theorem. Assume thus $V = \mathbb{R}$ and consider the trivial line bundle $\widetilde{\mathsf{U}\Gamma} \times \mathbb{R}$ equipped with the bundle automorphisms

$$\gamma \cdot (p, s) := (\gamma p, e^{-c(\gamma,(x,y))}s),$$

where $p = (x, y, t)$. Denote by $\mathsf{F} \to \mathsf{U}\Gamma$ the quotient line bundle. It is equipped with a flow $(\hat{\mathsf{g}}_t : \mathsf{F} \to \mathsf{F})_{t\in\mathbb{R}}$ by bundle automorphisms, induced on the quotient by

$$t \cdot (p, s) = (\mathsf{g}_t p, s).$$

Let $|\ |$ be a Euclidean metric on F and define, for $v \in \mathsf{F}_p$,

$$\mathcal{T}(p, t) = \log \frac{|\hat{\mathsf{g}}_t v|}{|v|}. \tag{11}$$

It is a translation cocycle over g, indeed since F_p is one dimensional, the choice of v does not matter, and since $\hat{\mathsf{g}}$ is a flow, one has

$$\log \frac{|\hat{\mathsf{g}}_{t+s}(v)|}{|v|} = \log \frac{|\hat{\mathsf{g}}_t(\hat{\mathsf{g}}_s v)|}{|v|}\frac{|\hat{\mathsf{g}}_s(v)|}{|\hat{\mathsf{g}}_s(v)|} = \log \frac{|\hat{\mathsf{g}}_t(\hat{\mathsf{g}}_s v)|}{|\hat{\mathsf{g}}_s(v)|} + \log \frac{|\hat{\mathsf{g}}_s(v)|}{|v|}.$$

By Corollary 2.2.3, there exists a Hölder-continuous function $\mathcal{J}_c : \mathsf{U}\Gamma \to \mathbb{R}$ such that $\mathcal{T}$ and $k_{\mathcal{J}_c}$ are Livšic-cohomologous. We end the proof by a period computation. For every hyperbolic $\gamma \in \Gamma$, one has, for all $s \in \mathbb{R}$, that $\gamma(\gamma_-, \gamma_+, s) = (\gamma_-, \gamma_+, e^{-\ell_c(\gamma)}s)$, or equivalently, for any $x \in [\gamma] \subset \mathsf{U}\Gamma$ and $v \in \mathsf{F}_x$,

$$\ell_c(\gamma) = \log \frac{|\hat{\mathsf{g}}_{p(\gamma)} v|}{|v|} = \ell_{[\gamma]}(\mathcal{T}) = \int_{[\gamma]} \mathcal{J}_c,$$

where $p(\gamma)$ is the period of $[\gamma]$ for g. Since cohomologous cocycles have the same marked spectrum, the associated functions have the same periods and are thus Livšic-cohomolgous by Theorem 2.2.2. □

Definition 3.1.2. We say that $\mathcal{J}_c$ is *a Ledrappier potential* of c over g.

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3.2. *Real cocycles and reparameterizations.* Assume now $V=\mathbb{R}$ and consider a cocycle c with non-negative (and not all vanishing) periods. Define its *entropy* by

$$\hbar_c=\limsup_{t\to\infty}\frac{1}{t}\log\#\{[\gamma]\in[\Gamma]\text{ hyperbolic}:\ell_c(\gamma)\leq t\}.$$

Remark 3.2.1. It follows from Proposition 3.1.1 and Remark 2.2.6 that $\hbar_c>0$.

For such a cocycle, consider the action of Γ on $\partial^2\Gamma\times\mathbb{R}$ via c:

$$\gamma\cdot(x,y,t)=(\gamma x,\gamma y,t-c(\gamma,(x,y))). \tag{12}$$

Let us denote by χ^c the quotient space $\chi^c=\Gamma\backslash(\partial^2\Gamma\times\mathbb{R})$. The following is reported by Sambarino [65] for fundamental groups of closed negatively curved manifolds and by Carvajales [23] for the refraction cocycle of a Δ-Anosov representation (see Definition 5.3.1).

THEOREM 3.2.2. *If c is a Hölder cocycle with non-negative periods and finite entropy, then its Ledrappier potential is Livšic-cohomologous to a strictly positive function. Moreover, the above action of Γ on $\partial^2\Gamma\times\mathbb{R}$ is properly discontinuous and co-compact, and the flow $\phi^c=(\phi^c_t:\chi^c\to\chi^c)_{t\in\mathbb{R}}$ induced on the quotient by the $\mathbb{R}$-translation flow is Hölder-conjugated to the reparameterization of g by $\mathcal{J}_c$.*

The topological entropy of ϕ^c is thus $\hbar_c$. (When Γ has torsion elements, this fact requires some work, see the work of Carvajales *et al* [24, §5] for details.)

Proof. The first assertion follows at once from Lemma 2.2.7. For the remaining statements, we continue as in the proof of the proposition but for the cocycle $-c$. Observe first that $-\mathcal{J}_c=\mathcal{J}_{-c}$. Since $\mathcal{T}$ and $k_{\mathcal{J}_{-c}}$ are Livšic-cohomologous, there exists $U:\mathsf{U}\Gamma\to\mathbb{R}$ such that for all $t\in\mathbb{R}$, $p\in\mathsf{U}\Gamma$, and $v\in\mathsf{F}_p$, one has (recall equation (11))

$$\log\frac{|\hat{\mathsf{g}}_t v|}{|v|}-\int_0^t\mathcal{J}_{-c}=U(\mathsf{g}_t p)-U(x).$$

Since $\mathcal{J}_{-c}=-\mathcal{J}_c$ is Livšic-cohomologous to a strictly negative function, the above equation implies that the flow $\hat{\mathsf{g}}$ is contracting on F, that is, there exist positive C and μ such that for all $v\in\mathsf{F}$ and $t\in\mathbb{R}$, one has

$$|\hat{\mathsf{g}}_t v|\leq Ce^{-\mu t}|v|.$$

A standard procedure (see for example [41] or [17, Lemma 4.3]) provides a Euclidean metric $\|\ \|$ on F such that the constant C equals 1. We denote also by $\|\ \|$ the lift of this metric to $\widetilde{\mathsf{U}\Gamma}\times\mathbb{R}$, it is a Γ-invariant family.

Given then $(x,y,t)\in\widetilde{\mathsf{U}\Gamma}$, we let $v_{(x,y,t)}\in(\mathbb{R}-\{0\})/\pm$ be such that $\|v_{(x,y,t)}\|=1$. As in [17, Proposition 4.2], the map

$$\begin{aligned}\widetilde{\mathsf{U}\Gamma}\to\partial^2\Gamma\times(\mathbb{R}-\{0\}/\pm)&\to\partial^2\Gamma\times\mathbb{R},\\ \xi:(x,y,t)\mapsto(x,y,v_{(x,y,t)})&\mapsto(x,y,\log v_{(x,y,t)}),\end{aligned} \tag{13}$$

is Γ-equivariant and an orbit equivalence between the $\mathbb{R}$-actions. $\square$

Definition 3.2.3. The flow ϕ^c will be called the *refraction flow* of c.

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3.3. *Patterson–Sullivan measures.* Let us consider now Hölder cocycles with $V=\mathbb{R}$ and only depending on one variable, that is, $c:\Gamma\times\partial\Gamma\to\mathbb{R}$. Assume moreover that c has non-negative periods and finite entropy. A cocycle $\bar{c}:\Gamma\times\partial\Gamma\to\mathbb{R}$ is *dual to c* if for every hyperbolic $\gamma\in\Gamma$, one has

$$\ell_{\bar{c}}(\gamma)=\ell_c(\gamma^{-1}).$$

Definition 3.3.1

- A *Patterson–Sullivan measure for c of exponent* $\delta\in\mathbb{R}_+$ is a probability measure μ on $\partial\Gamma$ such that for every $\gamma\in\Gamma$, one has
$$\frac{d\gamma_*\mu}{d\mu}(\cdot)=e^{-\delta c(\gamma^{-1},\cdot)}. \tag{14}$$
- A *Gromov product* for the ordered pair $(\bar{c},c)$ is a function $[\cdot,\cdot]:\partial^2\Gamma\to\mathbb{R}$ such that for all $\gamma\in\Gamma$ and $(x,y)\in\partial^2\Gamma$, one has
$$[\gamma x,\gamma y]-[x,y]=-(\bar{c}(\gamma,x)+c(\gamma,y)).$$

Consider a pair of dual cocycles $(\bar{c},c)$ and assume a Patterson–Sullivan measure of exponent δ exists for each c and $\bar{c}$, denoted by μ and $\bar{\mu}$, respectively. Assume moreover that a Gromov product for the pair $(\bar{c},c)$ exists. The measure

$$\widetilde{m}=e^{-\delta[\cdot,\cdot]}\bar{\mu}\otimes\mu\otimes\, dt \tag{15}$$

on $\partial^2\Gamma\times\mathbb{R}$ is hence Γ-invariant (the action being via c, as in equation (12) and $\mathbb{R}$-invariant. Passing to the quotient, one obtains a measure m on χ^c invariant under the flow ϕ^c. Observe that we can write $\widetilde{m}$ as

$$d\widetilde{m}(x,y,t)=\int_{\partial\Gamma\times\mathbb{R}}e^{-\delta t}\,d\mu(y)\,dt\bigg(\int_{\partial\Gamma}e^{\delta t}e^{-\delta[x,y]}d\bar{\mu}(x)\bigg).$$

The measure $e^{-\delta t}\,d\mu\,dt$ is Γ-invariant on $\partial\Gamma\times\mathbb{R}$ (indeed, if $f:\partial\Gamma\times\mathbb{R}\to\mathbb{R}$ is continuous, then

$$\begin{aligned}\int f(\gamma(y,t))e^{-\delta t}\,d\mu\,dt,&=\int f(\gamma y,t-c(\gamma,y))e^{-\delta t}\,d\mu\,dt\\ &=\int f(\gamma y,t)e^{-\delta(t-c(\gamma,y))}\,d\mu\,dt &&\text{(by translation invariance)}\\ &=\int f(y,t)e^{-\delta(t-c(\gamma,\gamma^{-1}y))}e^{-\delta c(\gamma^{-1},y)}\,d\mu\,dt &&\text{(by definition of }\mu)\\ &=\int f(y,t)e^{-\delta t}\,d\mu\,dt)\end{aligned}$$

and the family

$$m_{(y,t)}=e^{\delta t}e^{-\delta[\cdot,y]}d\bar{\mu}$$

is Γ-equivariant.

This decomposition induces then a local decomposition of m as a product of measures on the laminations induced on the quotient by $\widetilde{\mathcal{W}}^{cu}=\{(x,\cdot,\cdot):x\in\partial\Gamma\}$ and $\widetilde{\mathcal{V}}=\{(\cdot,y,t):y\in\partial\Gamma,\ t\in\mathbb{R}\}$. The local product structure induced by $\partial\Gamma\times\partial\Gamma\times\mathbb{R}$ permits

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to transport the measures $m_{(y,t)}$, parallel to the central stable leaf $W^{\mathrm{cu}}[(x, y, t)]$, to the stable leaf of $[(x, y, t)]$ for ϕ^c to obtain a family of measures at each strong stable leaf ν_p^{s} such that

$$\frac{d(\phi^c_{-t})_*\nu^{\mathrm{s}}_p}{d\nu^{\mathrm{s}}_{\phi^c_t p}} = e^{-\delta t}. \tag{16}$$

Margulis's description of the measure maximizing entropy (§2.3.4) then gives that $\delta = \hbar_c$ and that $\mathscr{m}$ maximizes entropy for ϕ^c. Thus, subject to the existence of the Patterson–Sullivan measures and the Gromov product, we summarize the above discussion in the following proposition.

PROPOSITION 3.3.2. *The measure $\mathscr{m}$ on χ^c maximizes entropy for the flow ϕ^c. The exponent δ necessarily equals the topological entropy of ϕ^c, $\hbar_c$, if ν is another Patterson–Sullivan measure for c then $\nu = \mu$.*

Let us consider again the measure $\widetilde{\mathscr{m}}$ on $\partial^2\Gamma \times \mathbb{R}$ from equation (15) but let us instead study the Γ-action on the $\mathbb{R}$-coordinate by the Gromov–Mineyev cocycle, so that $\Gamma\backslash(\partial^2\Gamma \times \mathbb{R}) = \mathsf{U}\Gamma$ and the induced flow is g. The quotient measure, $\mathscr{a}$, is g-invariant and the orbit equivalence ξ from equation (13) preserves, by the way it is defined, zero flow-invariant sets between $\mathscr{a}$ and $\mathscr{m}$. Since ξ is a conjugation between $\mathrm{g}^{\mathcal{J}_c}$ and ϕ^c, and $\mathscr{m}$ maximizes entropy for ϕ^c, the measure $\xi_*\mathscr{m}$ is $\mathrm{g}^{\mathcal{J}_c}$-invariant, maximizes entropy for $\mathrm{g}^{\mathcal{J}_c}$, and has the same zero sets as $\mathscr{a}$.

One concludes that the Abramov transform in equation (3) $\mathscr{a}^\#$ (with respect to $\mathcal{J}_c$) is a measure of maximal entropy of the flow $\mathrm{g}^{\mathcal{J}_c}$. Lemma 2.1.1 implies then that $\mathscr{a}/|\mathscr{a}|$ is the equilibrium state of $-\hbar_c\mathcal{J}_c$. Let us summarize in the following remark.

Remark 3.3.3. The probability measure $\mathscr{a}/|\mathscr{a}|$ on $\mathsf{U}\Gamma = \Gamma\backslash\widetilde{\mathsf{U}\Gamma}$ induced by $\widetilde{\mathscr{m}}$ is the equilibrium state of $-\hbar_c\mathcal{J}_c$.

Since the zero sets of an equilibrium state are uniquely determined by the Livšic-cohomology class of the associated potential up to an additive constant (Theorem 2.3.3), one concludes the following corollary.

COROLLARY 3.3.4. *Let $\bar{\kappa}, \kappa$ be a pair of dual cocycles with non-negative periods and finite entropy. Assume Patterson—Sullivan measures, ν and $\bar{\nu}$, exist for κ and $\bar{\kappa}$, respectively, together with a Gromov product for the pair. If ν has the same zero sets as μ, then the (scaled) Ledrappier potentials $\hbar_c\mathcal{J}_c$ and $\hbar_\kappa\mathcal{J}_\kappa$ are Livšic-cohomologous and $\nu = \mu$.*

Proof. A final argument is required, indeed from §2.3.3, there exists a constant K such that $\hbar_c\mathcal{J}_c$ and $\hbar_\kappa\mathcal{J}_\kappa + K$ are Livšic-cohomologous. However, since $\hbar_c$ is the topological entropy of $\mathrm{g}^{\mathcal{J}_c}$, Lemma 2.1.1 gives

$$0 = P(\mathrm{g}, -\hbar_c\mathcal{J}_c) = P(\mathrm{g}, -\hbar_\kappa\mathcal{J}_\kappa + K) = K,$$

where the second equality holds by Remark 2.1.2, and the third equality by the definition of P (equation (2)). □

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The above corollary was reported by Ledrappier [44] when Γ is the fundamental group of a negatively curved closed manifold. The proof uses also a disintegration argument. One may also check [39] and Babillot's survey [4] specifically for the Buseman cocycle (Γ as in Ledrappier's aforementioned situation), and [23, Appendix] for the refraction cocycle β_φ (see §5.3 for the definition) of a Δ-Anosov representation of an arbitrary word-hyperbolic group.

3.4. *Vector-valued cocycles I: the critical hypersurface.* Let now $c : \Gamma \times \partial\Gamma \to V$ be a Hölder cocycle and consider the compact convex set $\mathcal{M}^{\mathrm{g}}(\mathcal{J}_c) \subset V$. Since this set depends on the base flow g, it is natural to consider the *limit cone* of c:

$$\mathscr{L}_c = \overline{\bigcup_{\gamma\in\Gamma} \mathbb{R}_+ \cdot \ell_c(\gamma)} = \mathbb{R}_+ \cdot \mathcal{M}^{\mathrm{g}}(\mathcal{J}_c).$$

Remark 3.4.1. Up to Livšic-cohomology, we can assume that $\mathcal{J}_c$ has values in the vector space $V' = \langle \mathcal{M}^{\mathrm{g}}(\mathcal{J}_c)\rangle$. We can moreover choose a reparameterization g^f of g so that if $\mathcal{J}_c^{\mathrm{g}^{\mathrm{f}}} : \mathsf{U}\Gamma \to V'$ is the Ledrappier potential for c over g^f, then the flow g^f together with the potential $\mathcal{J}_c^{\mathrm{g}^{\mathrm{f}}}$ verify Assumption A from §2.6.

Proof. By Remark 2.2.7, the space of Livšic-cohomology classes over g is infinite dimensional, so the remark follows. □

We will work from now on with flow g^f given by the remark and the Ledrappier potential $\mathcal{J}_c^{\mathrm{g}^{\mathrm{f}}}$. We will rename these by g and $\mathcal{J}_c$ though as to not overcharge the paper with notation and keep in mind that when we restrict the image of $\mathcal{J}_c$ to V', Assumption A is verified.

Let $(\mathscr{L}_c)^* = \{\psi \in V^* : \psi|\mathscr{L}_c \geq 0\}$ be the *dual cone* of c. For $\psi \in V^*$, denote $c_\psi = \psi \circ c : \Gamma \times \partial\Gamma \to \mathbb{R}$ and $\hslash_\psi = \hslash_{c_\psi}$.

Assumption C. There exists $\psi \in (\mathscr{L}_c)^*$ such that c_ψ has finite entropy.

LEMMA 3.4.2. *In this case,* $0 \notin \mathcal{M}^{\mathrm{g}}(\mathcal{J}_c)$, $(\mathscr{L}_c)^*$ *has non-empty interior, and* $\mathrm{int}(\mathscr{L}_c)^* = \{\varphi \in (\mathscr{L}_c)^* : \hslash_\varphi < \infty\}$.

Proof. The lemma follows essentially from §3.1 and Lemma 2.2.7, indeed since $\hslash_\psi = \hslash_{\psi(\mathcal{J}_c)} < \infty$, there exists $\kappa > 0$ such that for all hyperbolic $\gamma \in \Gamma$, it holds $\psi(\ell_c(\gamma)/p(\gamma)) > \kappa$; by density of periodic orbits on $\mathcal{M}^{\mathrm{g}}$, one has $\inf\{\psi(\mathcal{M}^{\mathrm{g}}(\mathcal{J}_c))\} > \kappa > 0$. The remaining statements follow similarly. □

Since $0 \notin \mathcal{M}^{\phi^{c_\psi}}(\mathcal{J}_c)$, we can apply Corollary 2.6. Denote by

$$\mathcal{Q}_c = \{\varphi \in \mathrm{int}(\mathscr{L}_c)^* : \hslash_\varphi = 1\},$$
$$\mathcal{D}_c = \{\varphi \in \mathrm{int}(\mathscr{L}_c)^* : \hslash_\varphi \in (0,1)\} \subset \left\{\varphi \in V^* : \sum_{[\gamma]\in[\Gamma]} e^{-\ell_{c_\varphi}(\gamma)} < \infty\right\}$$

respectively the *critical hypersurface* and the *convergence domain* of c.

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Since we have not required the cone $\mathscr{L}_c$ to have non-empty interior, consider its annihilation space

$$\mathrm{Ann}(\mathscr{L}_c) = \{\psi \in V^* : \mathscr{L}_c \subset \ker \psi\}.$$

If $\varphi \in \mathrm{int}(\mathscr{L}_c)^*$ and $\psi \in \mathrm{Ann}(\mathscr{L}_c)$, then the potentials $\varphi(\mathcal{J}_c)$ and $(\varphi + \psi)(\mathcal{J}_c)$ are Livšic-cohomologous. Let $\pi^c : V^* \to V^*/\mathrm{Ann}(\mathscr{L}_c)$ be the quotient projection.

We also import the concept of dynamical intersection of §2.7 to this setting using the Ledrappier potential of c. For $\varphi \in Q_c$, define the *dynamical intersection map* associated to c by $\mathbf{I}_\varphi = \mathbf{I}^c_\varphi : V^* \to \mathbb{R}$ be defined by

$$\mathbf{I}_\varphi(\psi) = \mathbf{I}(\varphi(\mathcal{J}_c), \psi(\mathcal{J}_c)).$$

By definition, $\mathbf{I}(\varphi(\mathcal{J}_c), \psi(\mathcal{J}_c))$ only depends on the Livšic-cohomology classes of $\varphi(\mathcal{J}_c)$ and $\psi(\mathcal{J}_c)$, so we may freely consider $\mathbf{I}$ as defined on $Q_c \times V^*$ or on $\pi^c(Q_c) \times V^*/\mathrm{Ann}(\mathscr{L}_c)$. One has the following corollary.

COROLLARY 3.4.3. *Under Assumption C, one has that $\pi^c(\mathcal{D}_c)$ is a strictly convex set whose boundary is $\pi^c(Q_c)$. The latter is an analytic co-dimension-one sub-manifold. The map $\mathsf{u} : \pi^c(Q_c) \to \mathbb{P}(\mathrm{span}\{\mathscr{L}_c\})$ defined by*

$$\varphi \mapsto \mathsf{u}_\varphi := \mathsf{T}_\varphi \pi^c(Q_c) = \ker \mathbf{I}_\varphi$$

is an analytic diffeomorphism between $\pi^c(Q_c)$ and directions in the relative interior of $\mathscr{L}_c$.

Proof. By Remark 3.4.1, we can apply §2.6 to $\mathcal{J}_c$, the equality $\mathsf{T}_\varphi Q_c = \ker \mathbf{I}_\varphi$ follows from Corollary 2.7.2. □

3.5. *Vector-valued cocycles II: skew-product structure.* We remain in the situation of §3.4, that is, we keep Assumption C. It follows at once from Theorem 3.2 that the Γ-action $\partial^2\Gamma \times V$

$$\gamma(x, y, v) = (\gamma x, \gamma y, v - c(\gamma, y))$$

is properly discontinuous. We aim to give a description of the V-action on the quotient space $\Gamma\backslash(\partial^2\Gamma \times V)$.

By Lemma 3.4.2 and Theorem 3.2.2, we can, for every $\varphi \in \mathrm{int}(\mathscr{L}_c)^*$, consider the refraction flow $\phi^{c_\varphi} = (\phi^{c_\varphi}_t : \chi^{c_\varphi} \to \chi^{c_\varphi})_{t\in\mathbb{R}}$. Such a φ is fixed from now on.

Remark 3.5.1. Let us still denote by $\mathcal{J}_c$ the Ledrappier potential for c over the flow ϕ^{c_φ}, that is, for every hyperbolic $\gamma \in \Gamma$, one has

$$\int_{[\gamma]} \mathcal{J}_c = \ell_c(\gamma) \in V.$$

Let $\not{p}$ be the probability measure of maximal entropy of ϕ^{c_φ}. The *growth direction of* φ is the line of V

$$\mathsf{u}_\varphi = \mathsf{T}_\varphi Q_c = \mathbb{R} \cdot \int_{\chi^{c_\varphi}} \mathcal{J}_c \, d\not{p} \tag{17}$$

(the last equality follows from equation (9) and Corollary 2.6.4). Consider also the projection $\pi^\varphi : V \to \ker\varphi$ parallel to u_φ and denote by $\mathcal{J}_c^\varphi : \chi^{c_\varphi} \to \ker\varphi$ the composition $\mathcal{J}_c^\varphi = \pi^\varphi \circ \mathcal{J}_c$. Observe that

$$\int_{\chi^{c_\varphi}} \mathcal{J}_c^\varphi \, d\mathscr{p} = 0. \tag{18}$$

Fix $u_\varphi \in \mathsf{u}_\varphi$ with $\varphi(u_\varphi) = 1$ and define the *directional flow* $(\omega_t^\varphi : \Gamma\backslash(\partial^2\Gamma \times V) \to \Gamma\backslash(\partial^2\Gamma \times V))_{t\in\mathbb{R}}$ by induction on the quotient of

$$t \cdot (x, y, v) = (x, y, v - tu_\varphi).$$

PROPOSITION 3.5.2. (Sambarino [66]) *There exists a (bi)-Hölder-continuous homeomorphism*

$$\overline{E} : \Gamma\backslash(\partial^2\Gamma \times V) \to \chi^{c_\varphi} \times \ker\varphi,$$

commuting with the $\ker\varphi$ *action, that conjugates the flow* ω^φ *with* $\Phi^\varphi = (\Phi_t^\varphi : \chi^{c_\varphi} \times \ker\varphi \to \chi^{c_\varphi} \times \ker\varphi)_{t\in\mathbb{R}}$

$$\Phi_t^\varphi(p, v) := \bigg(\phi_t^{c_\varphi}(p), v - \int_0^t \mathcal{J}_c^\varphi(\phi_s^{c_\varphi} p)\, ds\bigg).$$

Proof. This is the first item of Sambarino [66, Proposition 3.5] when Γ is the fundamental group of a closed negatively curved manifold. The proof adapts *mutatis mutandis* once §3.2 is established. □

Let us moreover place ourselves in the existence assumptions of §3.3 for the cocycle c_φ, that is, assume there exists:

- a dual cocycle $\bar{c}_\varphi$ together with a Gromov product $[\cdot,\cdot]$ for the pair $(\bar{c}_\varphi, c_\varphi)$;
- a Patterson–Sullivan measure for each c_φ and $\bar{c}_\varphi$, denoted by μ and $\bar{\mu}$, respectively. Recall from Proposition 3.3.2 that, necessarily, the exponent of both μ and $\bar{\mu}$ is $\hbar_{c_\varphi}$, the topological entropy of ϕ^{c_φ}.

By Proposition 3.3.2, the measure $\mathscr{m}$ maximizes entropy for ϕ^{c_φ}. One has then $\mathscr{p} = \mathscr{m}/|\mathscr{m}|$. Consider the φ*-Bowen–Margulis* measure Ω^φ on $\Gamma\backslash(\partial^2\Gamma \times V)$ defined as induction on the quotient by

$$e^{-\hbar_{c_\varphi}[\cdot,\cdot]}\bar{\mu} \otimes \mu \otimes \mathrm{Leb}_V,$$

for a Lebesgue measure on V. One has the following result.

PROPOSITION 3.5.3. (Sambarino [66]) *The (bi)-Hölder-continuous homeomorphism from Proposition 3.5.2 is a measurable isomorphism between* Ω^φ *and* $\mathscr{m} \otimes \mathrm{Leb}_{\ker\varphi}$.

Proof. This follows again as in [66, Proposition 3.5] once Proposition 3.3.2 is established. □

3.6. *Vector-valued cocycles III: dynamical consequences.* We remain in the situation of §3.4. Let us say that c is *non-arithmetic* if the periods of c span a dense subgroup in V.

One concludes at once the following consequences.

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COROLLARY 3.6.1. (Ergodicity dichotomy) *Assume c is non-arithmetic, then ω^{φ} is mixing as in Theorem 2.5.2, consequently if* $\dim V \geq 4$, *then* $\mathcal{K}(\omega^{\varphi})$ *has zero* Ω^{φ}*-measure. If* $\dim V \leq 2$, *then the directional flow* ω^{φ} *is* Ω^{φ}*-ergodic.*

Proof. By Proposition 3.5.2, the flow ω^{φ} is Hölder-conjugated to the skew-product of $\phi^{c_{\varphi}}$ with the Ledrappier potential $\mathcal{J}_c^{\varphi}$, the fiber being $\ker \varphi$ and thus $\dim V - 1$-dimensional. Proposition 3.5.3 describes the desired measures in terms of the skew-product structure; non-arithmeticity of c and equation (18) allow us to apply Corollary 2.5.5. □

4. *Algebraic semi-simple groups over a local field*

This section is a collection of necessary language and basic results needed for the following section. Most of the material covered here can be found in, for example, Borel's book [12] and/or in the book by Benoist and Quint [9].

Let $\mathbb{K}$ be a local field. If $\mathbb{K}$ is non-Archimedean, let us denote by q the cardinality of its residue field, $u \in \mathbb{K}$ a uniformizing element, and choose the norm $|\ |$ on $\mathbb{K}$ so that $|u| = q^{-1}$. In this case, log denotes the logarithm on base q, so that $\log q = 1$. If $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, then $|\ |$ is the standard modulus, $q := e$, and log is the usual logarithm.

Let G be the $\mathbb{K}$-points of a connected semi-simple $\mathbb{K}$-group, A the $\mathbb{K}$-points of a maximal $\mathbb{K}$-split torus, and $\mathbf{X}(\mathsf{A})$ the group of its $\mathbb{K}^*$-characters. Consider the real vector space $\mathsf{E}^* = \mathbf{X}(\mathsf{A}) \otimes_{\mathbb{Z}} \mathbb{R}$ and E its dual. For every $\chi \in \mathbf{X}(\mathsf{A})$, we denote by χ^{ω} the corresponding linear form on E.

Let Φ be the set of restricted roots of A in $\mathfrak{g}$, the set Φ^{ω} is a root system of E^*. Let $(\Phi^{\omega})^+$ be a system of positive roots, E^+ the associated Weyl chamber, and Φ^+ and $\Delta \subset \Phi$ the corresponding system of positive roots and simple roots, respectively.

Let $\nu : \mathsf{A} \to \mathsf{E}$ be defined, for $z \in \mathsf{A}$, as the unique vector in E such that for every $\chi \in \mathbf{X}(\mathsf{A})$, one has

$$\chi^{\omega}(\nu(z)) = \log |\chi(z)|.$$

Denote $\mathsf{A}^+ = \nu^{-1}(\mathsf{E}^+)$.

Let $\mathcal{W}$ be the Weyl group of Φ. It is isomorphic to the quotient of the normalizer $N_{\mathsf{G}}(\mathsf{A})$ of A in G by its centralizer $Z_{\mathsf{G}}(\mathsf{A})$. Let $\mathrm{i} : \mathsf{E} \to \mathsf{E}$ be the opposition involution: if $u : \mathsf{E} \to \mathsf{E}$ is the unique element in the Weyl group with $u(\mathsf{E}^+) = -\mathsf{E}^+$, then $\mathrm{i} = -u$.

4.1. *Restricted roots and parabolic groups.*

Consider $\vartheta \subset \Delta$ and let P_{ϑ}, respectively $\check{\mathsf{P}}_{\vartheta}$, be the normalizers in G of, respectively the Lie algebras

$$\bigoplus_{\alpha \in <\Delta - \vartheta>} \mathfrak{g}_{-\alpha} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_{\alpha},$$
$$\bigoplus_{\alpha \in <\Delta - \vartheta>} \mathfrak{g}_{\alpha} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_{-\alpha}.$$

The ϑ*-flag space* is $\mathcal{F}_{\vartheta} = \mathsf{G}/\mathsf{P}_{\vartheta}$. The orbit $\mathsf{G} \cdot ([\check{\mathsf{P}}_{\vartheta}], [\mathsf{P}_{\vartheta}]) \subset \mathcal{F}_{\mathrm{i}\vartheta} \times \mathcal{F}_{\vartheta}$ is the unique open orbit on this product space, we will denote it by $\mathcal{F}_{\vartheta}^{(2)}$ and say that $(x, y) \in \mathcal{F}_{\mathrm{i}\vartheta} \times \mathcal{F}_{\vartheta}$ are *transverse* if in fact $(x, y) \in \mathcal{F}_{\vartheta}^{(2)}$.

Denote by $(\cdot, \cdot)$ a $\mathcal{W}$-invariant inner product on E, $(\cdot, \cdot)$ the induced inner product on E^*, define $\langle , \rangle$ on E^* by

$$\langle \chi, \psi \rangle = \frac{2(\chi, \psi)}{(\psi, \psi)},$$

and let $\{\varpi_\alpha\}_{\alpha \in \Pi}$ be *the fundamental weights* of Φ, defined by the equations $\langle \varpi_\alpha, \sigma \rangle = d_\alpha \delta_{\alpha\sigma}$, where $d_\alpha = 1$ if $2\alpha \notin \Phi$ and $d_\alpha = 2$ otherwise.

4.2. *The center of the Levi group.* We now consider the vector subspace

$$\mathsf{E}_\vartheta = \bigcap_{\alpha \in \Delta - \vartheta} \ker \alpha^\omega$$

together with the unique projection $p_\vartheta : \mathsf{E} \to \mathsf{E}_\vartheta$ invariant under the subgroup of the Weyl group $\mathcal{W}_\vartheta = \{w \in \mathcal{W} : w|\mathsf{E}_\vartheta = \mathrm{id}\}$. The dual space $(\mathsf{E}_\vartheta)^*$ sits naturally as the subspace of E^* of p_ϑ-invariant linear forms

$$(\mathsf{E}_\vartheta)^* = \{\varphi \in \mathsf{E}^* : \varphi \circ p_\vartheta = \varphi\}.$$

It is spanned by the fundamental weights $\{\varpi_\sigma|\mathsf{E}_\vartheta : \sigma \in \vartheta\}$.

4.3. *Cartan decomposition.* Let $\mathsf{K} \subset \mathsf{G}$ be a compact group that contains a representative for every element of the Weyl group $\mathcal{W}$, this is to say, such that the normalizer $N_\mathsf{G}(\mathsf{A})$ verifies $N_\mathsf{G}(\mathsf{A}) = (N_\mathsf{G}(\mathsf{A}) \cap \mathsf{K})\mathsf{A}$. One has $\mathsf{G} = \mathsf{K}\mathsf{A}^+\mathsf{K}$ and if $z, w \in \mathsf{A}^+$ are such that $z \in \mathsf{K}w\mathsf{K}$, then $\nu(z) = \nu(w)$. There exists thus a function

$$a : \mathsf{G} \to \mathsf{E}^+$$

such that for every $g_1, g_2 \in \mathsf{G}$, one has $g_1 \in \mathsf{K}g_2\mathsf{K}$ if and only if $a(g_1) = a(g_2)$. It is called the *Cartan projection* of G.

4.4. *Jordan decomposition.* Recall that the Jordan decomposition states that every $g \in \mathsf{G}$ has a power g^k ($k = 1$ if $\mathbb{K}$ is Archimedean) that can be written as a commuting product $g = g_e g_h g_n$, where g_e is elliptic, g_h is semi-simple over $\mathbb{K}$, and g_n is unipotent. The component g_h is conjugate to an element $z_g \in \mathsf{A}^+$ and we let

$$\lambda(g) = (1/k)\nu(z_g) \in \mathsf{E}^+.$$

The map $\lambda : \mathsf{G} \to \mathsf{E}^+$ is *the Jordan projection* of G. We will also denote by $\lambda_\vartheta : \mathsf{G} \to \mathsf{E}_\vartheta$ the composition $p_\vartheta \circ \lambda$. For $\mathsf{G} = \mathsf{PGL}_d(\mathbb{K})$, we will denote by $\lambda_1(g)$ the logarithm of the spectral radius of g.

4.5. *Representations of* G. Let V be a finite-dimensional $\mathbb{K}$-vector space and $\phi : \mathsf{G} \to \mathsf{PGL}(\mathsf{V})$ be an algebraic irreducible representation. Then the *weight space* associated to $\chi \in \mathbf{X}(\mathsf{A})$ is the vector space

$$\mathsf{V}_\chi = \{v \in \mathsf{V} : \phi(a)v = \chi(a)v \text{ for all } a \in \mathsf{A}\}$$

and if $V_\chi \neq 0$, then we say that $\chi^\omega \in \mathsf{E}^*$ is a *restricted weight* of ϕ. Theorem 7.2 of Tits [69] states that the set of weights has a unique maximal element with respect to the

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order $\chi \geq \psi$ if $\chi - \psi$ is a sum of simple roots with non-negative coefficients. This is called *the highest weight* of ϕ and denoted by χ_ϕ.

We denote by $\| \ \|_\phi$ a norm on V invariant under $\phi\mathsf{K}$ and such that $\phi\mathsf{A}$ consists on semi-homotheties (that is, diagonal on an orthonormal basis $\mathcal{E}$ of V, in the classical sense if $\mathbb{K}$ Archimedean, and such that $\| \sum_{e\in\mathcal{E}} v_e e\| = \max\{|v_e|\}$ if $\mathbb{K}$ is non-Archimedean). If $\mathbb{K}$ is Archimedean, the existence of such a norm is classical (see for example [9, Lemma 6.33]), if $\mathbb{K}$ is non-Archimedean, then this is the content of [60, Théorème 6.1] due to Quint.

For every $g \in \mathsf{G}$, one has then

$$\log \|\phi g\|_\phi = \chi_\phi^\omega(a(g)),$$
$$\log \lambda_1(\phi g) = \chi_\phi^\omega(\lambda(g)). \tag{19}$$

Denote by W_{χ_ϕ} the $\phi\mathsf{A}$-invariant complement of V_{χ_ϕ}. The stabilizer in G of W_{χ_ϕ} is $\check{\mathsf{P}}_{\vartheta,\mathbb{K}}$, and thus one has a map of flag spaces

$$(\Xi_\phi, \Xi_\phi^*) : \mathcal{F}_{\vartheta_\phi}^{(2)}(\mathsf{G}) \to \mathrm{Gr}^{(2)}_{\dim \mathsf{V}_{\chi_\phi}}(\mathsf{V}), \tag{20}$$

where $\vartheta_\phi = \{\sigma \in \Delta : \chi_\phi - \sigma \text{ is a weight of } \phi\}$. This is a proper embedding which is a homeomorphism onto its image. Here, $\mathscr{G}^{(2)}_{\dim \mathsf{V}_{\chi_\phi}}(\mathsf{V})$ is the open $\mathsf{PGL}(\mathsf{V})$-orbit in the product of the Grassmannians of $(\dim \mathsf{V}_{\chi_\phi})$-dimensional and $(\dim \mathsf{V} - \dim \mathsf{V}_{\chi_\phi})$-dimensional subspaces.

One has the following proposition from Tits [69] that guarantees existence of certain representations of G. We say that ϕ is *proximal* if $\dim \mathsf{V}_{\chi_\phi} = 1$.

PROPOSITION 4.5.1. (Tits [69]) *For every $\sigma \in \Delta$, there exists an irreducible proximal representation of G whose highest restricted weight is $l_\sigma \varpi_\sigma$ for some $l_\sigma \in \mathbb{Z}_{\geq 1}$.*

Definition 4.5.2. We will fix and denote by $\phi_\sigma : \mathsf{G} \to \mathsf{GL}(\mathsf{V}_\sigma)$ such a set of representations.

4.6. *Buseman–Iwasawa cocycle.* The *Iwasawa decomposition* of G states that every $g \in \mathsf{G}$ can be written as a product lzu with $l \in \mathsf{K}$, $z \in \mathsf{A}$, and $u \in \mathsf{U}_\Delta$, where U_Δ is the unipotent radical of P_Δ. When $\mathbb{K}$ is non-Archimedean, the Iwasawa decomposition is not unique; however, if $z_1, z_2 \in \mathsf{A}$ are such that $z_1 \in \mathsf{K}z_2\mathsf{U}_\Delta$, then $\nu(z_1) = \nu(z_2)$.

The *Buseman–Iwasawa cocycle* of G, $\beta : \mathsf{G} \times \mathcal{F} \to \mathsf{E}$, is defined, for all $g \in \mathsf{G}$ and $k[\mathsf{P}_\Delta] \in \mathcal{F}$, if $gk = lzu$ is an Iwasawa decomposition of gk, by $\beta(g, k[\mathsf{P}_\Delta]) = \nu(z)$. Quint proved the following lemma.

LEMMA 4.6.1. (Quint [62, Lemmas 6.1 and 6.2]) *The function $\beta_\vartheta = p_\vartheta \circ \beta$ factors as a cocycle $\beta_\vartheta : \mathsf{G} \times \mathcal{F}_\vartheta \to \mathsf{E}_\vartheta$.*

The Buseman–Iwasawa cocycle can also be read from the representations of G. Indeed, Quint [62, Lemme 6.4] states that for every $g \in \mathsf{G}$ and $x \in \mathcal{F}_\vartheta$, one has

$$l_\sigma \varpi_\sigma(\beta(g, x)) = \log \frac{\|\phi_\sigma(g)v\|_\phi}{\|v\|_\phi}, \tag{21}$$

where $v \in \Xi_{\phi_\sigma}(x) \in \mathbb{P}(\mathsf{V}_\sigma)$ is non-zero.

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4.7. *Gromov product.* As in [66], the *Gromov product* $\mathscr{G}_\vartheta : \mathcal{F}_\vartheta^{(2)} \to \mathsf{E}_\vartheta$ is defined such that, for every $(x, y) \in \mathcal{F}_\vartheta^{(2)}$ and $\sigma \in \vartheta$, one has

$$l_\sigma \varpi_\sigma(\mathscr{G}_\vartheta(x, y)) = \log \frac{|\varphi(v)|}{\|\varphi\|_{\phi_\sigma} \|v\|_{\phi_\sigma}},$$

where $\varphi \in \Xi^*_{\phi_\sigma}(x)$ and $v \in \Xi_{\phi_\sigma}(y)$ are the equivariant maps from equation (20).

Remark 4.7.1. Observe that the limiting situation $l_\sigma \varpi_\sigma(\mathscr{G}_\vartheta(x, y)) = -\infty$ occurs when $v \in \ker \varphi$, that is, when x and y are no longer transverse flags, so a statement of the form $\varpi_\sigma \mathscr{G}_\vartheta(x, y) \geq -\kappa$ for all $\sigma \in \vartheta$ is a quantitative version (that depends on K) of the transversality between x and y.

A straightforward computation ([66, Lemma 4.12]) gives, for all $g \in \mathsf{G}$ and $(x, y) \in \mathcal{F}_\vartheta^{(2)}$,

$$\mathscr{G}_\vartheta(gx, gy) - \mathscr{G}_\vartheta(x, y) = -(\mathrm{i}\, \beta_{\mathrm{i}\,\vartheta}(g, x) + \beta_\vartheta(g, y)). \tag{22}$$

4.8. *Proximality.* Recall that $g \in \mathsf{PGL}_d(\mathsf{V})$ is *proximal* if it has a unique eigenvalue with maximal modulus and that the multiplicity of this eigenvalue in the characteristic polynomial of g is 1. The associated eigenline is denoted by $g^+ \in \mathbb{P}(\mathsf{V})$ and g^- is its g-invariant complementary subspace.

We say then that $g \in \mathsf{G}$ is ϑ*-proximal* if for every $\sigma \in \vartheta$, one has $\phi_\sigma(g)$ is proximal. In this situation, there exists a pair $(g_\vartheta^-, g_\vartheta^+) \in \mathcal{F}_\vartheta^{(2)}$, defined by, for every $\sigma \in \vartheta$, $\Xi_{\phi_\sigma}(g_\vartheta^+) = \phi_\sigma(g)^+$, and every flag $x \in \mathcal{F}_\vartheta$ in general position with g_ϑ^- verifies $g^n x \to g_\vartheta^+$.

It is also useful to consider a quantified version of proximality. Given r, ε positive, we say that g is (r, ε)*-proximal* if it is proximal,

$$\varpi_\sigma \mathscr{G}_\vartheta(g_\vartheta^-, g_\vartheta^+) \geq -r$$

for all $\sigma \in \vartheta$, and for every $x \in \mathcal{F}_\vartheta$ with $\min_{\sigma \in \vartheta} \varpi_\sigma \mathscr{G}_\vartheta(g_\vartheta^-, x) \geq -\varepsilon^{-1}$, one has $d_{\mathcal{F}_\vartheta}(gx, g_\vartheta^+) \leq \varepsilon$. More details on the following can be found in [65, Lemma 5.6].

PROPOSITION 4.8.1. (Benoist [6, Corollaire 6.3]) *For every $\delta > 0$, there exist $r, \varepsilon > 0$ such that if $g \in \mathsf{G}$ is (r, ε)-proximal, then*

$$\|a_\vartheta(g) - \lambda_\vartheta(g) - \mathscr{G}_\vartheta(g_\vartheta^-, g_\vartheta^+)\| \leq \delta.$$

4.9. *Cartan attractors.* Consider $g \in \mathsf{G}$ and let $g = k_g z_g l_g$ be a Cartan decomposition. We say that $g \in \mathsf{G}$ has *a gap at* ϑ if for all $\sigma \in \vartheta$, one has

$$\sigma(a(g)) > 0.$$

In that case, the *Cartan attractor of g in $\mathcal{F}_\vartheta$*

$$U_\vartheta(g) = k_g[\mathsf{P}_\vartheta]$$

is well defined: uniquely defined if $\mathbb{K}$ is Archimedian; defined up to a ball of radius $q^{-\min_{\sigma \in \vartheta} \sigma(g)}$ if $\mathbb{K}$ is non-Archimedean (see [59, Remark 2.4]).

Remark 4.9.1. For every $\sigma \in \vartheta$, one has $\Xi_{\phi_\sigma}(U_\vartheta(g)) = U_1(\phi_\sigma(g))$.

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LEMMA 4.9.2. (Bochi, Potrie, and Sambarino [10, Lemma A.5]) *Consider* $g, h \in \mathsf{G}$ *such that* h *and* gh *have gaps at every* $\sigma \in \vartheta$, *then one has*

$$d(U_\vartheta(gh), gU_\vartheta(h)) \leq q^{-\min_{\sigma\in\vartheta} \sigma(h)} \cdot \max_{\sigma\in\vartheta}\{\|\phi_\sigma(g)\|\|\phi_\sigma(g^{-1})\|\}.$$

The *Cartan basin of* g is defined, for $\alpha > 0$, by (recall Remark 4.7.1)

$$B_{\vartheta,\alpha}(g) = \{x \in \mathcal{F}_\vartheta : \varpi_\sigma(\mathscr{G}_\vartheta(U_{\mathrm{i}\,\vartheta}(g^{-1})), x) > -\alpha \text{ for all } \sigma \in \vartheta\}.$$

It is clear from the definition that given $\alpha > 0$, there exists a constant K_α such that if $y \in \mathcal{F}_\vartheta$ belongs to $B_{\vartheta,\alpha}(g)$, then one has

$$\|a_\vartheta(g) - \beta_\vartheta(g, y)\| \leq K_\alpha. \tag{23}$$

LEMMA 4.9.3. (Quint [62, Lemme 6.6]) *For every* $g \in \mathsf{G}$, *one has* $a_\vartheta(gh) - a_\vartheta(h) - \beta_\vartheta(g, U_\vartheta(h)) \to 0$ *as* $\min_{\sigma\in\vartheta} \sigma(a(h)) \to \infty$.

4.10. *General facts on discrete subgroups.* We record here some facts related to the title that we will need in the following work.

LEMMA 4.10.1. *Let* $\Delta \subset \mathsf{G}$ *be a discrete subgroup. Then for every* $\varphi \in \mathsf{E}^*$ *strictly positive on* E^+, *the exponential rate*

$$\delta_\Delta^\varphi = \limsup_{t\to\infty} \frac{1}{t} \log \#\{g \in \Delta : \varphi(a(g)) \leq t\}$$

is finite.

Proof. The proof follows from a computation of the Haar measure of G, to be found in [37] for the Archimedean case and in [50, §3.2.7] for the non-Archimedean case, see [61, §4] for details. □

We record also the following theorem from Benoist [8].

THEOREM 4.10.2. (Benoist [8]) *Assume* $\mathbb{K} = \mathbb{R}$ *and let* $\Delta \subset \mathsf{G}$ *be a Zariski-dense subgroup. Then, the group spanned by the Jordan projections* $\{\lambda(g) : g \in \Delta\}$ *is dense in* E.

5. *Anosov representations*

Anosov representations were introduced by Labourie [43] for fundamental groups of negatively curved manifolds and extended to arbitrary finitely generated hyperbolic groups by Guichard and Wienhard [35]. They originated as a tool to study higher rank Teichmüller theory, and are nowadays considered as the higher-rank analog of what is known in pinched negative curvature as *convex co-compact* groups.

Notation. If $\rho : \Gamma \to \mathsf{G}$ is a representation, we will simplify notation and denote, for $\gamma \in \Gamma$, by $\gamma_\rho = \rho(\gamma)$.

5.1. *Real projective-Anosov representations.* We begin by recalling Labourie's original approach. Let Γ be a finitely generated word-hyperbolic group. If $\rho : \Gamma \to \mathsf{PGL}_d(\mathbb{R})$ is a representation, then we can consider the *natural flat bundle automorphism* defined as follows. Consider the flat bundle $\mathbb{R}^d \to \mathsf{E}_\rho \to \mathsf{U}\Gamma$ defined by $\widetilde{\mathsf{U}\Gamma} \times \mathbb{R}^d/_{\sim}$, where $(p, v) \sim (\gamma p, \gamma_\rho v)$, and define $\hat{\mathrm{g}} = (\hat{\mathrm{g}}_t : \mathsf{E}_\rho \to \mathsf{E}_\rho)_{t\in\mathbb{R}}$ as the induction on the quotient by $t \cdot (p, v) = (\widetilde{\mathrm{g}}_t p, v)$.

Definition 5.1.1. The representation ρ is *projective-Anosov* if there exists a pair of continuous ρ-equivariant maps

$$\xi^1 : \partial\Gamma \to \mathbb{P}(\mathbb{R}^d),$$
$$\xi^{d-1} : \partial\Gamma \to \mathbb{P}((\mathbb{R}^d)^*)$$

such that:

- for every $(x, y) \in \partial^2\Gamma$, one has $\ker \xi^{d-1}(x) \oplus \xi^1(y) = \mathbb{R}^d$. This induces a $\hat{\mathrm{g}}$-invariant decomposition $\Xi \oplus \Theta = \mathsf{E}_\rho$;
- the decomposition $\mathsf{E}_\rho = \Xi \oplus \Theta$ is a *dominated splitting* for $\hat{\mathrm{g}}$, that is, there exist c, α positive such that for every $v \in \Xi_p$ and $w \in \Theta_p$, one has

$$\frac{\|\hat{\mathrm{g}}_t v\|}{\|\hat{\mathrm{g}}_t w\|} \leq ce^{-\alpha t}\frac{\|v\|}{\|w\|}.$$

One has the following standard consequences, see for example [35, Lemma 3.1] or [17, Lemma 2.5 and Proposition 2.6]. Recall from §4.8 that $g \in \mathsf{PGL}_d(\mathbb{R})$ is *proximal* if the Jordan block associated to the eigenvalues with maximal modulus is one-dimensional.

LEMMA 5.1.2. *If ρ is projective-Anosov, then for every hyperbolic γ, one has γ_ρ is proximal with attracting line $\xi^1(\gamma_+)$. In particular, the entropy*

$$\lim_{t\to\infty} \frac{1}{t} \log \#\{[\gamma] \in [\Gamma] \textit{ hyperbolic} : \lambda_1(\gamma_\rho) \leq t\} \in [0, \infty).$$

The equivariant maps ξ^1 and ξ^{d-1} are Hölder-continuous.

Proof. Let us add a word on finiteness of entropy. Recall from Bowditch [13] that, since Γ is hyperbolic, its action on the space of pairwise distinct triples $\partial^{(3)}\Gamma$ is properly discontinuous and co-compact. If $\gamma \in \Gamma$ is hyperbolic, one can choose then $\eta \in [\gamma]$ (the conjugacy class of γ) whose fixed points are far apart by a constant independent of γ. Since the image γ_ρ is proximal and the equivariant maps are continuous, one has that η_ρ is (r, ε)-proximal for constants r, ε independent of $[\gamma]$. By Proposition 4.8.1, one has then $|\log \|\eta_\rho\| - \lambda_1(\eta_\rho)| < K$ for some K independent of η. If follows then that for every $t \in \mathbb{R}_+$,

$$\#\{[\gamma] \in [\Gamma] \text{ hyperbolic} : \lambda_1(\gamma_\rho) \leq t\} \leq \#\{\gamma \in \Gamma : \log \|\gamma_\rho\| \leq t + K\}.$$

Finiteness of entropy then follows from Lemma 4.10.1. □

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We use the equivariant maps to construct a bundle $\mathbb{R} \to \widetilde{\mathsf{F}} \to \partial^2\Gamma$ whose fiber at $(x, y) \in \partial^2\Gamma$ is

$$\widetilde{\mathsf{F}}_{(x,y)} = \{(\varphi, v) \in \xi^{d-1}(x) \times \xi^1(y) : \varphi(v) = 1\}/\sim,$$

where $(\varphi, v) \sim (-\varphi, -v)$. This bundle is equipped with a Γ-action $\gamma(\varphi, v) = (\varphi \circ \gamma_\rho^{-1}, \gamma_\rho v)$ and an $\mathbb{R}$-action $(\widetilde{\mathsf{g}^\rho}_t : \widetilde{\mathsf{F}} \to \widetilde{\mathsf{F}})_{t\in\mathbb{R}}$ defined by $\widetilde{\mathsf{g}^\rho}_t \cdot (\varphi, v) = (e^t\varphi, e^{-t}v)$. Let $\mathsf{F} = \Gamma\backslash\widetilde{\mathsf{F}}$ and denote by $\mathsf{g}^\rho = (\mathsf{g}^\rho_t : \mathsf{F} \to \mathsf{F})_{t\in\mathbb{R}}$ the induced flow on the quotient. It is usually called *the geodesic flow of* ρ.

THEOREM 5.1.3. (Bridgeman *et al* [17]) *The above* Γ*-action is properly discontinuous and co-compact. The flow* g^ρ *is Hölder-continuous and metric-Anosov with stable/unstable laminations (induced on the quotient by)*

$$\tilde{W}^{\mathrm{s}}((x, y, (\varphi, v)) = \{(x, \cdot, (\varphi, \cdot)) \in \widetilde{\mathsf{F}}\},$$
$$W^{\mathrm{u}}((x, y, (\varphi, v)) = \{(\cdot, y, (\cdot, v)) \in \widetilde{\mathsf{F}}\}.$$

It is moreover Hölder-conjugated to the Gromov–Mineyev geodesic flow g *of* Γ*, consequently, this latter flow is also metric-Anosov.*

Therefore, hyperbolic groups admitting a real projective-Anosov representation verify Assumption B and are thus subject of a Ledrappier correspondence (§3). It is established by Carvajales [22, Appendix] that g^ρ is topologically mixing (regardless of the Zariski closure of ρ) and thus mixing for any equilibrium state.

5.2. *Arbitrary* G*, coarse geometry viewpoint.* Let G be as in §4. We use freely the notation introduced there and fix from now on a subset $\vartheta \subset \Delta$ of simple roots.

Let Γ be a finitely generated group and denote, for $\gamma \in \Gamma$, by $|\gamma|$ the word length with respect to a fixed finite symmetric generating set of Γ.

Definition 5.2.1. A representation $\rho : \Gamma \to \mathsf{G}$ is ϑ*-Anosov* if there exist c, μ positive such that for all $\gamma \in \Gamma$ and $\sigma \in \vartheta$, one has

$$\sigma(a(\gamma_\rho)) \geq \mu|\gamma| - c. \tag{24}$$

The constants c and μ will be referred to as *the domination constants* of ρ.

Theorem 5.2.2 follows from the main result by Kapovich, Leeb, and Porti [40] and the standard facts from representation theory stated in §4.5, a proof was also reported by Bochi, Potrie, and Sambarino [10].

THEOREM 5.2.2. *If* $\rho : \Gamma \to \mathsf{G}$ *is* ϑ*-Anosov, then* Γ *is word-hyperbolic. If moreover* $\mathbb{K} = \mathbb{R}$*, then for every* $\sigma \in \vartheta$*, the representation* $\phi_\sigma \circ \rho : \Gamma \to \mathsf{PGL}(V_\sigma)$ *is projective-Anosov (as in §5.1).*

The following lemma is essentially a consequence of [10, Lemma 4.9]. See [59, Proposition 3.5] for details concerning the non-Archimedean case. The last assertion is classical.

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PROPOSITION 5.2.3. (Bochi, Potrie, and Sambarino [10, Proposition 4.9]) *If $\rho : \Gamma \to \mathsf{G}$ is ϑ-Anosov, then for any geodesic ray $\{\alpha_n\}_0^\infty$ with endpoint x, the limits*

$$\xi_\rho^\vartheta(x) := \lim_{n\to\infty} U_\vartheta(\rho(\alpha_n)), \quad \xi_\rho^{\mathrm{i}\,\vartheta}(x) := \lim_{n\to\infty} U_{\mathrm{i}\,\vartheta}(\rho(\alpha_n))$$

exist and do not depend on the ray; they define continuous ρ-equivariant transverse maps $\xi^\vartheta : \partial\Gamma \to \mathcal{F}_\vartheta$, $\xi^{\mathrm{i}\,\vartheta} : \partial\Gamma \to \mathcal{F}_{\mathrm{i}\,\vartheta}$. If $\gamma \in \Gamma$ is hyperbolic, then γ_ρ is ϑ-proximal with attracting point $\xi^\vartheta(\gamma^+) = (\gamma_\rho)_\vartheta^+$.

Proposition 5.2.3 readily implies the following lemma (recall Remark 4.7.1).

LEMMA 5.2.4. *Let $\rho : \Gamma \to \mathsf{G}$ be ϑ-Anosov, $\{\gamma_n\} \subset \Gamma$ a divergent sequence, and $x \in \partial\Gamma$. Then, as $n \to \infty$, one has*

$$\begin{aligned}\gamma_n \to x &\Leftrightarrow U_\vartheta(\rho(\gamma_n)) \to \xi^\vartheta(x)\\ &\Leftrightarrow \text{ there exists } \sigma \in \vartheta \text{ such that } \varpi_\sigma \mathscr{G}_\vartheta(U_{\mathrm{i}\,\vartheta}(\rho(\gamma_n)), \xi^\vartheta(x)) \to -\infty.\end{aligned}$$

We finally record the following useful lemma.

LEMMA 5.2.5. (Pozzetti, Sambarino, and Wienhard [59, Lemma 3.6]) *Let $\rho : \Gamma \to \mathsf{G}$ be ϑ-Anosov, then for every $\varepsilon > 0$, there exists L such that*

$$\overline{\bigcup_{\gamma:|\gamma|>L} U_\vartheta(\gamma_\rho)} \subset \mathcal{N}_\varepsilon(\xi^\vartheta(\partial\Gamma)),$$

where $\mathcal{N}_\varepsilon$ denotes the ε-tubular-neighborhood.

*Remark 5.2.6. (*Non-Archimedean case) The existence of continuous ρ-equivariant maps implies, when $\mathbb{K}$ is non-Archimedean, that the boundary of Γ is necessarily a Cantor set and thus Γ is virtually free. The Gromov–Mineyev of Γ is thus a suspension of a sub-shift of finite type and is, hence, metric-Anosov.

Setting. A ϑ-Anosov representation $\rho : \Gamma \to \mathsf{G}$ is fixed from now on. By §5.1 for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and the preceding paragraph for non-Archimedean $\mathbb{K}$, the Gromov–Mineyev flow g of Γ satisfies Assumption B.

5.3. *The refraction cocycle verifies Assumption C.* By the equivariant boundary maps of ρ, one can pullback the Buseman–Iwasawa cocycle of G to obtain a Hölder-cocycle on the boundary of Γ.

Definition 5.3.1. The *refraction cocycle* of ρ is $\beta : \Gamma \times \partial\Gamma \to \mathsf{E}_\vartheta$,

$$\beta(\gamma, x) = \beta^\rho(\gamma, x) = \beta_\vartheta(\gamma_\rho, \xi_\rho^\vartheta(x)).$$

The limit cone of β will be denoted by $\mathscr{L}_{\vartheta,\rho}$ and referred to as *the ϑ-limit cone of ρ*. The period computation below implies it is the smallest closed cone of E_ϑ that contains the projections $\{\lambda_\vartheta(\gamma_\rho) : \gamma \in \Gamma\}$. We prove moreover that β verifies Assumption C from §3.4.

LEMMA 5.3.2. *The periods of β are $\beta(\gamma, \gamma_+) = \lambda_\vartheta(\gamma_\rho)$, consequently Assumption C holds for β, in particular,* $\mathrm{int}(\mathscr{L}_{\vartheta,\rho})^* = \{\varphi \in (\mathscr{L}_{\vartheta,\rho})^* : \hbar_\varphi < \infty\}$.

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Proof. The first assertion follows from Proposition 5.2.3. To prove Assumption C holds, one considers any $\sigma \in \vartheta$ and the representation ϕ_σ. By Theorem 5.2.2, the composition $\phi_\sigma \rho : \Gamma \to \mathsf{GL}(\mathsf{V}_\sigma)$ is projective-Anosov and thus, by equation (19) and Lemma 5.1.2, the form $\varpi_\sigma \in (\mathsf{E}_\vartheta)^*$ has finite entropy. The last assertion follows from Lemma 3.4.2. $\square$

Lemma 3.4.2 and Theorem 3.2.2 give then the following corollary.

COROLLARY 5.3.3. *There exists a Hölder-continuous function $\mathcal{J}_{\vartheta,\rho} : \mathsf{U}\Gamma \to \mathsf{E}_\vartheta$ such that for every hyperbolic $\gamma \in \Gamma$, one has $\int_{[\gamma]} \mathcal{J}_{\vartheta,\rho} = \lambda_\vartheta(\gamma_\rho)$. For every $\varphi \in \operatorname{int}(\mathscr{L}_{\vartheta,\rho})^*$, the Γ-action on $\partial^2\Gamma \times \mathbb{R}$ defined by*

$$\gamma \cdot (x, y, t) = (\gamma x, \gamma y, t - \beta_\varphi(\gamma, y)) \tag{25}$$

is properly discontinuous and co-compact. The $\mathbb{R}$-translation flow induces on the quotient a flow $\phi^\varphi = (\phi_t^\varphi : \chi^\varphi \to \chi^\varphi)_{t\in\mathbb{R}}$ (bi)-Hölder-conjugated to the reparameterization of g *by $\varphi \circ \mathcal{J}_{\vartheta,\rho}$.*

Definition 5.3.4. The function $\mathcal{J}_{\vartheta,\rho}$ will be referred to as *the Ledrappier potential* of ρ. The flow ϕ^φ will be called the *φ-refraction flow of ρ.*

5.4. *The ϑ-limit cone.* We mimic some celebrated results by Benoist [7] for Zariski-dense subgroups and $\vartheta = \Delta$.

LEMMA 5.4.1. (Benoist [6, Proposition 5.1]) *For every compact set $L \subset \mathsf{G}$, there exists a compact set $H \subset \mathsf{E}$ such that for every $g \in \mathsf{G}$, one has $a(LgL) \subset a(g) + H$.*

Let us also denote $a_\vartheta = p_\vartheta \circ a$.

PROPOSITION 5.4.2. *Let $\rho : \Gamma \to \mathsf{G}$ be a ϑ-Anosov representation. Then there exists a compact set $D \subset \mathsf{E}_\vartheta$ such that $a_\vartheta(\rho(\Gamma)) \subset \lambda_\vartheta(\rho(\Gamma)) + D$.*

Proof. As Γ is finitely generated and word-hyperbolic, there exist $\kappa > 0$ and two elements $u, v \subset \Gamma$ such that for every non-torsion $\gamma \in \Gamma$, there exists $f \in \{u, v\}$ such that $f\gamma$ verifies

$$d_{\partial\Gamma}((f\gamma)^+, (f\gamma)^-) > \kappa.$$

As ρ is ϑ-Anosov, the above equation implies the element $\rho(f\gamma)$ is (r, ε)-proximal on ϑ for some r only depending on κ. By Proposition 4.8.1, one has

$$\|a_\vartheta(\rho(f\gamma)) - \lambda_\vartheta(\rho(f\gamma))\| \leq K,$$

for some K only depending on κ and ρ. We consider the compact set H from the lemma above applied to $L = \rho(\{u^{-1}, v^{-1}\})$ and we let $D := p_\vartheta(H) + B(0, K)$. $\square$

We will mainly use the following direct consequence.

COROLLARY 5.4.3. *If $\varphi \in \operatorname{int}(\mathscr{L}_{\vartheta,\rho})^*$, then the exponential rate*

$$\delta^\varphi := \limsup_{t\to\infty} \frac{1}{t} \log \#\{\gamma \in \Gamma : \varphi(a(\gamma_\rho)) \leq t\} < \infty.$$

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Proof. If $\sigma \in \vartheta$ then, since both intersections $\ker \varpi_\sigma \cap \mathscr{L}_{\vartheta,\rho}$ and $\ker \varphi \cap \mathscr{L}_{\vartheta,\rho}$ vanish (the first one always does, the second one by the assumption on φ), the function φ/ϖ_σ is bounded below away from zero on $\mathscr{L}_{\vartheta,\rho}$. By Proposition 5.4.2, there exist positive c and C such that for all hyperbolic $\gamma \in \Gamma$, one has

$$\varphi(a(\gamma_\rho)) \geq c\varpi_\sigma(a(\gamma_\rho)) - C.$$

Lemma 4.10.1 gives then the desired result. □

5.5. *Patterson–Sullivan theory along the Anosov roots: existence.* In this section, we will construct, for each $\varphi \in \operatorname{int}(\mathscr{L}_{\vartheta,\rho})^*$, a β_φ-Patterson–Sullivan measure. (The ϑ-Anosov property is not really used until the uniqueness corollary. The existence presented here works for any discrete group whose limit cone on E does not intersect any wall associated to ϑ and replacing $\xi^\vartheta(\partial\Gamma)$ by

$$\bigcap_{n\in\mathbb{N}} \overline{\{U_\vartheta(g) : g \in \Delta \text{ with } \min_{\sigma\in\vartheta} \sigma(a(g)) \geq n\}}.)$$

The procedure is standard and follows the original idea by Patterson.

We begin by considering the Dirichlet series

$$\mathcal{P}^\varphi(s) = \sum_{\gamma\in\Gamma} q^{-s\varphi(a(\gamma_\rho))}.$$

It is convergent for every $s > \delta^\varphi$ and divergent for every $s < \delta^\varphi$. As it is customary when constructing Patterson–Sullivan measures, we can assume throughout this subsection that $\mathcal{P}^\varphi(\delta^\varphi) = \infty$, otherwise, one would consider the series

$$s \mapsto \sum_{\gamma\in\Gamma} h(\varphi(a(\gamma_\rho)))q^{-s\varphi(a(\gamma_\rho))}$$

for some real function h defined, for example, as by Quint [62, Lemma 8.5].

For $s > \delta^\varphi$, consider the probability measure on $\mathcal{F}_\vartheta$ defined by

$$\nu_s = \frac{1}{\mathcal{P}^\varphi(s)} \sum_{\gamma\in\Gamma} q^{-s\varphi(a(\gamma_\rho))}\delta_{U_\vartheta(\gamma_\rho)}.$$

LEMMA 5.5.1. *For every $\eta \in \Gamma$, the signed measure*

$$\varepsilon(\eta, s) = (\eta_\rho)_*\nu_s - \frac{1}{\mathcal{P}^\varphi(s)} \sum_{\gamma\in\Gamma} q^{-s\varphi(a(\gamma_\rho))}\delta_{U_\vartheta(\eta_\rho\gamma_\rho)}$$

weakly converges to 0 *as* $s \searrow \delta^\varphi$.

This is a standard argument that can be found, for example, in [59, Lemma 5.11].

Proof. It is sufficient to check the convergence for continuous functions. If $f : \mathcal{F}_\vartheta \to \mathbb{R}$ is continuous, then

$$|\varepsilon(\eta, s)(f)| \leq \frac{1}{\mathcal{P}^\varphi(s)} \sum_{\gamma\in\Gamma} q^{-s\varphi(a(\gamma_\rho))}|f(\eta_\rho U_\vartheta(\gamma_\rho)) - f(U_\vartheta(\eta_\rho\gamma_\rho))|.$$

By Lemma 4.9.2 and uniform continuity of f, the convergence follows. □

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LEMMA 5.5.2. *Let ν^{φ} be any weak-star limit of ν_s when $s \searrow \delta^{\varphi}$. Then, the support of ν^{φ} is contained in $\xi^{\vartheta}(\partial\Gamma)$. Moreover, for every $\eta \in \Gamma$, one has*

$$\frac{d\rho(\eta)_*\nu^{\varphi}}{d\nu^{\varphi}}(x) = q^{-\varphi(\beta_{\vartheta}(\eta_{\rho}^{-1},x))}.$$

Proof. The first statement follows at once from Lemma 5.2.5 since we assumed $\mathcal{P}^{\varphi}(\delta^{\varphi}) = \infty$. For the second statement, consider a sequence $s_k \searrow \delta^{\varphi}$ such that $\nu_{s_k} \to \nu^{\varphi}$. One then has

$$\begin{aligned}\rho(\eta)_*\nu^{s_k} &= \varepsilon(\eta, s_k) + \frac{1}{\mathcal{P}^{\varphi}(s_k)} \sum_{\gamma\in\Gamma} q^{-s_k\varphi(a(\gamma_\rho))}\delta_{U_\vartheta(\eta_\rho\gamma_\rho)} \\ &= \varepsilon(\eta, s_k) + \frac{1}{\mathcal{P}^{\varphi}(s_k)} \sum_{\gamma\in\Gamma} q^{-s_k\varphi(a(\eta_\rho^{-1}\gamma_\rho))}\delta_{U_\vartheta(\gamma_\rho)} \\ &= \varepsilon(\eta, s_k) + \frac{1}{\mathcal{P}^{\varphi}(s_k)} \sum_{\gamma\in\Gamma} q^{-s_k\varphi(a(\eta_\rho^{-1}\gamma_\rho)-a(\gamma_\rho))}q^{-s_k\varphi(a(\gamma_\rho))}\delta_{U_\vartheta(\gamma_\rho)} \\ &= \varepsilon(\eta, s_k) + \frac{1}{\mathcal{P}^{\varphi}(s_k)} \sum_{\gamma\in\Gamma} q^{-s_k\varphi(\beta_\vartheta(\eta_\rho^{-1},U_\vartheta(\gamma_\rho))+\varepsilon'(\eta,\gamma))}q^{-s_k\varphi(a(\gamma_\rho))}\delta_{U_\vartheta(\gamma_\rho)},\end{aligned}$$

where by Quint's Lemma 4.9.3 and the fact that $\varphi \circ p_\vartheta = \varphi$, one has $\varepsilon'(\eta, \gamma) \to 0$ as $\min_{\sigma\in\vartheta} \sigma(a(\gamma_\rho)) \to \infty$. Taking the limit as $s_k \searrow \delta^{\varphi}$, one has, since we assumed $\mathcal{P}^{\varphi}(\delta^{\varphi}) = \infty$, that only elements $\gamma \in \Gamma$ with arbitrary big $|\gamma|$ count in the sum. Since ρ is ϑ-Anosov, this is equivalent to considering elements $\gamma \in \Gamma$ such that

$$\min_{\sigma\in\vartheta} \sigma(a(\gamma_\rho))$$

is arbitrarily big. The result then follows as $\varepsilon(\eta, s_k) \to 0$ by Lemma 5.5.2 and $\varepsilon'(\eta, \gamma)$ is arbitrarily small. □

Since Assumption C holds for β (Lemma 5.3.2), §3.3 is applied to give the following corollary.

COROLLARY 5.5.3. *For every $\varphi \in \mathrm{int}(\mathscr{L}_{\vartheta,\rho})^*$, there exists a β_φ-Patterson–Sullivan measure $\mu^{\varphi} := (\xi^{\vartheta})_*\nu^{\varphi}$ of exponent δ^{φ}. Such a measure is ergodic and, moreover, one has $\delta^{\varphi} = \hbar_\varphi$. If $\psi \in \mathrm{int}(\mathscr{L}_{\vartheta,\rho})^*$ is such that $\mu^{\psi} \ll \mu^{\varphi}$, then for every hyperbolic $\gamma \in \Gamma$, one has*

$$\hbar_\varphi\varphi(\lambda_\vartheta(\gamma_\rho)) = \hbar_\psi\psi(\lambda_\vartheta(\gamma_\rho))$$

and, in particular, $\mu^{\psi} = \mu^{\varphi}$.

The above corollary was previously established by Dey and Kapovich [**29**, Main Theorem] for real algebraic groups, i-invariant functionals $\varphi \in \mathrm{int}(\mathfrak{a}^+)^*$, and i-invariant subsets ϑ. The equality $\delta^{\varphi} = \hbar_\varphi$, together with more information, can also be found in Glorieux, Monclair, and Tholozan's work [**32**, Theorem 2.31(2)] for real groups.

Remark 5.5.4. We conclude by remarking that for $\varphi \in \mathrm{int}(\mathscr{L}_{\vartheta,\rho})^*$, the existence assumptions of §3.3 are guaranteed for β_φ. Indeed, Proposition 5.5.3 states the existence of a

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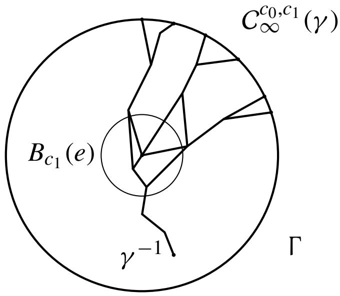

FIGURE 1. The coarse cone type at infinity, the black broken lines are (c_0, c_1)-quasi-geodesics.

Patterson–Sullivan measure μ^φ for β_φ. The cocycle

$$\bar{\beta}(\gamma, x) = \mathrm{i}\,\beta_{\mathrm{i}\,\vartheta}(\gamma_\rho, \xi^{\mathrm{i}\,\vartheta}(x))$$

is dual to β and moreover, from equation (22), the function $[\cdot,\cdot]_\varphi : \partial^2\Gamma \to \mathbb{R}$

$$[x, y]_\varphi = \varphi(\mathscr{G}_\vartheta(\xi^{\mathrm{i}\,\vartheta}(x), \xi^\vartheta(y))) \tag{26}$$

is a Gromov product for the pair $(\bar{\beta}_\varphi, \beta_\varphi)$. Finally, exchanging ϑ with $\mathrm{i}\,\vartheta$, Proposition 5.5.3 provides a Patterson–Sullivan measure for $\bar{\beta}_\varphi$. We can thus apply the results from §§3.4 and 3.5.

5.6. *Cartan's basins have controlled overlaps.* The job of understanding the overlaps of Cartan's basins for Anosov representations has been carried out by Pozzetti, Sambarino, and Wienhard [58]. The idea is to compare the Cartan's basins of elements γ_ρ, for hyperbolic $\gamma \in \Gamma$, with the *coarse cone type* of γ.

Let c_0, c_1 be positive and $I \subset \mathbb{Z}$ an interval. Then, a (c_0, c_1)-*quasigeodesic* is a sequence $\{\alpha_i\}_{i\in I} \in \Gamma$ such that for every pair j, l in the interval I, one has

$$\frac{1}{c_0}|j-l| - c_1 \le d_\Gamma(\alpha_j, \alpha_l) \le c_0|j-l| + c_1.$$

The *coarse cone type at infinity* of $\gamma \in \Gamma$ consists of endpoints on $\partial\Gamma$ of quasi geodesic rays based at γ^{-1} passing through the identity (see Figure 1):

$$C_\infty^{c_0,c_1}(\gamma) = \{[\{\alpha_j\}_0^\infty] \in \partial\Gamma : \ \{\alpha_i\}_0^\infty \text{ is a } (c_0, c_1)\text{-quasi-geodesic with } \alpha_0 = \gamma^{-1}, e \in \{\alpha_j\}\}.$$

Pozzetti, Sambarino, and Wienhard [58, Proposition 3.3] together with Bochi, Potrie, and Sambarino [10, Lemma 2.5] (see also [59, Proposition 3.3]) give the following. The last statement can be found in [58, Proposition 3.5].

PROPOSITION 5.6.1. (Pozzetti, Sambarino, and Wienhard [58, Proposition 3.3]) *For a given $\alpha > 0$, there exist c_0, c_1, depending on α and the domination constants of ρ, such that for every hyperbolic $\gamma \in \Gamma$, one has*

$$(\xi^\vartheta)^{-1}(B_{\theta,\alpha}(\gamma_\rho)) \subset C_\infty^{c_0,c_1}(\gamma).$$

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Reciprocally, there exists α'; only depending on c_0, c_1 and the domination constants of ρ, such that

$$C_\infty^{c_0,c_1}(\gamma) \subset (\xi^\vartheta)^{-1}(B_{\theta,\alpha'}(\gamma_\rho)).$$

There exists then $N \in \mathbb{N}$, only depending on c_0, c_1 and the domination constants of ρ such that, for all $t \in \mathbb{N}$, the family

$$\mathcal{U}_t = \{\gamma_\rho B_{\vartheta,\alpha}(\gamma_\rho) : t \leq |\gamma| \leq t+1\}$$

is an open covering of $\xi^\vartheta(\partial\Gamma)$ and such that every element $\xi^\vartheta(x)$ belongs to at most N elements of the covering $\mathcal{U}_t$.

5.7. *Sullivan's shadow lemma.* We establish now a version of Sullivan's shadow lemma.

LEMMA 5.7.1. *Consider $\varphi \in (\mathsf{E}_\vartheta)^*$ and let μ be a β_φ-Patterson–Sullivan measure of exponent δ. Let $\nu = \xi_*^\vartheta \mu$. Then given $\alpha > 0$, there exist constants C, C', and $L \in \mathbb{N}$ such that for every $\gamma \in \Gamma$ with $|\gamma| \geq L$, one has*

$$q^{-\delta\cdot\varphi(a(\gamma_\rho))}C' \leq \nu(\gamma_\rho B_{\vartheta,\alpha}(\gamma_\rho)) \leq C q^{-\delta\cdot\varphi(a(\gamma_\rho))}.$$

Proof. It suffices to establish that there exist α and $\kappa > 0$ such that for all large enough $\gamma \in \Gamma$, one has $\nu(B_{\vartheta,\alpha}(\gamma_\rho)) \geq \kappa$. Indeed, using this fact, the lemma follows from the defining equations (27) and (23).

To establish the desired lower bound, we suppose by contradiction that there exists $\alpha_n \to \infty$, $\gamma_n \to \infty$ such that $\nu(B_{\vartheta,\alpha_n}((\gamma_n)_\rho)) \to 0$ as $n \to \infty$. We can then extract a subsequence (γ_{n_k}) such that

$$U_{\mathrm{i}\,\vartheta}((\gamma_{n_k}^{-1})_\rho) \to Y \in \mathcal{F}_{\mathrm{i}\,\vartheta},\ k \to \infty.$$

Moreover, since ρ is ϑ-Anosov, Lemma 5.2.5 guarantees that $Y = \xi^{\mathrm{i}\,\vartheta}(y)$ for some $y \in \partial\Gamma$. Also, since $\nu(B_{\vartheta,\alpha_{n_k}}((\gamma_{n_k})_\rho)) \to 0$ and $\alpha_{n_k} \to \infty$, we get that the complement

$$\begin{aligned}&(B_{\vartheta,\alpha_{n_k}}((\gamma_{n_k})_\rho))^c\\ &\quad = \{X \in \mathcal{F}_\vartheta : \text{ there exists } \sigma \in \vartheta \text{ such that } \varpi_\sigma \mathscr{G}_\vartheta(U_{\mathrm{i}\,\vartheta}((\gamma_{n_k}^{-1})_\rho), X) \leq -\alpha_n\}\end{aligned}$$

converges to the subset of $\mathcal{F}_\vartheta$

$$\{X \in \mathcal{F}_\vartheta : (X, \xi^{\mathrm{i}\,\vartheta}(y)) \notin \mathcal{F}_\vartheta^{(2)}\},$$

and that this subset has total ν-mass. Since the support of ν is contained in $\xi^\vartheta(\partial\Gamma)$ and the equivariant maps are transverse (Proposition 5.2.3), one has that

$$\{\xi^\vartheta(y)\} = \{\xi^\vartheta(x) : (\xi^\vartheta(x), \xi^{\mathrm{i}\,\vartheta}(y)) \notin \mathcal{F}_\vartheta^{(2)}\}$$

has total ν-mass. However, considering $\gamma \in \Gamma$ with $\gamma y \neq y$, we get, since

$$\frac{d(\gamma_\rho)_*\nu}{d\nu}(\cdot) = q^{-\delta\cdot\varphi(\beta_\theta(\gamma_\rho^{-1},\cdot))}, \tag{27}$$

that $\nu\{\xi^1(\gamma y)\} > 0$, contradicting that $\{\xi^\vartheta(y)\}$ has total ν-mass. $\square$

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COROLLARY 5.7.2. *For every $\varphi \in \operatorname{int}(\mathscr{L}_{\vartheta,\rho})^*$, one has* $\sum_{\gamma\in\Gamma} q^{-\delta^\varphi\varphi(a(\gamma_\rho))} = \infty$.

Proof. We apply Sullivan's shadow Lemma 5.7.1 to the measure ν^φ of Lemma 5.5.2. Indeed, considering the coverings of $\xi^\vartheta(\partial\Gamma)$ given by Proposition 5.6.1, one has

$$1 = \nu^\varphi(\xi^\vartheta(\partial\Gamma)) \leq \sum_{t\leq|\gamma|\leq t+1} \nu^\varphi(\gamma_\rho B_{\vartheta,\alpha}(\gamma_\rho)) \leq C \sum_{t\leq|\gamma|\leq t+1} q^{-\delta^\varphi\varphi(a(\gamma_\rho))}$$

for all large enough t, giving divergence of the desired series. □

5.8. *Patterson–Sullivan theory along the Anosov roots: surjectivity.* We prove here surjectivity of the map $\varphi \mapsto \mu^\varphi$ defined in §5.5.

The following proposition should be compared with [**58**, Theorem 5.14], where a similar result is obtained for measures on the flag space $\mathcal{F}_\theta$, for θ not necessarily equal to ϑ but assuming that $\vartheta \cap \theta \neq \emptyset$.

PROPOSITION 5.8.1. *Consider $\varphi \in (\mathsf{E}_\vartheta)^*$. If there exists a β_φ-Patterson–Sullivan measure μ of exponent δ, then $\varphi \in \operatorname{int}(\mathscr{L}_{\vartheta,\rho})^*$, $\delta = \delta^\varphi$, and $\mu = \mu^\varphi$.*

Proof. We let $\nu = \xi^\vartheta_*\mu$. Using Proposition 5.6.1, we get a family of coverings $\mathcal{U}_t$ with bounded overlap. In combination with Lemma 5.7.1, one has for t large enough that

$$1 = \nu(\xi^\vartheta(\partial\Gamma)) \geq K \sum_{\gamma:t\leq|\gamma|\leq t+1} e^{-\delta\varphi(a(\gamma_\rho))},$$

for some constant $K > 0$. This is to say, there exists $\kappa > 0$ such that for all $t \in \mathbb{R}_+$ large, one has $\sum_{\gamma:t\leq|\gamma|\leq t+1} e^{-\delta\varphi(a(\gamma_\rho))} \leq \kappa$, which gives in turn that

$$\sum_{\gamma:|\gamma|\leq t} e^{-\delta\varphi(a(\gamma_\rho))} \leq \kappa t.$$

A standard argument (using for example §4.8) permits to replace Cartan projections with Jordan projections giving

$$\sum_{[\gamma]:p([\gamma])\leq t} e^{-\delta\varphi(\lambda(\gamma_\rho))} = \sum_{[\gamma]:p([\gamma])\leq t} e^{-\ell_{\delta\mathcal{J}^\varphi_{\vartheta,\rho}}(\gamma)} \leq \kappa' t,$$

for a suitable κ', where $p([\gamma])$ is the g-period of the periodic orbit associated to $[\gamma]$, and $\mathcal{J}^\varphi_{\vartheta,\rho}$ is the Ledrappier potential of β_φ. Equation (5) for the pressure function gives then

$$P(-\delta\mathcal{J}^\varphi_{\vartheta,\rho}) \leq 0.$$

Consequently, Lemma 2.2.7 gives that $\mathcal{J}^\varphi_{\vartheta,\rho}$ is Livšic-cohomologous to a positive function, this is to say, $\varphi \in \operatorname{int}(\mathscr{L}_{\vartheta,\rho})^*$. Finally, since Remark 5.5.4 guarantees the existence assumptions of §3.3 for β_φ, the remaining two equalities in the statement follow from Corollary 3.3.2. □

5.9. *The critical hypersurface parameterizes Patterson–Sullivan measures.* By Lemma 5.3, Assumption C holds for β and thus §3.4 applies. Define the ϑ-*critical hypersurface*, respectively ϑ-*convergence domain*, of ρ by

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$$\mathcal{Q}_{\vartheta,\rho} := \mathcal{Q}_{\beta} = \{\varphi \in \operatorname{int}(\mathscr{L}_{\vartheta,\rho})^* : \hbar_\varphi = 1\},$$
$$\mathcal{D}_{\vartheta,\rho} := \mathcal{D}_{\beta} = \{\varphi \in \operatorname{int}(\mathscr{L}_{\vartheta,\rho})^* : \hbar_\varphi \in (0,1)\}.$$

Moreover, by Corollary 5.5.3, one has $\delta^\varphi = \hbar_\varphi$, so one has the equalities

$$\begin{aligned}\mathcal{Q}_{\vartheta,\rho} &= \{\varphi \in \operatorname{int}(\mathscr{L}_{\vartheta,\rho})^* : \delta^\varphi = 1\},\\ \mathcal{D}_{\vartheta,\rho} &= \{\varphi \in \operatorname{int}(\mathscr{L}_{\vartheta,\rho})^* : \delta^\varphi \in (0,1)\}\\ &= \{\varphi \in (\mathsf{E}_\vartheta)^* : \sum_{\gamma\in\Gamma} e^{-\varphi(a(\gamma_\rho))} < \infty\},\end{aligned}$$

where the last equality comes from Corollary 5.7.2.

For $\varphi \in \mathcal{Q}_{\vartheta,\rho}$, we consider the *dynamical intersection* map $\mathbf{I}_\varphi = \mathbf{I}^{\beta}_\varphi : (\mathsf{E}_\vartheta)^* \to \mathbb{R}$, associated to the cocycle β as in §3.4 and defined by

$$\mathbf{I}_\varphi(\psi) = \mathbf{I}^{\beta}_\varphi(\psi) = \lim_{t\to\infty} \frac{1}{\#\mathsf{R}_t(\varphi)} \sum_{\gamma\in\mathsf{R}_t(\varphi)} \frac{\psi(\lambda(\gamma_\rho))}{\varphi(\lambda(\gamma_\rho))},$$

where $\mathsf{R}_t(\varphi) = \{\gamma \in \Gamma \text{ hyperbolic} : \varphi(\lambda(\gamma_\rho)) \leq t\}$. Let $\operatorname{Ann}(\mathscr{L}_{\vartheta,\rho})$ be the annihilator of the ϑ-limit cone and denote by

$$\pi^\vartheta_\rho : (\mathsf{E}_\vartheta)^* \to (\mathsf{E}_\vartheta)^*/\operatorname{Ann}(\mathscr{L}_{\vartheta,\rho})$$

the quotient projection. As before, the map $\mathbf{I}^\beta$ is also well defined on $\pi^\vartheta_\rho(\mathcal{Q}_{\vartheta,\rho}) \times (\mathsf{E}_\vartheta)^*/\operatorname{Ann}(\mathscr{L}_{\vartheta,\rho})$.

Some statements in the following corollary were previously established by Sambarino [64] for $\mathbb{K} = \mathbb{R}$ and Zariski-dense ϑ-Anosov representations of closed negatively curved manifolds.

COROLLARY 5.9.1. *The sets $\mathcal{Q}_{\vartheta,\rho}$ and $\pi^\vartheta_\rho(\mathcal{Q}_{\vartheta,\rho})$ are closed co-dimension-one analytic sub-manifolds. The latter bounds the strictly convex set $\pi^\vartheta_\rho(\mathcal{D}_{\vartheta,\rho})$. The map*

$$\varphi \mapsto \mathsf{T}_\varphi \pi^\vartheta_\rho(\mathcal{Q}_{\vartheta,\rho}) = \ker \mathbf{I}_\varphi$$

is an analytic diffeomorphism between $\pi^\vartheta_\rho(\mathcal{Q}_{\vartheta,\rho})$ and directions in the relative interior of $\mathscr{L}_{\vartheta,\rho}$.

We now prove the following proposition.

PROPOSITION 5.9.2. *The map $\varphi \mapsto \mu^\varphi$ is an analytic homeomorphism from the manifold $\pi^\vartheta_\rho(\mathcal{Q}_{\vartheta,\rho})$ to the space of Patterson–Sullivan measures supported on $\xi^\vartheta(\partial\Gamma)$.*

Proof. By uniqueness in Corollary 5.5.3, the map $\varphi \mapsto \mu^\varphi$ is well defined and injective. Regularity follows from Remark 3.3.3 and analytic variation of equilibrium states (Theorem 2.3.3). Surjectivity follows from Proposition 5.8.1. □

Proposition 5.9.2 was previously established by Lee and Oh [46, Theorem 1.3] for $\mathbb{K} = \mathbb{R}$ and Δ-Anosov Zariski-dense representations. The convergence domain $\mathcal{D}_{\Delta,\rho}$ is dual to Quint's *growth indicator function* [61].

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Remark 5.10. Observe that, by definition, a β_φ-Patterson–Sullivan measure has its support on $\partial\Gamma$, and thus on $\xi^\vartheta(\partial\Gamma)$ when pushed to $\mathcal{F}_\vartheta$. One could more generally study measures on $\mathcal{F}_\vartheta$ verifying

$$\frac{d(\gamma_\rho)_*\nu}{d\nu}(\cdot) = q^{-\delta\cdot\varphi(\,\beta_\theta(\gamma_\rho^{-1},\cdot))}, \tag{28}$$

without imposing conditions on their support. Such measures exist, for example, the K-invariant measure on $\mathcal{F}_\vartheta$, but their exponent is too large. The question would be totally settled if the following had an affirmative answer: Is the support of a measure verifying equation (28) with $\delta = \delta^\varphi$ necessarily contained on $\xi^\vartheta(\partial\Gamma)$?

5.11. *Variation of the critical hypersurface.* We record the following consequence of Bridgeman *et al* [17, §6.3]

COROLLARY 5.11.1. (Bridgeman et al [17]) *Let $\{\rho_u : \Gamma \to \mathsf{G}\}_{u\in D}$ be an analytic family of ϑ-Anosov representations. Then, Livšic-cohomology class of the Ledrappier potential $\mathcal{J}_{\beta^{\rho_u}} : \widetilde{\mathsf{U}\Gamma} \to \mathsf{E}_\vartheta$ associated varies analytically with u.*

Consequently, we can apply Corollary 2.6.5 to obtain the following corollary.

COROLLARY 5.11.2. *Let $\{\rho_u : \Gamma \to \mathsf{G}\}_{u\in D}$ be an analytic family of ϑ-Anosov representations. Then the critical hypersurface $\mathcal{Q}_{\vartheta,\rho_u}$ varies analytically (on compact sets of E_ϑ) with the representation u.*

5.12. *Consequences of the skew-product structure.* Consider $\varphi \in \operatorname{int}(\mathscr{L}_{\vartheta,\rho})^*$. By Remark 5.5.4, we can freely apply results from §§3.4 and 3.5 to the cocycle β_φ.

Let $\mathsf{u}_\varphi = \mathsf{T}_{\hbar_\varphi\varphi}\mathcal{Q}_{\vartheta,\rho} \in \mathbb{P}(\mathscr{L}_{\vartheta,\rho})$ be the *growth direction of* φ. By §3.4, the half-line $\mathsf{u}_\varphi \cap \mathscr{L}_{\vartheta,\rho}$ lies in the relative interior of $\mathscr{L}_{\vartheta,\rho}$ (and every direction in this relative interior is obtained in this fashion).

Consider the *φ-Bowen–Margulis* measure Ω^φ on $\Gamma\backslash(\partial^2\Gamma \times \mathsf{E}_\vartheta)$, defined as induction on the quotient by

$$e^{-\delta^\varphi[\cdot,\cdot]_\varphi}\bar{\mu}^\varphi \otimes \mu^\varphi \otimes d\,\mathrm{Leb}. \tag{29}$$

Consider $u_\varphi \in \mathsf{u}_\varphi$ with $\varphi(u_\varphi) = 1$ and denote by $\omega^\varphi = (\omega_t^\varphi : \Gamma\backslash(\partial^2\Gamma \times \mathsf{E}_\vartheta) \to \Gamma\backslash(\partial^2\Gamma \times \mathsf{E}_\vartheta))_{t\in\mathbb{R}}$ the *directional flow* induced on the quotient of

$$t \cdot (x, y, v) = (x, y, v - tu_\varphi).$$

The ergodic dichotomy from §3.6 then gives the following theorem.

THEOREM 5.12.1. *Assume $\mathbb{K} = \mathbb{R}$ and ρ is Zariski-dense, and let $\varphi \in \operatorname{int}(\mathscr{L}_{\vartheta,\rho})^*$. If $|\vartheta| \leq 2$, then the directional flow ω^φ is ergodic with respect to Ω^φ, in particular, $\mathcal{K}(\omega^\varphi)$ has total mass. If $|\vartheta| \geq 4$, then $\mathcal{K}(\omega^\varphi)$ has measure* 0.

Proof. The non-arithmeticity assumption for β holds by Benoist's Theorem 4.10.2 and thus Corollary 3.6.1 applies. □

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5.13. *Directional conical points.* The present task is to study the set of points on $\partial\Gamma$ that are *conical* in the direction u_φ.

Consider $y \in \partial\Gamma$ and a sequence $\{\gamma_n\} \subset \Gamma$ with $\gamma_n \to y$. Then we say that γ_n converges *conically* to y if for every $z \in \partial\Gamma - \{y\}$, the sequence $\gamma_n^{-1}(z, y)$ remains on a compact subset of $\partial^2\Gamma$.

Remark 5.13.1. Equivalently, since any compact subset of $\partial^2\Gamma$ is contained in a compact subset of the form $\{(a, b) : d_{\partial\Gamma}(a, b) \geq \kappa\}$ for a fixed κ, one has that $\gamma_n \to y$ conically if and only if there exists a geodesic ray $\{\alpha_i\}_0^\infty$ on Γ, converging to y, such that $\{\gamma_n\}$ is at bounded Hausdorff distance from $\{\alpha_i\}_0^\infty$. It follows then the existence of constants, c_0, c_1, such that for all n, one has

$$\gamma_n^{-1} y \in C_\infty^{c_0,c_1}(\gamma_n). \tag{30}$$

Let us fix an (auxiliary) Euclidean norm on E_ϑ and denote by $B(v, r)$ the associated ball of radius r about v. The *tube* of size r about u_φ is the tubular neighborhood:

$$\mathbb{T}_r(\mathsf{u}_\varphi) = \{v \in \mathsf{E}_\vartheta : B(v, r) \cap \mathsf{u}_\varphi \neq \emptyset\}.$$

Definition 5.13.2. We say that $y \in \partial\Gamma$ is (r, φ)-*conical* if there exists a conical sequence $\{\gamma_n\} \subset \Gamma$ converging to y such that for all n,

$$a_\vartheta((\gamma_n)_\rho) \in \mathbb{T}_r(\mathsf{u}_\varphi).$$

We say that y is φ-conical if it is (r, φ)-conical for some r.

Let us denote by $\partial_{r,\varphi}\Gamma \subset \partial\Gamma$ the set of (r, φ)-conical points and by $\partial_\varphi\Gamma$ the set of φ-conical points. We now establish the following dichotomy.

THEOREM 5.13.3. *Assume $\mathbb{K} = \mathbb{R}$ and that ρ is Zariski-dense. If $|\vartheta| \leq 2$, then $\mu^\varphi(\partial_\varphi\Gamma) = 1$; if $|\vartheta| \geq 4$, then $\mu^\varphi(\partial_\varphi\Gamma) = 0$.*

Theorem 5.13.3 follows directly from Theorem 5.12.1 and the following proposition. Let us denote by $\mathsf{p} : \partial^2\Gamma \times \mathsf{E}_\vartheta \to \Gamma\backslash(\partial^2\Gamma \times \mathsf{E}_\vartheta)$ the quotient projection.

PROPOSITION 5.13.4. *A point $y \in \partial\Gamma$ belongs to $\partial_\varphi\Gamma$ if and only if for every pair $(x, v) \in (\partial\Gamma - \{y\}) \times \mathsf{E}_\vartheta$, one has $\mathsf{p}(x, y, v) \in \mathcal{K}(\omega^\varphi)$.*

Proof. If $(x, y, v) \in \partial^2\Gamma \times \mathsf{E}_\vartheta$ is such that $y \in \partial_\varphi\Gamma$, then consider $r > 0$ and $\gamma_n \to y$ conically such that $a((\gamma_n)_\rho) \in \mathbb{T}_r(\mathsf{u}_\varphi)$. By equation (30), there exists ε given by Proposition 5.6.1 (only depending on c_0 and c_1) such that for all n,

$$\xi^\vartheta(y) \in (\gamma_n)_\rho B_{\vartheta,\varepsilon}((\gamma_n)_\rho).$$

Consequently, equation (23) gives

$$\|\beta(\gamma_n^{-1}, y) + a_\vartheta(\rho(\gamma_n))\| = \| - \beta(\gamma_n, \gamma_n^{-1} \cdot y) + a_\vartheta(\rho(\gamma_n))\| < K_\varepsilon. \tag{31}$$

By assumption, $a_\vartheta(\rho(\gamma_n)) \in \mathbb{T}_r(\mathsf{u}_\varphi)$ and one finds thus a divergent sequence $t_n \in \mathbb{R}_+$ such that

$$\|\beta(\gamma_n^{-1}, y) + t_n u_\varphi\| < K', \tag{32}$$

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for some K' only depending on r and ε. The sequence

$$\omega^{\varphi}_{-t_n}\gamma_n^{-1}(x, y, v) = (\gamma_n^{-1}x, \gamma_n^{-1}y, v - \beta(\gamma_n^{-1}, y) - t_n u_{\varphi})$$

is thus contained in $\{(z, w) \in \partial^2\Gamma : d_{\partial\Gamma}(z, w) > \kappa\} \times B(v, K')$, for some κ only depending on $d_{\partial\Gamma}(x, y)$, in particular, $\mathsf{p}(x, y, v) \in \mathcal{K}(\omega^{\varphi})$ as desired.

Reciprocally, if $\mathsf{p}(x_0, y_0, v_0) \in \Gamma\backslash(\partial^2\Gamma \times \mathsf{E}_{\vartheta})$ belongs to $\mathcal{K}(\omega^{\varphi})$, let B be a bounded open set to which the ω^{φ}-orbit of $\mathsf{p}(x_0, y_0, v_0)$ returns to unbounded. Considering an accumulation point of the orbit points in B, we can assume that $\mathrm{B} = \mathsf{p}(\tilde{\mathrm{B}})$ for some $\tilde{\mathrm{B}}$ of the form

$$\{(z, w) \in \partial^2\Gamma : d_{\partial\Gamma}(z, w) \geq \kappa'\} \times B(v, c).$$

We obtain thus divergent sequences $\{\gamma_n\} \subset \Gamma$ and $\{t_n\} \subset \mathbb{R}^+$ such that for all n,

$$d_{\partial\Gamma}(\gamma_n^{-1}x_0, \gamma_n^{-1}y_0) > \kappa' \quad \text{and} \quad \|\beta(\gamma_n^{-1}, y_0) + t_n u_{\varphi}\| \leq K''. \tag{33}$$

Considering subsequences, we can assume that $\gamma_n^{-1}x_0 \to x_\infty$ and $\gamma_n^{-1}y_0 \to y_\infty$. Necessarily, $x_\infty \neq y_\infty$ since they are at least κ' apart. The sequence $\{\gamma_n\}$ is thus conical, but it is still to be determined whether it converges to x_0 or to y_0.

Using the last inequality in equation (33), we deduce, since $t_n \to +\infty$, that for all $\sigma \in \vartheta$,

$$\varpi_\sigma(\beta(\gamma_n^{-1}, y_0)) \to -\infty.$$

By definition of β and the interpretation of the Buseman–Iwasawa cocycle via representations (equation (21)), one has $\log(\|\phi_\sigma\rho(\gamma_n^{-1})v\|/\|v\|) \to -\infty$ for any non-vanishing $v \in \Xi_{\phi_\sigma}(\xi^{\vartheta}(\gamma_n^{-1}y_0))$, or equivalently, as $n \to \infty$,

$$\frac{\|\phi_\sigma\rho(\gamma_n^{-1})v\|}{\|v\|} \to 0.$$

We now use a standard linear algebra computation to conclude that

$$\sin\measuredangle(\Xi_{\phi_\sigma}(\xi^{\vartheta}(y_0)), U_{d-1}(\phi_\sigma(\gamma_n)_\rho)) \to 0.$$

LEMMA 5.13.5. (Bochi, Potrie, and Sambarino [10, Lemma A.3]) *Let $A \in \mathsf{GL}_d(\mathbb{R})$ have a gap at α_1, then for every $v \in \mathbb{R}^d$ one has*

$$\frac{\|Av\|}{\|v\|} \geq \|A\| \sin\measuredangle(\mathbb{R}\cdot v, U_{d-1}(A^{-1})).$$

By Lemma 5.2.4, one concludes that

$$U_1(\phi_\sigma(\rho(\gamma_n))) \to \Xi_{\phi_\sigma}(\xi^{\vartheta}(y_0)),$$

as $n \to \infty$ for all $\sigma \in \vartheta$. Again by Lemma 5.2.4, one has $\gamma_n \to y_0$ as $n \to \infty$ (in $\Gamma \cup \partial\Gamma$) and thus, by conicality of $\{\gamma_n\}$, that for all $z \in \partial\Gamma - \{y_0\}$, it holds $\gamma_n^{-1}z \to x_\infty$. It follows then that

$$U_{d-1}(\phi_\sigma\rho(\gamma_n)^{-1}) \to \Xi^*_{\phi_\sigma}(\xi^{\mathrm{i}\,\vartheta}(x_\infty)),$$

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and, since $\gamma_n^{-1}y_0 \to y_\infty \neq x_\infty$, that

$$\measuredangle(\Xi_{\phi_\sigma}(\xi^\vartheta(\gamma_n^{-1}y_0)), U_{d-1}(\phi_\sigma\rho(\gamma_n)^{-1})) > \kappa'.$$

Since the latter lower bound holds for all $\sigma \in \vartheta$, one concludes that $\xi^\vartheta(\gamma_n^{-1}y_0)$ belongs to the Cartan basin $B_{\vartheta,\kappa''}(\rho(\gamma_n))$. Thus, as in equation (31), one has

$$\|\beta(\gamma_n^{-1}, y_0) + a_\vartheta(\rho(\gamma_n))\| \leq K,$$

for some K only depending on κ''. The latter, together with the second inequality from equation (33), implies that y_0 is φ-conical, as desired. □

Proof of Theorem 5.13.3. Consider a positive ε. Fix $y \in \partial_\varphi\Gamma$, $x \in \partial\Gamma - \{y\}$ and two neighborhoods A^- and A^+ of x and y, respectively, so that for all $(z, w) \in A^-, A^+$, one has $|[z, w]_\varphi - [x, y]_\varphi| < \varepsilon$. Pick also an arbitrary $T > 0$ so that the quotient projection p is injective on $\tilde{\mathrm{B}} = A^- \times A^+ \times B(0, T)$. We can thus compute the measure of $\mathrm{B} = \mathsf{p}(\tilde{\mathrm{B}})$ by equation (29).

If we let $\tilde{\mathcal{K}}(\omega^\varphi) = \mathsf{p}^{-1}(\mathcal{K}(\omega^\varphi))$, then the lemma above asserts that

$$A^- \times (A^+ \cap \partial_\varphi\Gamma) \times B(0, T) = \tilde{\mathcal{K}}(\omega^\varphi) \cap \tilde{\mathrm{B}}.$$

If $|\vartheta| \leq 2$, Theorem 5.12.1 states that $\Omega^\varphi(\tilde{\mathrm{B}}) = \Omega^\varphi(\tilde{\mathcal{K}}(\omega^\varphi) \cap \tilde{\mathrm{B}})$, which implies, up to $e^{-\delta^\varphi\varepsilon}$, that

$$\mu^\varphi(A^+) = \mu^\varphi(A^+ \cap \partial_\varphi\Gamma).$$

Since ε is arbitrary, one concludes $\mu^\varphi(\partial_\varphi\Gamma) = 1$. However, if $|\vartheta| \geq 4$, then we have $\Omega^\varphi(\tilde{\mathcal{K}}(\omega^\varphi)) = 0$, so $\mu^\varphi(A^+ \cap \partial_\varphi\Gamma) = 0$ and the theorem is proved. □

A. *Appendix. Ergodicity of skew-products with values on* $\mathbb{R}$

We freely use notation from Proposition 2.4.2 which we intend to prove. The proof presented here is mainly a collection of results.

We say that K is *recurrent* if for every measurable set $A \subset \Sigma$ with $\nu(A) > 0$ and every neighborhood $N(0)$ of 0 in V, there exists $n \in \mathbb{Z} - \{0\}$ such that one has

$$\nu\bigg(A \cap \sigma^{-n}A \cap \{x : \sum_{k=0}^{n} \mathsf{K}(\sigma^i x) \in N(0)\}\bigg) > 0.$$

It is proven by Schmidt [**67**, Theorem 5.5] that K is recurrent if and only if the skew-product $f^{\mathsf{K}} : \Sigma \times \mathbb{R} \to \Sigma \times \mathbb{R}$ is conservative (see Aaronson's book [**1**, §1.1] for the definition). It is moreover a general fact that mean-zero cocycles over the reals are conservative, see [**1**, Corollary 8.1.5] from which we state here a particular case.

COROLLARY A.1. *Since, by assumption,* $\int \mathsf{K}\, d\nu = 0$, *the cocycle* f^{K} *is conservative and so* K *is recurrent.*

The proof of Proposition 2.4.2 ends with the following theorem of Coelho (obtained by the combination of Example 2.4 and Corollary 3.4 of Coelho [**26**]), specific to sub-shifts and equilibrium states.

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THEOREM A.0.1. (Coelho [26]) *Assume* K *is non-arithmetic and let* ν *be an equilibrium state of* σ *for a Hölder potential. Then,* f *is ergodic with respect to* Ω_ν *if and only if* K *is recurrent.*

B. *Appendix. Mixing*

In this appendix, we give a quick outline of the proof of Theorem 2.5.2. We use small modifications of classical computations performed by Babillot [3] as well as by Babillot and Ledrappier [5], Ledrappier and Sarig [45], and more recently by Oh and Pan [52] and Chow and Sarkar [25], where an extra parameter (an holonomy with values on a compact group) has been added to the Ruelle operator. We thank M. Chow and P. Sarkar for pointing out an issue in the argument presented by Sambarino [66], Ledrappier for suggesting the reference [45], and Oh for pinpointing the reference [52].

It is first convenient to straighten the flow action by means of twisting the r-action.

LEMMA B.1. *Let* $U = \mathbb{R} \times W$, *and let* $k : \Sigma \to U$ *be* $k(x) = (r(x), \int_0^{r(x)} K(x,s)\,ds)$, *then:*

- *there exists* $\varphi \in U^*$ *such that* $\varphi(k) = r > 0$,
- $\int k d\nu = (\int r d\nu, 0) \neq 0$;
- *there exists a bi-Hölder homeomorphism* $E : \Sigma_r \times W \to \Sigma \times U/\hat{k}$, *where*
$$\hat{k}(x,u) = (\sigma(x), u - k(x)),$$
which is a measurable isomorphism between $\bar{\Omega}$ *and* $\nu \otimes \mathrm{Leb}_U/\hat{k}$, *that conjugates* ψ *with the flow induced on the quotient by*
$$(x,u) \mapsto (x, u - t\tau),$$
where $\tau \in \int k d\nu$ *is such that* $\varphi(\tau) = 1$.

Proof. Define $E : \Sigma_r \times W \to \Sigma \times U/\hat{k}$ by $E((x,t),w) = (x,(t, w + \int_0^t K(x,s)\,ds))$. It is well defined since the above formula is equivariant. Indeed, one has

$$\begin{aligned}\int_0^{t-r(x)} K(\sigma(x),s)\,ds &= \int_{r(x)}^t K(\sigma(x), s - r(x))\,ds\\ &= \int_{r(x)}^t K(x,s)\,ds \qquad\qquad (K \text{ is } \hat{r}\text{-invariant})\\ &= \int_0^t K(x,s)\,ds - \int_0^{r(x)} K(x,s)\,ds,\end{aligned}$$

which implies

$$\begin{aligned}E(\hat{r}(x,t),w) &= \bigg(\sigma(x),(t - r(x), w + \int_0^{t-r(x)} K(\sigma(x),s)\,ds)\bigg)\\ &= \bigg(\sigma(x),(t-r(x), w + \int_0^t K(x,s)\,ds - \int_0^{r(x)} K(x,s)\,ds)\bigg)\\ &= \hat{k}(E((x,t),w)),\end{aligned}$$

as desired. The remaining assertions follow similarly. □

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We will work from now on with this latter flow, still denoted by ψ so as not to overcharge with notation. Up to Livšic-cohomology, we may assume that k is defined on Σ^+.

By measure-theoretic arguments, we consider $F, G : \Sigma^+ \times U \to \mathbb{R}$ that we can assume have separated variables, that is, can be written as $F(x, u) = p_F(x)v_F(u)$, with p_F and p_G Hölder-continuous, and v_F and v_G smooth with compact support. We have to show that, as $t \to \infty$,

$$t^{\dim W/2}\bar{\Omega}(F \cdot G \circ \psi_t) \to \bar{\Omega}(F)\bar{\Omega}(G).$$

Tracing back the definitions, one is brought up to understanding the limit as $t \to \infty$ of

$$t^{\dim W/2}\bigg(\int_{\Sigma^+\times U} \sum_{n\in\mathbb{N}} F(x, u)G(\sigma^n x, u - S_n k(x) - t\tau)\, d\nu\, d\,\mathrm{Leb}_U\bigg), \tag{B.1}$$

where $S_n k(x) = \sum_{i=0}^{n} k(\sigma^i(x))$ is the Birkhoff sum. We focus on the integral between brackets, only to multiply at the very end of our computation by $\sqrt{t}^{d-1}$, where $d = \dim U$. The above integral becomes

$$\int_{\Sigma^+\times U} \sum_{n\in\mathbb{N}} p_F(x)p_G(\sigma^n x)v_F(u)v_G(u - S_n k(x) - t\tau)\, d\nu(x)\, d\,\mathrm{Leb}_U(u).$$

Recall that, by assumption, there is $\varphi \in U^*$ so that $\varphi(k) = hr > 0$ and that $P(-hr) = 0$, so up to Livšic-cohomology, we can assume that $-\varphi(k) = -hr$ is *normalized*, that is, so that the Ruelle operator, defined by

$$\mathcal{L}_\varphi\Phi(x) = \sum_{y:\sigma(y)=x} e^{-\varphi(k(y))}\Phi(y),$$

verifies $\mathcal{L}_\varphi^*\nu = \nu$, in particular, for every pair of Hölder-continuous functions j, l on Σ^+, one has $\int_{\Sigma^+} j(\sigma x)l(x)\, d\nu = \int_{\Sigma^+} j(x)(\mathcal{L}_\varphi l)(x)\, d\nu$.

Denote by Leb_{U^*} the Lebesgue measure on U^* defined by the Fourier inversion formula

$$v_G(w) = \int_{U^*} e^{i\psi(w)}\mathcal{F}v_G(\psi)\, d\,\mathrm{Leb}_{U^*}(\psi)$$

for the Fourier transform $\mathcal{F}v_G$ if v_G. As in [5, §2.3] we can, and will, assume that $\mathcal{F}v_G$ is of class C^N for some $N > (d-1)/2$ and has compact support.

We will suppress the notation ν, Leb_U, and Leb_{U^*} from the integrals from now on. The desired integral, equation (B.1), then becomes

$$\begin{aligned}
&\int_{\Sigma^+\times U} \sum_{n\in\mathbb{N}} (\mathcal{L}_\varphi^n p_F)(x)p_G(x)v_F(u)v_G(u - S_n k(x) - t\tau)\, dx\, du \\
&= \int_{\Sigma^+\times U} \sum_{n\in\mathbb{N}} (\mathcal{L}_\varphi^n p_F)(x)p_G(x)v_F(u) \int_{U^*} e^{i\psi(u - S_n k(x) - t\tau)}\mathcal{F}v_G(\psi)\, d\psi\, dx\, du, \\
&= \int_{\Sigma^+\times U} \int_{U^*} \sum_{n\in\mathbb{N}} (\mathcal{L}_{\varphi+i\psi}^n p_F)(x)p_G(x)v_F(u)e^{i\psi(u-t\tau)}\mathcal{F}v_G(\psi)\, d\psi\, dx\, du \\
&= \int_{\Sigma^+\times U} p_G(x)v_F(u) \int_{U^*} \sum_{n\in\mathbb{N}} (\mathcal{L}_{\varphi+i\psi}^n p_F)(x)e^{i\psi(u-t\tau)}\mathcal{F}v_G(\psi)\, d\psi\, dx\, du.
\end{aligned} \tag{B.2}$$

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We seek thus to understand the nature of $\psi \mapsto \sum_{n\in\mathbb{N}} \mathcal{L}^n_{\varphi+i\psi}$, for which one is brought to understand the spectral radius r_ψ of $\mathcal{L}_{\varphi+i\psi}$. Applying [53, Ch. 4], we obtain that for every $\psi \in U^*$, one has $\mathsf{r}_\psi \in (0, 1]$. We then distinguish two situations.

The spectral radius r_ψ *is smaller than* 1. In which case

$$\eta \mapsto \sum_{n\in\mathbb{N}} \mathcal{L}^n_{\varphi+i\eta} = (1-\mathcal{L}_{\varphi+i\eta})^{-1},$$

is analytic on a neighborhood of ψ.

The spectral radius r_ψ *equals* 1. One has then the following theorem.

THEOREM B.1. (Parry and Pollicott [53, Ch. 4]) *One has $\mathsf{r}_\psi = 1$ if and only if there exists a Hölder-continuous $w_\psi : \Sigma^+ \to \mathbb{S}^1$ such that for all $x \in \Sigma^+$, one has*

$$e^{i\psi(k(x))} = \lambda_\psi \frac{w_\psi(\sigma(x))}{w_\psi(x)}.$$

In this situation, the function w_ψ is unique up to scalars.

Applying moreover [53, Theorem 4.5] and the perturbation theorem [53, Proposition 4.6], there exists a neighborhood O_ψ of ψ such that for all $\eta \in O_\psi$, one has

$$\mathcal{L}_{\varphi+i\eta} = \lambda_\eta Q_\eta + N_\eta, \tag{B.3}$$

where Q_η is a rank-one projector, N_η is an operator with spectral radius strictly smaller than $\mathsf{r}_\eta = |\lambda_\eta|$, and such that $Q_\eta N_\eta = N_\eta Q_\eta = 0$. The above objects are analytic on O_ψ. The operator $M_\psi = \sum_{n\in\mathbb{N}} N^n_\psi$ is hence well defined and analytic on O_ψ. Observe that, as we have assumed $-\varphi(k)$ to be normalized, one has

$$Q_0(f)(x) = \bigg(\int_{\Sigma^+} f\,d\nu\bigg)\cdot 1. \tag{B.4}$$

Remark B.2. Since k has a dense group of periods on U, $\psi(k) : \Sigma^+ \to \mathbb{R}$ has a dense group of periods as soon as $\psi \neq 0$, and [53, Theorem 4.5] implies that $\{\lambda^n_\psi : n \in \mathbb{Z}\}$ is dense in $\mathbb{S}^1 = \partial\mathbb{D}$. In particular, $\lambda_\psi \neq 1$, unless $\psi = 0$. Consequently, if $\psi \neq 0$, then for all $\eta \in O_\psi$, the operator

$$\frac{Q_\eta}{1-\lambda_\eta} - M_\eta$$

is well defined and analytic on O_ψ.

LEMMA B.3. *If $\mathsf{r}_\psi = 1$, then for all $\eta \in O_\psi - \{\psi\}$, it holds $\mathsf{r}_\eta < 1$. Consequently,*

$$\eta \mapsto \sum_{n\in\mathbb{N}} \mathcal{L}^n_{\varphi+i\eta} = (1-\mathcal{L}_{\varphi+i\eta})^{-1} = \frac{Q_\eta}{1-\lambda_\eta} + M_\eta$$

is analytic on $O_\psi - \{\psi\}$ and, if $\psi \neq 0$, it extends analytically to O_ψ, as the right-hand-side of the equation is well defined on ψ.

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Proof. As $\eta \mapsto \lambda_\eta$ is analytic on $\mathcal{O}_\psi$, together with the above density result, it follows that there is a neighborhood (possibly smaller but) still denoted by $\mathcal{O}_\psi$ such that if $|\lambda_\eta| = 1$, then $\lambda_\eta = \lambda_\psi$. One has then, applying Theorem B.1, that

$$e^{i(\psi-\eta)(k(x))} = \lambda_\psi \frac{w_\psi(\sigma(x))}{w_\psi(x)} \lambda_\eta^{-1} \frac{w_\eta(x)}{w_\eta(\sigma(x))} = \frac{(w_\psi/w_\eta)(\sigma(x))}{(w_\psi/w_\eta)(x)}.$$

The remark and Theorem B.1 give $\psi - \eta = 0$ and so the lemma is established. □

One obtains then that the operator $\sum_{n\in\mathbb{N}} \mathcal{L}^n_{\varphi+i\psi}$ is well defined and varies analytically on ψ except at $\psi = 0$. One is thus taken to localize the integral in equation (B.2) about 0. To that end, one considers an auxiliary C^∞ function $\kappa : U^* \to \mathbb{R}$, supported on the neighborhood $\mathcal{O}_0$, where equation (B.3) holds, with $\kappa(0) = 1$, and we seek to understand the modified integral over U^* on equation (B.2) given, for $(x, u) \in \Sigma^+ \times U$, by

$$\int_{U^*} (1 - \kappa(\psi)) \sum_{n\in\mathbb{N}} (\mathcal{L}^n_{\varphi+i\psi} p_F)(x) e^{i\psi(u-t\tau)} \mathcal{F}v_G(\psi)\, d\psi. \tag{B.5}$$

Consider from §2.6 the critical hypersurface $\mathcal{Q}_k \subset U^*$. It follows from Babillot and Ledrappier [5] that one has $\mathcal{Q}_k = \mathbf{P}^{-1}(0)$ and the tangent space $\mathsf{T}_\varphi\mathcal{Q}_k = \{\psi \in U^* : \psi(\tau) = 0\}$. For $\psi \in U^*$, let

$$\psi = s_\psi \varphi + \psi_0 \tag{B.6}$$

be its decomposition along $U^* = \mathbb{R}\varphi \oplus \mathsf{T}_\varphi\mathcal{Q}_k$. Decomposing $d\psi$ as $ds\,d\psi_0$, the integral in equation (B.5) becomes

$$\int_{\mathbb{R}} e^{-its} \int_{\mathsf{T}_\varphi\mathcal{Q}_k} e^{i\psi(u)} (1 - \kappa(\psi)) \sum_{n\in\mathbb{N}} (\mathcal{L}^n_{\varphi+i\psi} p_F)(x) \mathcal{F}v_G(\psi)\, d\psi_0\, ds = O(t^{-N}) \tag{B.7}$$

as it is the Fourier transform of the integral over $\mathsf{T}_\varphi\mathcal{Q}_k$, which is, by Lemma B.3, as regular as $\mathcal{F}v_G(\psi)$, and this function was chosen to be of class C^N for some $N > (d-1)/2$.

We can thus focus on the integral from equation (B.2) localized about 0, so it becomes, using again Lemma B.3,

$$\begin{aligned}\int_{\Sigma^+\times U} \int_{U^*} \kappa(\psi) \bigg(\frac{(Q_\psi p_F)x}{1 - \lambda_\psi} + (M_\psi p_F)(x) \bigg) p_G(x) v_F(u) e^{i\psi(u-t\tau)} \mathcal{F}v_G(\psi)\, d\psi\, dx\, du \\ + O(t^{-N}).\end{aligned} \tag{B.8}$$

We first treat the term containing M_ψ, which is dealt with as we did with equation (B.5). Indeed, for the same reasons, one has for all $(x, u) \in \Sigma^+ \times U$, the integral

$$\begin{aligned}&\int_{U^*} \kappa(\psi)(M_\psi p_F)(x) e^{i\psi(u-t\tau)} \mathcal{F}v_G(\psi)\, d\psi \\ &\quad = \int_{\mathbb{R}} e^{-its} \int_{\mathsf{T}_\varphi\mathcal{Q}_k} e^{i\psi(u)} \kappa(\psi)(M_\psi p_F)(x) \mathcal{F}v_G(\psi)\, d\psi_0\, ds \\ &\quad = O(t^{-N}).\end{aligned} \tag{B.9}$$

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We attempt then to understand the integral

$$\int_{U^*} \kappa(\psi) \frac{(Q_\psi p_F)x}{1-\lambda_\psi} e^{i\psi(u-t\tau)} \mathcal{F}v_G(\psi)\, d\psi, \tag{B.10}$$

the issue being the singularity at $\psi = 0$ of $1/(1-\lambda_\psi)$. To that end, consider the function $\mathbf{Q} : \mathsf{T}_\varphi Q_k \to \mathbb{R}$ defined implicitly by the equation

$$\mathbf{Q}(\psi_0)\varphi + \psi_0 \in Q_k.$$

It is analytic, critical at 0 with $\mathbf{Q}(0) = 1$, and has positive-definite Hessian at 0, $\mathrm{Hess}_0\, \mathbf{Q}$. Using Taylor expansion, one writes

$$\mathbf{Q}(\psi_0) = 1 + (1/2)\, \mathrm{Hess}_0\, \mathbf{Q}(\psi_0) + O(\|\psi_0\|^2).$$

One applies the Weierstrass preparation theorem [38, Theorem 7.5.1] to express $1 - \lambda_\psi$ about 0 as

$$1 - \lambda_\psi = a(\psi)(is_\psi - (1/2)\, \mathrm{Hess}_0\, \mathbf{Q}(\psi_0) - O(\|\psi_0\|^2)),$$

(recall the decomposition of ψ from equation (B.6)) where a is real-analytic and $a(0) = h \int r\, d\nu$ (see [5, p. 37] or [45, p. 17] for details). Whence, as in [5, Lemma 2.3], using the formula $1/z = -\int_0^\infty e^{Tz} dT$, one has

$$\frac{1}{1-\lambda_\psi} = -\frac{1}{a(\psi)} \int_0^\infty e^{T(is_\psi - (1/2)\, \mathrm{Hess}_0\, \mathbf{Q}(\psi_0) - O(\|\psi_0\|^2))}\, dT,$$

and equation (B.10) becomes, denoting $C(\psi, x) = \kappa(\psi)(Q_\psi p_F)(x)/a(\psi)\mathcal{F}v_G(\psi)$ to lighten the notation,

$$\begin{aligned} &\int_{U^*} e^{i\psi(u-t\tau)} \kappa(\psi) \frac{(Q_\psi p_F)(x)}{a(\psi)} \mathcal{F}v_G(\psi) \int_0^\infty e^{T(is_\psi - (1/2)\, \mathrm{Hess}_0\, \mathbf{Q}(\psi_0) - O(\|\psi_0\|^2))}\, dT\, d\psi_0\, ds \\ &\quad = \int_0^\infty \int_{\mathbb{R}} e^{-i(t-T)s} \int_{\mathsf{T}_\varphi Q_k} C(\psi, x) e^{i\psi(u) - T(\mathrm{Hess}_0\, \mathbf{Q}(\psi_0)/2 + O(\|\psi_0\|^2))}\, d\psi_0\, ds\, dT. \end{aligned} \tag{B.11}$$

Using Taylor series on the ψ-variable, we may, and will, assume that $C(\psi, x)$ is of the form $c_x(s_\psi)b(\psi_0)$. Equation (B.11) now becomes

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}} e^{-i(t-T-\varphi(u))s} c_x(s)\, ds \int_{\mathsf{T}_\varphi Q_k} b(\psi_0) e^{i\psi_0(u) - T(\mathrm{Hess}_0\, \mathbf{Q}(\psi_0)/2 + O(\|\psi_0\|^2))}\, d\psi_0\, dT \\ &\quad = \int_0^\infty \mathcal{F}c_x(t - T - \varphi(u)) \int_{\mathsf{T}_\varphi Q_k} b(\psi_0) e^{i\psi_0(u) - T(\mathrm{Hess}_0\, \mathbf{Q}(\psi_0)/2 + O(\|\psi_0\|^2))}\, d\psi_0\, dT \\ &\quad = \int_{\varphi(u)}^\infty \mathcal{F}c_x(t - T) \int_{\mathsf{T}_\varphi Q_k} b(\psi_0) e^{i\psi_0(u) - (T-\varphi(u))(\mathrm{Hess}_0\, \mathbf{Q}(\psi_0)/2 + O(\|\psi_0\|^2))}\, d\psi_0\, dT, \\ &\quad = \int_{\max(t/2, \varphi(u))}^\infty \mathcal{F}c_x(t - T) \\ &\qquad \times \int_{\mathsf{T}_\varphi Q_k} b(\psi_0) e^{i\psi_0(u) - (T-\varphi(u))(\mathrm{Hess}_0\, \mathbf{Q}(\psi_0)/2 + O(\|\psi_0\|^2))} d\psi_0\, dT + O(t^{-N}), \end{aligned} \tag{B.12}$$

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where the last equality is, as in [5, p. 27], essentially due to the fact that c_x has compact support and is of class C^N. Applying the change of variables $\psi_0 \mapsto \psi_0/\sqrt{T}$, the last integral becomes

$$\begin{aligned}
&\int_{\max(t/2,\varphi(u))}^{\infty} \frac{\mathcal{F}c_x(t-T)}{\sqrt{T}^{d-1}} \\
&\quad\times \int_{\mathsf{T}_\varphi \mathsf{Q}_\mathsf{k}} b\bigg(\frac{\psi_0}{\sqrt{T}}\bigg) e^{i(\psi_0/\sqrt{T})(u)-(1-\varphi(u)/T)(\mathrm{Hess}_0\,\mathbf{Q}(\psi_0)/2+O(\|\psi_0\|^2/T))} d\psi_0\, dT + O(t^{-N}) \\
&\quad= -\int_{-\infty}^{\min(t/2,t-\varphi(u))} \frac{\mathcal{F}c_x(S)}{\sqrt{t-S}^{d-1}} \int_{\mathsf{T}_\varphi \mathsf{Q}_\mathsf{k}} b\bigg(\frac{\psi_0}{\sqrt{t-S}}\bigg) \\
&\qquad\times e^{i(\psi_0/\sqrt{t-S})(u)-(1-\varphi(u)/(t-S))(\mathrm{Hess}_0\,\mathbf{Q}(\psi_0)/2+O(\|\psi_0\|^2/(t-S)))}\, d\psi_0\, dS + O(t^{-N}).
\end{aligned} \tag{B.13}$$

We finally multiply by $\sqrt{t}^{d-1}$, take the limit as $t \to \infty$, and trace back our definitions to get, as we assumed $N > (d-1)/2$, the convergence of equation (B.13) (and thus that of equation (B.10)) to

$$\begin{aligned}
\lim_{t\to\infty} \sqrt{t}^{d-1} &\int_{U^*} \kappa(\psi) \frac{(Q_\psi p_F)x}{1-\lambda_\psi} e^{i\psi(u-t\tau)} \mathcal{F}v_G(\psi)\, d\psi \\
&= b(0) \int_{-\infty}^{\infty} \mathcal{F}c_x(S)dS \int_{\mathsf{T}_\varphi \mathsf{Q}_\mathsf{k}} e^{-1/2\,\mathrm{Hess}_0\,\mathbf{Q}(\psi_0)}\, d\psi_0 \\
&= c_x(0)b(0) \int_{\mathsf{T}_\varphi \mathsf{Q}_\mathsf{k}} e^{-1/2\,\mathrm{Hess}_0\,\mathbf{Q}(\psi_0)}\, d\psi_0 \\
&= \frac{Q_0(p_F)(x)\mathcal{F}v_G(0)}{a(0)} \int_{\mathsf{T}_\varphi \mathsf{Q}_\mathsf{k}} e^{-1/2\,\mathrm{Hess}_0\,\mathbf{Q}(\psi_0)}\, d\psi_0 \\
&= \frac{\int_{\Sigma^+} p_F d\nu \int_U v_G(u)\, du}{h \int_{\Sigma^+} r d\nu} \int_{\mathsf{T}_\varphi \mathsf{Q}_\mathsf{k}} e^{-1/2\,\mathrm{Hess}_0\,\mathbf{Q}(\psi_0)}\, d\psi_0,
\end{aligned} \tag{B.14}$$

where we have used equation (B.4) and the formula $\mathcal{F}v_G(0) = \int_U v_G(u)\, du$. Observe that, as $\mathrm{Hess}_0\,\mathbf{Q}$ is positive-definite, the integral $\int_{\mathsf{T}_\varphi \mathsf{Q}_\mathsf{k}} e^{-1/2\,\mathrm{Hess}_0\,\mathbf{Q}(\psi_0)}\, d\psi_0$ is finite (and non-zero).

Remark B.4. If one modifies the action $(x, u) \mapsto (x, u - t\tau)$ to consider induction on the quotient by $(x, u) \mapsto (x, u - t\tau - \sqrt{t}w_0)$ for a fixed $w_0 \in \ker\varphi$, then tracing the computations, one readily sees that

$$\begin{aligned}
\lim_{t\to\infty} \sqrt{t}^{d-1} &\int_{U^*} \kappa(\psi) \frac{(Q_\psi p_F)x}{1-\lambda_\psi} e^{i\psi(u-t\tau-\sqrt{t}w_0)} \mathcal{F}v_G(\psi)\, d\psi \\
&= \frac{\int_{\Sigma^+} p_F d\nu \int_U v_G(u)\, du}{h \int_{\Sigma^+} r d\nu} \int_{\mathsf{T}_\varphi \mathsf{Q}_\mathsf{k}} e^{-i\psi_0(w_0)} e^{-1/2\,\mathrm{Hess}_0\,\mathbf{Q}(\psi_0)}\, d\psi_0 \\
&= \frac{\int_{\Sigma^+} p_F d\nu \int_U v_G(u)\, du}{h \int_{\Sigma^+} r d\nu} \mathcal{F}\mathrm{H}(w_0),
\end{aligned}$$

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where we have defined

$$\mathrm{H}(\psi_0) = e^{-(1/2)\,\mathrm{Hess}_0(\psi_0)}.$$

Observe that, as $\mathrm{H}(\psi_0) = \mathrm{H}(-\psi_0)$, the value of the Fourier transform $\mathcal{F}\mathrm{H}(w_0) \in \mathbb{R}$.

We finally group back the equations to get the desired result. Indeed, equation (B.1) is

$$\begin{aligned}
&\sqrt{t}^{d-1} \int_{\Sigma^+\times U} \sum_{n\in\mathbb{N}} F(x,u)G(\sigma^n x, u - S_n k(x) - t\tau)\, d\nu\, d\,\mathrm{Leb}_U \\
&= \sqrt{t}^{d-1} \int p_G(x)v_F(u) \int \kappa(\psi)\frac{(Q_\psi p_F)x}{1-\lambda_\psi} e^{i\psi(u-t\tau)}\mathcal{F}v_G(\psi)\, d\psi\, du\, dx + O(1/t) \\
&= \frac{\int_{\Sigma^+} p_F d\nu \int_U v_G(u)\, du}{h\int_{\Sigma^+} r d\nu} \int p_G(x)v_F(u)\, dx\, du \int_{\mathsf{T}_\varphi \mathsf{Q}_\mathsf{k}} e^{-1/2\,\mathrm{Hess}_0\,\mathbf{Q}(\psi_0)}\, d\psi_0 + O(1/t) \\
&= \frac{\mathcal{F}\mathrm{H}(0)}{h\int_{\Sigma^+} r d\nu} \int F d\nu\, d\,\mathrm{Leb} \int G d\nu\, d\,\mathrm{Leb} + O(1/t),
\end{aligned}$$

where we have used, in the first equality, equations (B.7) and (B.9) and the fact that $N > (1/2)(d-1)$ and, in the second equality, the convergence from equation (B.14). This completes the sketch of the proof.

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